Weak Markovian Bisimulation Equivalences and Exact CTMC-Level Aggregations

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Markovian Behavioral Equivalences

- Tools for relating and manipulating formal models with an underlying continuous-time Markov chain (CTMC) semantics.
- Markovian bisimilarity: two processes are equivalent whenever they are able to mimic each other's functional and performance behavior step by step.
- Markovian testing equivalence: two processes are equivalent whenever an external observer is not able to distinguish between them from a functional or performance viewpoint by interacting with them by means of tests and comparing their reactions.
- Markovian trace equivalence: two processes are equivalent whenever they are able to perform computations with the same functional and performance characteristics.

Abstracting from Internal Actions

- When comparing nondeterministic processes, internal actions can be abstracted away via *weak behavioral equivalences*: $a \,.\, \tau \,.\, b \,.\, \underline{0} \approx a \,.\, b \,.\, \underline{0}$.

- Abstraction not always possible when comparing Markovian processes.

- Immediate internal actions: invisible and take no time [Her,Ret,MT,AB].

- An exponentially timed internal action is invisible but takes time.

- $<a, \lambda>.<\tau, \gamma>.<b, \mu>.\underline{0}$ is not equivalent to $<a, \lambda>.<b, \mu>.\underline{0}$ because a nonzero delay can be observed between a and b in the first case.

- However $<a, \lambda>.<\tau, \gamma_1>.<\tau, \gamma_2>.<b, \mu>.\underline{0} \approx <a, \lambda>.<\tau, \gamma>.<b, \mu>.\underline{0}$ if the average duration of the sequence of the two τ-actions on the left is equal to the average duration of the τ-action on the right: $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{\gamma}$ or equivalently $\gamma = \frac{\gamma_1 \cdot \gamma_2}{\gamma_1 + \gamma_2}$.

- To what extent can we abstract from exp. timed internal actions? Only from sequences (of at least two) or also from branches?
- How to define a Markovian behavioral equivalence abstracting from exponentially timed internal actions?
- Will it be a congruence with respect to typical operators?
- Will it have a sound and complete axiomatization?
- Will it induce an exact CTMC-level aggregation?

Sequential Markovian Process Calculus

- *Interested in investigating congruence and axiomatization.*
- Representation of all the CTMCs.
- Durational actions, dynamic operators, and recursion.
- $Name_{\mathrm{v}}$: set of visible action names.
- $Name = Name_{\mathrm{v}} \cup \{\tau\}$: set of all action names.
- $Act_{\mathrm{M}} = Name \times \mathbb{R}_{>0}$: set of exponentially timed actions.
- Var: set of process variables.

- Process term syntax for process language $\mathcal{PL}_{\mathrm{M}}$:

$$
\begin{array}{llll}
P ::= & \underline{0} & \text{inactive process} & \\
\mid & <a, \lambda>.P & \text{exp. timed action prefix} & (a \in \mathit{Name}, \lambda \in \mathbb{R}_{>0}) \\
\mid & P + P & \text{alternative composition} & \\
\mid & X & \text{process variable} & (X \in \mathit{Var}) \\
\mid & \mathrm{rec}\, X : P & \text{recursion} & (X \in \mathit{Var})
\end{array}
$$

- $\mathbb{P}_{\mathrm{M}}$: set of closed and guarded process terms.

- Race policy: whenever several exponentially timed actions are enabled, the action that is executed is the fastest one.

- State transition graph expressing all computations and branching points and accounting for transition multiplicity ($<a, \lambda>.\underline{0} + <a, \lambda>.\underline{0}$ vs. $<a, \lambda>.\underline{0}$).
- Every $P \in \mathbb{P}_{\mathrm{M}}$ is mapped to a labeled multitransition system $[\![P]\!]_{\mathrm{M}}$:
  - Each state corresponds to a process term into which P can evolve.
  - The initial state corresponds to P.
  - Each transition from a source state to a target state is labeled with the action that determines the corresponding state change.
- Every $P \in \mathbb{P}_{\mathrm{M}}$ is mapped to a CTMC:
  - Dropping action names from all transitions of $[\![P]\!]_{\mathrm{M}}$.
  - Collapsing all the transitions between any two states of $[\![P]\!]_{\mathrm{M}}$ into a single transition by summing up the rates of the original transitions.

- Operational semantic rules:

$$(\textsc{Pre}) \quad \frac{}{<a,\lambda>.P \xrightarrow{a,\lambda}_{\mathrm{M}} P}$$

$$(\textsc{Alt}_1) \quad \frac{P_1 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{P_1 + P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'} \qquad (\textsc{Alt}_2) \quad \frac{P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{P_1 + P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}$$

$$(\textsc{Rec}) \quad \frac{P\{rec\, X : P \hookrightarrow X\} \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{rec\, X : P \xrightarrow{a,\lambda}_{\mathrm{M}} P'}$$

Markovian Bisimilarity

- Based on the comparison of process term exit rates.

- The exit rate of a process term $P \in \mathbb{P}_{\rm M}$ is the rate at which P can execute actions of a certain name $a \in \mathit{Name}$ that lead to a certain destination $D \subseteq \mathbb{P}_{\rm M}$:

$$\boxed{\mathit{rate}(P, a, D) \;=\; \sum \{\!| \, \lambda \in \mathbb{R}_{>0} \mid \exists P' \in D.\, P \xrightarrow{a,\lambda}_{\rm M} P' \, |\!\}}$$

- Summation stems from the adoption of the race policy.

- The total exit rate of P is the reciprocal of the average sojourn time associated with P:

$$\boxed{\mathit{rate}_{\rm t}(P) \;=\; \sum_{a \in \mathit{Name}} \mathit{rate}(P, a, \mathbb{P}_{\rm M})}$$

- An equivalence relation $\mathcal{B}$ over $\mathbb{P}_{\mathrm{M}}$ is a Markovian bisimulation iff, whenever $(P_1, P_2) \in \mathcal{B}$, then for all action names $a \in \mathit{Name}$ and equivalence classes $D \in \mathbb{P}_{\mathrm{M}}/\mathcal{B}$:

$$\mathit{rate}(P_1, a, D) \ = \ \mathit{rate}(P_2, a, D)$$

- Markovian bisimilarity, denoted $\sim_{\mathrm{MB}}$, is the union of all the Markovian bisimulations.

- $\sim_{\mathrm{MB}}$ can be proved to be a congruence with respect to all the operators of SMPC as well as typical static operators like parallel composition, hiding, and relabeling.

- Sound and complete axiomatization over the set of nonrecursive process terms of SMPC:

$$
\begin{array}{lrcl}
(\mathcal{A}_{\mathrm{MB},1}) & P_1 + P_2 & = & P_2 + P_1 \\
(\mathcal{A}_{\mathrm{MB},2}) & (P_1 + P_2) + P_3 & = & P_1 + (P_2 + P_3) \\
(\mathcal{A}_{\mathrm{MB},3}) & P + \underline{0} & = & P \\
(\mathcal{A}_{\mathrm{MB},4}) & <a, \lambda_1>.P + <a, \lambda_2>.P & = & <a, \lambda_1 + \lambda_2>.P
\end{array}
$$

- $\sim_{\mathrm{MB}}$ induces a CTMC-level aggregation that is consistent with ordinary lumping and hence it is exact both at steady state and at transient state.

Definition of Weak Markovian Bisimilarity

- Basic idea: weaken the distinguishing power of $\sim_{\mathrm{MB}}$ by viewing every sequence of exponentially timed τ-actions as a single exponentially timed τ-action with the same average duration as the sequence.

- $\mathbb{P}_{\mathrm{M,u}}$: set of unstable process terms, which can initially perform only exponentially timed τ-actions.

- $\mathbb{P}_{\mathrm{M,s}}$: set of stable process terms.

- A computation c having the form $P_1 \xrightarrow{\tau,\lambda_1}_{\mathrm{M}} P_2 \xrightarrow{\tau,\lambda_2}_{\mathrm{M}} \ldots \xrightarrow{\tau,\lambda_n}_{\mathrm{M}} P_{n+1}$ is reducible iff $P_i \in \mathbb{P}_{\mathrm{M,u}}$ for all $i = 1, \ldots, n$.

- Length-abstracting measure of a reducible computation c:

$$probtime(c) = \left(\prod_{i=1}^{n} \frac{\lambda_i}{rate_{\mathrm{t}}(P_i)} \right) \cdot \left(\sum_{i=1}^{n} \frac{1}{rate_{\mathrm{t}}(P_i)} \right)$$

- The first factor is the product of the execution probabilities of the transitions of c.

- The second factor is the sum of the average sojourn times of the states traversed by c.

- Convergent measure if there are no cycles of unstable states.

- $\mathbb{P}_{\mathrm{M,nd}}$: set of nondivergent process terms.

- The weak variant of $\sim_{\mathrm{MB}}$ should:
  - Work like $\sim_{\mathrm{MB}}$ over $\mathbb{P}_{\mathrm{M,nd,s}}$.
  - Abstract from the length of reducible computations while preserving their execution probability and average duration over $\mathbb{P}_{\mathrm{M,nd,u}}$.
- *redcomp*(P, D, t): multiset of reducible computations starting from process term $P \in \mathbb{P}_{\mathrm{M,nd}}$ and reaching destination $D \subseteq \mathbb{P}_{\mathrm{M,nd}}$ whose average duration is $t \in \mathbb{R}_{>0}$.
- Need to lift measure *probtime* from a single reducible computation to a multiset of reducible computations with the same origin and destination:

$$pbtm(P, D) \; = \; \{\!| \sum_{c \in redcomp(P, D, t)} probtime(c) \mid t \in \mathbb{R}_{>0} \,|\!\}$$

- An equivalence relation $\mathcal{B}$ over $\mathbb{P}_{\mathrm{M,nd}}$ is a weak Markovian bisimulation iff, whenever $(P_1, P_2) \in \mathcal{B}$, then:
  - If $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nd,s}}$, for all action names $a \in \mathit{Name}$ and equivalence classes $D \in \mathbb{P}_{\mathrm{M,nd}}/\mathcal{B}$:
$$\mathit{rate}(P_1, a, D) \; = \; \mathit{rate}(P_2, a, D)$$
  - If $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nd,u}}$, for all stable equivalence classes $D \in \mathbb{P}_{\mathrm{M,nd}}/\mathcal{B}$:
$$\mathit{pbtm}(P_1, D) \; = \; \mathit{pbtm}(P_2, D)$$
- Weak Markovian bisimilarity, denoted $\approx_{\mathrm{MB}}$, is the union of all the weak Markovian bisimulations.

- **Example 1** – Consider the two process terms:

$$\bar{P}_1 \equiv <\tau, \mu>.<\tau, \gamma>.Q$$
$$\bar{P}_2 \equiv <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.Q$$

  with $Q \in \mathbb{P}_{\mathrm{M,nd,s}}$.

- Then $\bar{P}_1 \approx_{\mathrm{MB}} \bar{P}_2$ because:

$$pbtm(\bar{P}_1, [Q]_{\approx_{\mathrm{MB}}}) = \{\!|\, (1 \cdot 1) \cdot (\frac{1}{\mu} + \frac{1}{\gamma}) \,|\!\} = $$
$$= \{\!|\, 1 \cdot \frac{\mu + \gamma}{\mu \cdot \gamma} \,|\!\} = pbtm(\bar{P}_2, [Q]_{\approx_{\mathrm{MB}}})$$

- In general, for $l \in \mathbb{N}_{>0}$ we have:

$$<\tau, \mu>.<\tau, \gamma_1>. \dots .<\tau, \gamma_l>.Q \approx_{\mathrm{MB}} <\tau, \left(\frac{1}{\mu} + \frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_l}\right)^{-1}>.Q$$

- **Example 2** – Consider the two process terms:

$$\bar{P}_3 \equiv <\tau, \mu>.(<\tau, \gamma_1>.Q_1 + <\tau, \gamma_2>.Q_2)$$

$$\bar{P}_4 \equiv <\tau, \frac{\gamma_1}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right)^{-1}>.Q_1 + <\tau, \frac{\gamma_2}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right)^{-1}>.Q_2$$

  with $Q_1, Q_2 \in \mathbb{P}_{\mathrm{M,nd,s}}$ and $Q_1 \not\approx_{\mathrm{MB}} Q_2$.

- Then $\bar{P}_3 \approx_{\mathrm{MB}} \bar{P}_4$ because:

$$pbtm(\bar{P}_3, [Q_1]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\gamma_1}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right) |\!\} = pbtm(\bar{P}_4, [Q_1]_{\approx_{\mathrm{MB}}})$$

$$pbtm(\bar{P}_3, [Q_2]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\gamma_2}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right) |\!\} = pbtm(\bar{P}_4, [Q_2]_{\approx_{\mathrm{MB}}})$$

- In general, for $n \in \mathbb{N}_{>0}$ we have:

$$\begin{aligned} <\tau, \mu>.(<\tau, \gamma_1>.Q_1 + \ldots + <\tau, \gamma_n>.Q_n) \approx_{\mathrm{MB}} & <\tau, \frac{\gamma_1}{\gamma_1+\ldots+\gamma_n} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\ldots+\gamma_n}\right)^{-1}>.Q_1 + \\ & \ldots + \\ & <\tau, \frac{\gamma_n}{\gamma_1+\ldots+\gamma_n} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\ldots+\gamma_n}\right)^{-1}>.Q_n \end{aligned}$$

- **Example 3** – Consider the two process terms:

$$\bar{P}_5 \equiv <\tau,\mu_1>.<\tau,\gamma>.Q_1 + <\tau,\mu_2>.<\tau,\gamma>.Q_2$$

$$\bar{P}_6 \equiv <\tau, \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right)^{-1}>.Q_1 + <\tau, \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right)^{-1}>.Q_2$$

  with $Q_1, Q_2 \in \mathbb{P}_{\mathrm{M,nd,s}}$ and $Q_1 \not\approx_{\mathrm{MB}} Q_2$.

- Then $\bar{P}_5 \approx_{\mathrm{MB}} \bar{P}_6$ because:

$$pbtm(\bar{P}_5, [Q_1]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right) |\!\} = pbtm(\bar{P}_6, [Q_1]_{\approx_{\mathrm{MB}}})$$

$$pbtm(\bar{P}_5, [Q_2]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right) |\!\} = pbtm(\bar{P}_6, [Q_2]_{\approx_{\mathrm{MB}}})$$

- In general, for $n \in \mathbb{N}_{>0}$ we have:

$$\begin{aligned} <\tau,\mu_1>.<\tau,\gamma>.Q_1 + \ldots + <\tau,\mu_n>.<\tau,\gamma>.Q_n \approx_{\mathrm{MB}} & <\tau, \frac{\mu_1}{\mu_1+\ldots+\mu_n} \cdot \left(\frac{1}{\mu_1+\ldots+\mu_n} + \frac{1}{\gamma}\right)^{-1}>.Q_1 + \\ & \ldots + \\ & <\tau, \frac{\mu_n}{\mu_1+\ldots+\mu_n} \cdot \left(\frac{1}{\mu_1+\ldots+\mu_n} + \frac{1}{\gamma}\right)^{-1}>.Q_n \end{aligned}$$

- **Example 4** – None of the variants of $\bar{P}_5$ related to actions $<\tau, \gamma>$ leads to a reduction.

- If we consider:

$$\begin{aligned} \bar{P}_7 &\equiv <\tau, \mu_1>.<\tau, \gamma_1>.Q_1 + <\tau, \mu_2>.<\tau, \gamma_2>.Q_2 \\ \bar{P}_8 &\equiv <\tau, \tfrac{\mu_1}{\mu_1+\mu_2} \cdot \left(\tfrac{1}{\mu_1+\mu_2} + \tfrac{1}{\gamma_1}\right)^{-1}>.Q_1 + <\tau, \tfrac{\mu_2}{\mu_1+\mu_2} \cdot \left(\tfrac{1}{\mu_1+\mu_2} + \tfrac{1}{\gamma_2}\right)^{-1}>.Q_2 \end{aligned}$$

  with $\gamma_1 \neq \gamma_2$, then $\bar{P}_7 \not\approx_{\mathrm{MB}} \bar{P}_8$.

- If we consider:

$$\begin{aligned} \bar{P}_9 &\equiv <\tau, \mu_1>.<\tau, \gamma>.Q_1 + <\tau, \mu_2>.Q_2 \\ \bar{P}_{10} &\equiv <\tau, \tfrac{\mu_1}{\mu_1+\mu_2} \cdot \left(\tfrac{1}{\mu_1+\mu_2} + \tfrac{1}{\gamma}\right)^{-1}>.Q_1 + <\tau, \mu_2>.Q_2 \end{aligned}$$

  then $\bar{P}_9 \not\approx_{\mathrm{MB}} \bar{P}_{10}$.

- Let:
  - $I \neq \emptyset$ be a finite index set.
  - $J_i \neq \emptyset$ be a finite index set for all $i \in I$.
  - $P_{i,j} \in \mathbb{P}_{\mathrm{M,nd}}$ for all $i \in I$ and $j \in J_i$.

  Whenever $\sum_{j \in J_1} \gamma_{i_1,j} = \sum_{j \in J_2} \gamma_{i_2,j}$ for all $i_1, i_2 \in I$, then:

$$\sum_{i \in I} {<}\tau, \mu_i{>}. \sum_{j \in J_i} {<}\tau, \gamma_{i,j}{>}.P_{i,j} \;\approx_{\mathrm{MB}}$$
$$\sum_{i \in I} \sum_{j \in J_i} {<}\tau, \frac{\mu_i}{\sum\limits_{k \in I} \mu_k} \cdot \frac{\gamma_{i,j}}{\sum\limits_{h \in J_i} \gamma_{i,h}} \cdot \left(\frac{1}{\sum\limits_{k \in I} \mu_k} + \frac{1}{\sum\limits_{h \in J_i} \gamma_{i,h}} \right)^{-1} {>}.P_{i,j}$$

Congruence Property

- $\approx_{\rm MB}$ is a congruence over $\mathbb{P}_{\rm M,nd,s}$.
- Let $P_1, P_2 \in \mathbb{P}_{\rm M,nd,s}$. Whenever $P_1 \approx_{\rm MB} P_2$, then:
  - $<a, \lambda>.P_1 \approx_{\rm MB} <a, \lambda>.P_2$ for all $<a, \lambda> \in Act_{\rm M}$ such that $a \neq \tau$.
  - $P_1 + P \approx_{\rm MB} P_2 + P$ for all $P \in \mathbb{P}_{\rm M,nd}$ such that $P_1 + P, P_2 + P \in \mathbb{P}_{\rm M,nd,s}$.
- Not a congruence with respect to the alternative composition operator when considering unstable process terms.
- For instance:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} \ \approx_{\rm MB} \ <\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0}$$

but:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} + <a, \lambda>.\underline{0} \ \not\approx_{\rm MB} \ <\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} + <a, \lambda>.\underline{0}$$

both for $a \neq \tau$ and for $a = \tau$.

- Apply the rate-based equality check also to unstable process terms, but the equivalence classes to consider are the ones with respect to $\approx_{\rm MB}$.

- Let $P_1, P_2 \in \mathbb{P}_{\rm M,nd}$. We say that P_1 is weakly Markovian bisimulation congruent to P_2, written $P_1 \simeq_{\rm MB} P_2$, iff for all action names $a \in Name$ and equivalence classes $D \in \mathbb{P}_{\rm M,nd} / \approx_{\rm MB}$:

$$rate(P_1, a, D) \; = \; rate(P_2, a, D)$$

- $\sim_{\rm MB} \subset \simeq_{\rm MB} \subset \approx_{\rm MB}$, with $\simeq_{\rm MB}$ and $\approx_{\rm MB}$ coinciding over $\mathbb{P}_{\rm M,nd,s}$.

- $\simeq_{\rm MB}$ is the coarsest congruence contained in $\approx_{\rm MB}$ over $\mathbb{P}_{\rm M,nd}$; i.e., it is a congruence and for all $P_1, P_2 \in \mathbb{P}_{\rm M,nd}$ it holds $P_1 \simeq_{\rm MB} P_2$ iff for all $P \in \mathbb{P}_{\rm M,nd}$ it holds $P_1 + P \approx_{\rm MB} P_2 + P$.

Sound and Complete Axiomatization

- $\simeq_{\mathrm{MB}}$ has a sound and complete axiomatization over the set $\mathbb{P}_{\mathrm{M,nr}}$ of nonrecursive – and hence nondivergent – process terms of $\mathbb{P}_{\mathrm{M}}$.

- Set of axioms (the first four coincide with those of $\sim_{\mathrm{MB}}$):

$$
\begin{array}{llcl}
(\mathcal{A}_{\mathrm{MB},1}) & P_1 + P_2 & = & P_2 + P_1 \\
(\mathcal{A}_{\mathrm{MB},2}) & (P_1 + P_2) + P_3 & = & P_1 + (P_2 + P_3) \\
(\mathcal{A}_{\mathrm{MB},3}) & P + \underline{0} & = & P \\
(\mathcal{A}_{\mathrm{MB},4}) & <a, \lambda_1>.P + <a, \lambda_2>.P & = & <a, \lambda_1 + \lambda_2>.P \\
(\mathcal{A}_{\mathrm{MB},5}) & <a, \lambda>.\sum_{i \in I} <\tau, \mu_i>.\sum_{j \in J_i} <\tau, \gamma_{i,j}>.P_{i,j} & = & \\
 & \multicolumn{3}{r}{<a, \lambda>.\sum_{i \in I}\sum_{j \in J_i} <\tau, \frac{\mu_i}{\mu} \cdot \frac{\gamma_{i,j}}{\gamma} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)^{-1}>.P_{i,j}}
\end{array}
$$

if: $I \neq \emptyset$ is a finite index set

$J_i \neq \emptyset$ is a finite index set for all $i \in I$

$\mu = \sum_{i \in I} \mu_i$

$\gamma = \sum_{j \in J_i} \gamma_{i,j}$ for all $i \in I$

- For proving completeness, we cannot resort to normal form saturation as this would alter the quantitative behavior.

- Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nr}}$. If $P_1 \approx_{\mathrm{MB}} P_2$ but $P_1 \not\simeq_{\mathrm{MB}} P_2$, then at least one between P_1 and P_2 – both of which must be unstable – is of the form:

$$\sum_{i \in I} <\tau, \mu_i>. \sum_{j \in J_i} <\tau, \gamma_{i,j}>.P_{i,j}$$

  where $I \neq \emptyset$ is a finite index set, $J_i \neq \emptyset$ is a finite index set for all $i \in I$, and one of the following two properties holds:

  - $\sum_{j \in J_{i_1}} <\tau, \gamma_{i_1,j}>.P_{i_1,j} \approx_{\mathrm{MB}} \sum_{j \in J_{i_2}} <\tau, \gamma_{i_2,j}>.P_{i_2,j}$ for all $i_1, i_2 \in I$.
  - $\sum_{j \in J_{i_1}} \gamma_{i_1,j} = \sum_{j \in J_{i_2}} \gamma_{i_2,j}$ for all $i_1, i_2 \in I$.

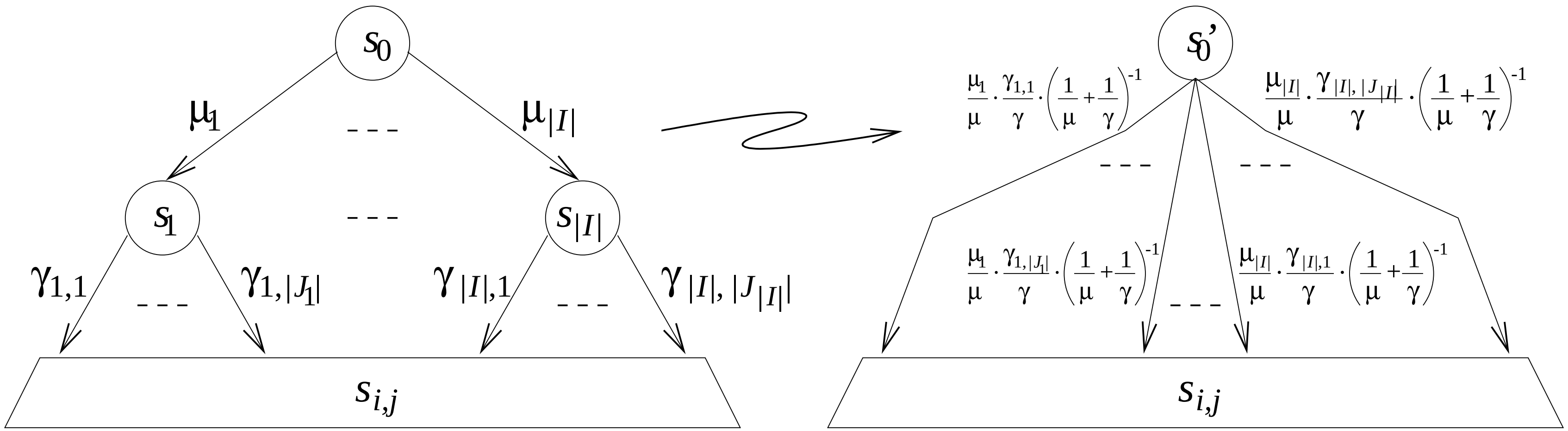
- Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nr}}$. Then $\mathcal{A}_{\mathrm{MB},1..5} \vdash P_1 = P_2 \iff P_1 \simeq_{\mathrm{MB}} P_2$.

Exactness of CTMC-Level Aggregation

- $\approx_{\mathrm{MB}}$ and $\simeq_{\mathrm{MB}}$ allow every sequence of exponentially timed τ-actions to be considered equivalent to a single exponentially timed τ-action having the same average duration.
- This amounts to approximating a hypoexponentially or Erlang distributed random variable with an exponentially distributed random variable having the same expected value.
- This can be exploited to assess more quickly properties expressed in terms of the mean time to certain events.
- Is there any other property that is preserved?

- Since $\sim_{\mathrm{MB}}$ is consistent with ordinary lumpability and the only new axiom is $\mathcal{A}_{\mathrm{MB},5}$, we can concentrate on this axiom.

- The induced CTMC-level aggregation, called W-lumpability, eliminates $|I|$ states and $|I|$ transitions by merging the first $1 + |I|$ states into a single one:



- W-lumpability is exact at steady state, i.e., the stationary probability of being in a macrostate of a CTMC obtained via W-lumpability is the sum of the stationary probabilities of being in one of the constituent microstates of the CTMC from which the reduced one has been derived.

- Unlike ordinary lumpability and T-lumpability, properties expressed in terms of transient state probabilities may not be preserved.

- Reconsider $\bar{P}_1 \equiv <\tau, \mu>.<\tau, \gamma>.Q$ and $\bar{P}_2 \equiv <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.Q$.

- The sum of the probabilities of being in one of the first two states of $[\![\bar{P}_1]\!]_{\mathrm{M}}$ at time $t \in \mathbb{R}_{>0}$ is $\frac{\gamma}{\gamma - \mu} \cdot \mathrm{e}^{-\mu \cdot t} - \frac{\mu}{\gamma - \mu} \cdot \mathrm{e}^{-\gamma \cdot t}$ for $\mu \neq \gamma$ or $(1 + \mu \cdot t) \cdot \mathrm{e}^{-\mu \cdot t}$ for $\mu = \gamma$.

- The probability of being in the first state of $[\![\bar{P}_2]\!]_{\mathrm{M}}$ at the same time instant is $\mathrm{e}^{-\frac{\mu \cdot \gamma}{\mu + \gamma} \cdot t}$, which reduces to $\mathrm{e}^{-\frac{\mu}{2} \cdot t}$ when $\mu = \gamma$.

Related and Future Work

- Problem originally addressed in [Hil1996] through a relation called weak isomorphism, from which we have taken the idea of preserving the average duration of internal action sequences.

- Congruence and steady-state exactness of weak isomorphism have been investigated, but no axiomatization is known.

- Different approach proposed in [Bra2002], where a variant of Markovian bisimilarity is defined that checks for exit rate equality with respect to all equivalence classes apart from the one including the process terms under examination.

- Congruence and axiomatization results have been provided, but nothing is said about exactness.

- Studying $\simeq_{\rm MB}$ over a calculus including parallel composition and hiding.