Weak Markovian Bisimulation Congruences and Exact CTMC-Level Aggregations for Sequential Processes

Marco Bernardo

Dipartimento di Scienze di Base e Fondamenti – Università di Urbino – Italy

Abstract. The Markovian behavioral equivalences defined so far treat exponentially timed internal actions like any other action. Since an exponentially timed internal action has a nonzero duration, it can be observed whenever it is executed between a pair of exponentially timed noninternal actions. However, no difference may be noted at steady state between a sequence of exponentially timed internal actions and a single exponentially timed internal action as long as their average durations coincide. We show that Milner's construction to derive a weak bisimulation congruence for nondeterministic processes can be extended to sequential Markovian processes in a way that captures the above situation. The resulting weak Markovian bisimulation congruence admits a sound and complete axiomatization, induces an exact CTMC-level aggregation at steady state, and is decidable in polynomial time for finite-state processes having no cycles of exponentially timed internal actions.

1 Introduction

System models with an underlying continuous-time Markov chain (CTMC) [23] semantics can be compared and manipulated by means of Markovian behavioral equivalences. Several of them have appeared in the literature – see [1] and the references therein – which are extensions of the traditional approaches to the definition of behavioral equivalences that take into account performance aspects too. A feature shared by relations like Markovian bisimilarity, Markovian testing equivalence, and Markovian trace equivalence is that of being strong. Only a few variants investigated in [12, 21, 17, 6] are capable of abstracting from internal immediate actions and/or purely probabilistic branchings.

Let us denote by τ the invisible or silent action. In a nondeterministic setting, a process that can perform action a followed by τ and action b and then terminates – written $a \,.\, \tau \,.\, b \,.\, \underline{0}$ – is weakly equivalent to a process that can perform action a followed by action b and then terminates – written $a \,.\, b \,.\, \underline{0}$. By contrast, in a setting where actions have exponentially distributed durations – uniquely identified by positive real numbers called rates – it is not necessarily the case that simplifications like the one above can be made.

For instance, if a has rate λ, b has rate μ, and τ has rate γ, the two resulting processes $<a, \lambda>.<\tau, \gamma>.<b, \mu>.\underline{0}$ and $<a, \lambda>.<b, \mu>.\underline{0}$ are not weakly equivalent. In fact, recalling that the average (i.e., expected) duration of an action coincides with the reciprocal of the rate of the action, the former process has a maximal computation whose average duration is $\frac{1}{\lambda} + \frac{1}{\gamma} + \frac{1}{\mu}$, whereas the latter process has a maximal computation whose average duration is $\frac{1}{\lambda} + \frac{1}{\mu}$. From another viewpoint, in the former case an external observer would see an a-action for an amount of time t_λ and a b-action for an amount of time t_μ with a delay t_γ in between, while in the latter case the external observer would not see any delay between the execution of a and the execution of b. Therefore, in a Markovian setting a τ-action executed between a pair of non-τ-actions cannot be abstracted away because it has a nonzero duration and hence can be observed.

As a different example, take now a process that can perform action a at rate λ followed by two τ-actions with rates γ_1 and γ_2, respectively, and then behaves as process P, i.e., $<a, \lambda>.<\tau, \gamma_1>.<\tau, \gamma_2>.P$. In this case, an observer may not be able to distinguish between the execution of the two τ-actions above and the execution of a single τ-action whose average duration is the sum of the average durations of the two original τ-actions, i.e., $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \frac{\gamma_1 + \gamma_2}{\gamma_1 \cdot \gamma_2}$. In other words, the process may be viewed as being weakly equivalent to $<a, \lambda>.<\tau, \frac{\gamma_1 \cdot \gamma_2}{\gamma_1 + \gamma_2}>.P$.

The two processes above are weakly equivalent from a functional standpoint. However, since the sum of the two exponential random variables quantifying the durations of the two original τ-actions has been approximated with a single average-preserving exponential random variable, it is not necessarily the case that the two processes have the same performance characteristics. This would be true if the equivalence induced an exact CTMC-level aggregation, i.e., an aggregation such that the transient/stationary probability of being in a macrostate of a reduced CTMC is the sum of the transient/stationary probabilities of being in one of the constituent microstates of the CTMC from which the reduced one has been obtained. This is the case with Markovian bisimilarity, which is in agreement with an exact CTMC-level aggregation called ordinary lumpability [14, 10], and Markovian testing and trace equivalences, which are consistent with a coarser exact CTMC-level aggregation called T-lumpability [4].

In this paper, we show that the construction used in [18] to derive a weak bisimulation congruence for nondeterministic processes can be extended to sequential Markovian processes. The resulting equivalence is weak in the sense that it is capable of abstracting from the number of consecutive exponentially timed τ-actions in a computation. It reduces any such sequence to a single exponentially timed τ-action preserving both the average duration and the execution probability of the original action sequence, which turns out to induce an exact CTMC-level aggregation at steady state. We also prove that the weak Markovian bisimulation congruence admits a sound and complete axiomatization and – in the absence of cycles of exponentially timed internal actions – is decidable in polynomial time for finite-state processes.

This paper is organized as follows. In Sect. 2, we introduce a process calculus for sequential Markovian processes with abstraction and we recall Markovian bisimilarity. In Sect. 3, we develop the weak variant of Markovian bisimilarity and we investigate its congruence, axiomatizability, exactness, and decidability properties. Finally, in Sect. 4 we conclude with related and future work.

2 Sequential Markovian Processes and Bisimilarity

In order to study properties like congruence and axiomatizability of the weak variant of Markovian bisimilarity, we need to define a Markovian process calculus (MPC for short). In particular, we introduce a calculus that generates all the CTMCs with as few operators as possible: the inactive process, exponentially timed action prefix, alternative composition, and recursion. Therefore, the resulting processes will be sequential Markovian processes governed by the race policy: if several exponentially timed actions are simultaneously enabled, the action that is executed is the one sampling the least duration. In addition to those operators, we include hiding because the behavioral equivalence we are going to propose is weak and hence we need a way for causing actions to become the internal action τ.

Definition 1. *Let* $Act_{\mathrm{M}} = Name \times \mathbb{R}_{>0}$ *be a set of actions, where* $Name = Name_{\mathrm{v}} \cup \{\tau\}$ *is a set of action names – ranged over by* a, b *– and* $\mathbb{R}_{>0}$ *is a set of action rates – ranged over by* λ, μ, γ. *Let* Var *be a set of process variables ranged over by* X, Y. *The process language* $\mathcal{PL}_{\mathrm{M}}$ *is generated by the following syntax:*

P ::=	$\underline{0}$	*inactive process*
|	$<a, \lambda>.P$	*exponentially timed action prefix*
|	$P + P$	*alternative composition*
|	X	*process variable*
|	$\mathrm{rec}\, X : P$	*recursion*
|	P/H	*hiding*

where $a \in Name$, $\lambda \in \mathbb{R}_{>0}$, $X \in Var$, *and* $H \subseteq Name_{\mathrm{v}}$. *We denote by* $\mathbb{P}_{\mathrm{M}}$ *the set of closed and guarded process terms of* $\mathcal{PL}_{\mathrm{M}}$ *– ranged over by* P, Q. ■

In order to distinguish between process terms like $<a, \lambda>.\underline{0} + <a, \lambda>.\underline{0}$ and $<a, \lambda>.\underline{0}$, the semantic model $[\![P]\!]_{\mathrm{M}}$ for a process term $P \in \mathbb{P}_{\mathrm{M}}$ is a labeled multitransition system that takes into account the multiplicity of each transition, intended as the number of different proofs for the transition derivation. The multitransition relation of $[\![P]\!]_{\mathrm{M}}$ is contained in the smallest multiset of elements of $\mathbb{P}_{\mathrm{M}} \times Act_{\mathrm{M}} \times \mathbb{P}_{\mathrm{M}}$ that satisfies the operational semantic rules below – where $\{\_ \hookrightarrow \_\}$ denotes syntactical replacement – and keeps track of all the possible ways of deriving each of its transitions:

$$(\textsc{Pre}_{\mathrm{M}}) \quad \frac{}{<a, \lambda>.P \xrightarrow{a,\lambda}_{\mathrm{M}} P} \qquad\qquad (\textsc{Rec}_{\mathrm{M}}) \quad \frac{P\{\mathrm{rec}\, X : P \hookrightarrow X\} \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{\mathrm{rec}\, X : P \xrightarrow{a,\lambda}_{\mathrm{M}} P'}$$

$$(\textsc{Alt}_{\mathrm{M},1}) \quad \frac{P_1 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{P_1 + P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'} \qquad\qquad (\textsc{Alt}_{\mathrm{M},2}) \quad \frac{P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{P_1 + P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}$$

$$(\textsc{Hid}_{\mathrm{M},1}) \quad \frac{P \xrightarrow{a,\lambda}_{\mathrm{M}} P' \qquad a \notin H}{P/H \xrightarrow{a,\lambda}_{\mathrm{M}} P'/H} \qquad\qquad (\textsc{Hid}_{\mathrm{M},2}) \quad \frac{P \xrightarrow{a,\lambda}_{\mathrm{M}} P' \qquad a \in H}{P/H \xrightarrow{\tau,\lambda}_{\mathrm{M}} P'/H}$$

The definition of bisimilarity for MPC is based on the comparison of exit rates [14, 13]. The exit rate of a process term $P \in \mathbb{P}_{\rm M}$ with respect to action name $a \in A$ and destination $D \subseteq \mathbb{P}_{\rm M}$ is the rate at which P can execute actions of name a that lead to D, i.e., $\mathit{rate}(P, a, D) = \sum \{\!| \lambda \in \mathbb{R}_{>0} \mid \exists P' \in D. P \xrightarrow{a,\lambda}_{\rm M} P' |\!\}$ where $\{\!|$ and $|\!\}$ are multiset delimiters and the summation is taken to be zero if its multiset is empty. By summing up the rates of all the actions of P, we obtain the total exit rate of P, i.e., $\mathit{rate}_{\rm t}(P) = \sum_{a \in \mathit{Name}} \mathit{rate}(P, a, \mathbb{P}_{\rm M})$, which is the reciprocal of the average (i.e., expected) sojourn time associated with P.

Definition 2. *An equivalence relation $\mathcal{B}$ over $\mathbb{P}_{\rm M}$ is a Markovian bisimulation iff, whenever $(P_1, P_2) \in \mathcal{B}$, then for all action names $a \in \mathit{Name}$ and equivalence classes $D \in \mathbb{P}_{\rm M}/\mathcal{B}$:*

$$\mathit{rate}(P_1, a, D) \;=\; \mathit{rate}(P_2, a, D)$$

Markovian bisimilarity $\sim_{\rm MB}$ is the union of all the Markovian bisimulations. ■

3 Abstracting from Exponentially Timed τ-Actions

In this section, we weaken the distinguishing power of $\sim_{\rm MB}$ in order to be able to abstract from sequences of exponentially timed τ-actions. As noted in Sect. 1, while it is not possible to get rid of an exponentially timed τ-action executed between a pair of exponentially timed non-τ-actions, a sequence of exponentially timed τ-actions may be indistinguishable at steady state from a single exponentially timed τ-action having the same average duration as the sequence.

We say that $P \in \mathbb{P}_{\rm M}$ is stable if $P \not\xrightarrow{\tau,\lambda}_{\rm M} P'$ for all λ and P', otherwise we say that P is unstable. In the latter case, we say that P is fully unstable iff, whenever $P \xrightarrow{a,\lambda}_{\rm M} P'$, then $a = \tau$. We denote by $\mathbb{P}_{\rm M,fu}$ and $\mathbb{P}_{\rm M,nfu}$ the sets of process terms of $\mathbb{P}_{\rm M}$ that are fully unstable and not fully unstable, respectively.

The most natural candidates as sequences of exponentially timed τ-actions to abstract are those labeling computations that traverse fully unstable states.

Definition 3. *Let $n \in \mathbb{N}_{>0}$ and $P_1, P_2, \dots, P_{n+1} \in \mathbb{P}_{\rm M}$. A computation c of length n from P_1 to P_{n+1} having the form $P_1 \xrightarrow{\tau,\lambda_1}_{\rm M} P_2 \xrightarrow{\tau,\lambda_2}_{\rm M} \dots \xrightarrow{\tau,\lambda_n}_{\rm M} P_{n+1}$ is reducible iff $P_i \in \mathbb{P}_{\rm M,fu}$ for all $i = 1, \dots, n$.* ■

The idea is that, if reducible, the computation c above can be reduced to a single exponentially timed τ-transition whose rate is obtained from the positive real value below:

$$\boxed{\mathit{probtime}(c) \;=\; \left(\prod_{i=1}^{n} \frac{\lambda_i}{\mathit{rate}(P_i, \tau, \mathbb{P}_{\rm M})} \right) \cdot \left(\sum_{i=1}^{n} \frac{1}{\mathit{rate}(P_i, \tau, \mathbb{P}_{\rm M})} \right)}$$

by leaving its first factor unchanged and taking the reciprocal of the second one. This value is a measure of the execution probability of c (first factor: product of the execution probabilities of the transitions of c) and the average duration of c (second factor: sum of the average sojourn times in the states traversed by c).

Notice that we consider only reducible computations of finite length. This will be enough to distinguish between fully unstable process terms that must be told apart. In fact, assuming $\lambda_1 \neq \lambda_2$, it makes sense to discriminate between $<\tau, \lambda_1>.P$ and $<\tau, \lambda_2>.P$ if P can reach a non-fully-unstable process term. By contrast, an external observer cannot see any difference between two divergent process terms like $\mathrm{rec}\, X : <\tau, \lambda_1>.X$ and $\mathrm{rec}\, X : <\tau, \lambda_2>.X$.

We are now ready to define a weak variant of $\sim_{\rm MB}$ such that (i) processes in $\mathbb{P}_{\rm M,nfu}$ are dealt with as in $\sim_{\rm MB}$ and (ii) the length of reducible computations from processes in $\mathbb{P}_{\rm M,fu}$ to processes in $\mathbb{P}_{\rm M,nfu}$ is abstracted away while preserving their execution probability and average duration. In the latter case, we need to lift measure *probtime* from individual reducible computations to multisets of reducible computations. More precisely, denoting by $\mathit{reducomp}(P, D, t)$ the multiset of reducible computations from $P \in \mathbb{P}_{\rm M,fu}$ to some P' in $D \subseteq \mathbb{P}_{\rm M}$ whose average duration is $t \in \mathbb{R}_{>0}$, we consider the following t-indexed multiset of sums of *probtime* measures:

$$\boxed{\mathit{pbtm}(P, D) \; = \bigcup_{t \in \mathbb{R}_{>0} \text{ s.t. } \mathit{reducomp}(P,D,t) \neq \emptyset} \{\!| \sum_{c \in \mathit{reducomp}(P,D,t)} \mathit{probtime}(c) \,|\!\}}$$

Notice that *pbtm* is not simply the multiset of the *probtime* measures of the various reducible computations from P to D. In that case, for example we would have $\mathit{pbtm}(<\tau, \lambda_1>.\underline{0} + <\tau, \lambda_2>.\underline{0}, \{\underline{0}\}) = \{\!| \frac{\lambda_1}{\lambda_1+\lambda_2} \cdot \frac{1}{\lambda_1+\lambda_2}, \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \frac{1}{\lambda_1+\lambda_2} |\!\}$ while $\mathit{pbtm}(<\tau, \lambda_1 + \lambda_2>.\underline{0}, \{\underline{0}\}) = \{\!| \frac{1}{\lambda_1+\lambda_2} |\!\}$, thus obtaining a behavioral equivalence that is not a conservative extension of $\sim_{\rm MB}$. On the other hand, *probtime* measures should be summed up only over reducible computations from P to D having the same average duration t. If this were not the case, then for instance we would have $\mathit{pbtm}(<\tau, \mu>.\underline{0}, \{\underline{0}\}) = \mathit{pbtm}(<\tau, \mu_1>.\underline{0} + <\tau, \mu_2>.<\tau, \gamma>.\underline{0})$ when $\frac{1}{\mu} = \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2}\right) + \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right)$, which would not be meaningful on the performance side as it would not induce an exact CTMC-level aggregation.

Definition 4. *An equivalence relation* $\mathcal{B} \subseteq (\mathbb{P}_{\rm M,nfu} \times \mathbb{P}_{\rm M,nfu}) \cup (\mathbb{P}_{\rm M,fu} \times \mathbb{P}_{\rm M,fu})$ *is a weak Markovian bisimulation iff for all* $(P_1, P_2) \in \mathcal{B}$*:*

- *If* $P_1, P_2 \in \mathbb{P}_{\rm M,nfu}$*, then for all* $a \in \mathit{Name}$ *and equivalence classes* $D \in \mathbb{P}_{\rm M}/\mathcal{B}$*:*
  $$\mathit{rate}(P_1, a, D) \; = \; \mathit{rate}(P_2, a, D)$$
- *If* $P_1, P_2 \in \mathbb{P}_{\rm M,fu}$*, then for all equivalence classes* $D \in \mathbb{P}_{\rm M,nfu}/\mathcal{B}$*:*
  $$\mathit{pbtm}(P_1, D) \; = \; \mathit{pbtm}(P_2, D)$$

Weak Markovian bisimilarity $\approx_{\rm MB}$ *is the union of all the weak Markovian bisimulations.* ■

Example 1. Consider the following two process terms:

$$\bar{P}_1 \; \equiv \; <\tau, \mu>.<\tau, \gamma>.Q \qquad (\text{or } \bar{P}_1 \; \equiv \; <\tau, \gamma>.<\tau, \mu>.Q)$$
$$\bar{P}_2 \; \equiv \; <\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.Q$$

with $Q \in \mathbb{P}_{\rm M,nfu}$. As anticipated in Sect. 1, it turns out that $\bar{P}_1 \approx_{\rm MB} \bar{P}_2$ because:

$$\mathit{pbtm}(\bar{P}_1, [Q]_{\approx_{\rm MB}}) \; = \; \{\!| (1 \cdot 1) \cdot (\tfrac{1}{\mu} + \tfrac{1}{\gamma}) |\!\} \; = \; \{\!| 1 \cdot \tfrac{\mu+\gamma}{\mu \cdot \gamma} |\!\} \; = \; \mathit{pbtm}(\bar{P}_2, [Q]_{\approx_{\rm MB}})$$

where $[Q]_{\approx_{\rm MB}}$ is the equivalence class of Q with respect to $\approx_{\rm MB}$.

In general, for $l \in \mathbb{N}_{>0}$ we have that $<\tau, \mu>.<\tau, \gamma_1>. \ldots .<\tau, \gamma_l>.Q$ is weakly Markovian bisimilar to $<\tau, \left(\frac{1}{\mu} + \frac{1}{\gamma_1} + \ldots + \frac{1}{\gamma_l}\right)^{-1}>.Q$. ■

Example 2. Consider the following two process terms:

$$\bar{P}_3 \equiv <\tau, \mu>.(<\tau, \gamma_1>.Q_1 + <\tau, \gamma_2>.Q_2)$$

$$\bar{P}_4 \equiv <\tau, \frac{\gamma_1}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right)^{-1}>.Q_1 + <\tau, \frac{\gamma_2}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right)^{-1}>.Q_2$$

with $Q_1, Q_2 \in \mathbb{P}_{\mathrm{M,nfu}}$ and $Q_1 \not\approx_{\mathrm{MB}} Q_2$. Unlike action $<\tau, \mu>$ of $\bar{P}_1$ in the previous example, action $<\tau, \mu>$ of $\bar{P}_3$ is followed by a choice between two exponentially timed τ-actions. It turns out that $\bar{P}_3 \approx_{\mathrm{MB}} \bar{P}_4$ because:

$$pbtm(\bar{P}_3, [Q_1]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\gamma_1}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right) |\!\} = pbtm(\bar{P}_4, [Q_1]_{\approx_{\mathrm{MB}}})$$

$$pbtm(\bar{P}_3, [Q_2]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\gamma_2}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right) |\!\} = pbtm(\bar{P}_4, [Q_2]_{\approx_{\mathrm{MB}}})$$

In general, for $n \in \mathbb{N}_{>0}$ we have that $<\tau, \mu>.(<\tau, \gamma_1>.Q_1 + \ldots + <\tau, \gamma_n>.Q_n)$ is weakly Markovian bisimilar to $<\tau, \frac{\gamma_1}{\gamma_1+\ldots+\gamma_n} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\ldots+\gamma_n}\right)^{-1}>.Q_1 + \ldots + <\tau, \frac{\gamma_n}{\gamma_1+\ldots+\gamma_n} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\ldots+\gamma_n}\right)^{-1}>.Q_n$. ■

Example 3. Consider the following two process terms:

$$\bar{P}_5 \equiv <\tau, \mu_1>.<\tau, \gamma>.Q_1 + <\tau, \mu_2>.<\tau, \gamma>.Q_2$$

$$\bar{P}_6 \equiv <\tau, \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right)^{-1}>.Q_1 + <\tau, \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right)^{-1}>.Q_2$$

with $Q_1, Q_2 \in \mathbb{P}_{\mathrm{M,nfu}}$ and $Q_1 \not\approx_{\mathrm{MB}} Q_2$ as before. Unlike $\bar{P}_1$ and $\bar{P}_3$ in the previous two examples, $\bar{P}_5$ starts with a choice between two exponentially timed τ-actions, each of which is followed by the same action $<\tau, \gamma>$. It turns out that $\bar{P}_5 \approx_{\mathrm{MB}} \bar{P}_6$ because:

$$pbtm(\bar{P}_5, [Q_1]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right) |\!\} = pbtm(\bar{P}_6, [Q_1]_{\approx_{\mathrm{MB}}})$$

$$pbtm(\bar{P}_5, [Q_2]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right) |\!\} = pbtm(\bar{P}_6, [Q_2]_{\approx_{\mathrm{MB}}})$$

In general, for $n \in \mathbb{N}_{>0}$ we have that $<\tau, \mu_1>.<\tau, \gamma>.Q_1 + \ldots + <\tau, \mu_n>.<\tau, \gamma>.Q_n$ is weakly Markovian bisimilar to $<\tau, \frac{\mu_1}{\mu_1+\ldots+\mu_n} \cdot \left(\frac{1}{\mu_1+\ldots+\mu_n} + \frac{1}{\gamma}\right)^{-1}>.Q_1 + \ldots + <\tau, \frac{\mu_n}{\mu_1+\ldots+\mu_n} \cdot \left(\frac{1}{\mu_1+\ldots+\mu_n} + \frac{1}{\gamma}\right)^{-1}>.Q_n$. The equivalence holds even if the derivative terms of actions $<\tau, \mu_i>$, $1 \leq i \leq n$, start with a choice among several exponentially timed τ-actions instead of a single exponentially timed τ-action, provided that all these derivative terms have the same total exit rate γ. ■

Example 4. We now examine all possible variants of $\bar{P}_5$ related to actions $<\tau, \gamma>$ and we show that none of these variants allows for any reduction because it is not possible to preserve execution probabilities or average durations.

Firstly, consider the following two process terms:

$$\bar{P}_7 \equiv <\tau, \mu_1>.<\tau, \gamma_1>.Q_1 + <\tau, \mu_2>.<\tau, \gamma_2>.Q_2$$

$$\bar{P}_8 \equiv <\tau, \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma_1}\right)^{-1}>.Q_1 + <\tau, \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma_2}\right)^{-1}>.Q_2$$

with $\gamma_1 \neq \gamma_2$. Then $\bar{P}_7 \not\approx_{\mathrm{MB}} \bar{P}_8$ because for instance:

$$
\begin{aligned}
pbtm(\bar{P}_7, [Q_1]_{\approx_{\mathrm{MB}}}) &= \{\!|\ \tfrac{\mu_1}{\mu_1+\mu_2} \cdot \left(\tfrac{1}{\mu_1+\mu_2} + \tfrac{1}{\gamma_1} \right)\ |\!\} \\
pbtm(\bar{P}_8, [Q_1]_{\approx_{\mathrm{MB}}}) &= \{\!|\ \frac{\frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma_1} \right)^{-1}}{\frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma_1} \right)^{-1} + \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma_2} \right)^{-1}} \\
&\quad \cdot \frac{1}{\frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma_1} \right)^{-1} + \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma_2} \right)^{-1}}\ |\!\}
\end{aligned}
$$

Secondly, consider the following two process terms:

$$
\begin{aligned}
\bar{P}_9 &\equiv <\tau, \mu_1>.<\tau, \gamma>.Q_1 + <\tau, \mu_2>.Q_2 \\
\bar{P}_{10} &\equiv <\tau, \tfrac{\mu_1}{\mu_1+\mu_2} \cdot \left(\tfrac{1}{\mu_1+\mu_2} + \tfrac{1}{\gamma} \right)^{-1}>.Q_1 + <\tau, \mu_2>.Q_2
\end{aligned}
$$

Then $\bar{P}_9 \not\approx_{\mathrm{MB}} \bar{P}_{10}$ because for instance:

$$
\begin{aligned}
pbtm(\bar{P}_9, [Q_2]_{\approx_{\mathrm{MB}}}) &= \{\!|\ \tfrac{\mu_2}{\mu_1+\mu_2} \cdot \tfrac{1}{\mu_1+\mu_2}\ |\!\} \\
pbtm(\bar{P}_{10}, [Q_2]_{\approx_{\mathrm{MB}}}) &= \{\!|\ \frac{\mu_2}{\frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma} \right)^{-1} + \mu_2} \cdot \frac{1}{\frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma} \right)^{-1} + \mu_2}\ |\!\}
\end{aligned}
$$
■

Proposition 1. *Let $I \neq \emptyset$ be a finite set, $J_i \neq \emptyset$ be a finite set for all $i \in I$, and $P_{i,j} \in \mathbb{P}_{\mathrm{M}}$ for all $i \in I$ and $j \in J_i$. Then:*

$$
\sum_{i \in I} <\tau, \mu_i>. \sum_{j \in J_i} <\tau, \gamma_{i,j}>.P_{i,j} \ \approx_{\mathrm{MB}} \ \sum_{i \in I} \sum_{j \in J_i} <\tau, \frac{\mu_i}{\sum\limits_{k \in I} \mu_k} \cdot \frac{\gamma_{i,j}}{\sum\limits_{h \in J_i} \gamma_{i,h}} \cdot \left(\frac{1}{\sum\limits_{k \in I} \mu_k} + \frac{1}{\sum\limits_{h \in J_i} \gamma_{i,h}} \right)^{-1}>.P_{i,j}
$$

whenever $\sum_{j \in J_{i_1}} \gamma_{i_1,j} = \sum_{j \in J_{i_2}} \gamma_{i_2,j}$ for all $i_1, i_2 \in I$. ■

3.1 Congruence Property

Let us investigate the compositionality of $\approx_{\mathrm{MB}}$ with respect to MPC operators.

Proposition 2. *Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M}}$. Whenever $P_1 \approx_{\mathrm{MB}} P_2$, then:*

1. *$<a, \lambda>.P_1 \approx_{\mathrm{MB}} <a, \lambda>.P_2$ for all $<a, \lambda> \in Act_{\mathrm{M}}$.*
2. *$P_1 / H \approx_{\mathrm{MB}} P_2 / H$ for all $H \subseteq Name_{\mathrm{v}}$.* ■

Similar to weak bisimilarity for nondeterministic processes, $\approx_{\mathrm{MB}}$ is not a congruence with respect to the alternative composition operator. The problem has to do with fully unstable process terms: e.g., $<\tau, \mu>.<\tau, \gamma>.\underline{0} \approx_{\mathrm{MB}} <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0}$ but $<\tau, \mu>.<\tau, \gamma>.\underline{0} + <a, \lambda>.\underline{0} \not\approx_{\mathrm{MB}} <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} + <a, \lambda>.\underline{0}$. In fact, if it were $a \neq \tau$ then we would have:

$$
\begin{aligned}
rate(<\tau, \mu>.<\tau, \gamma>.\underline{0} + <a, \lambda>.\underline{0}, \tau, [\underline{0}]_{\approx_{\mathrm{MB}}}) &= 0 \\
rate(<\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} + <a, \lambda>.\underline{0}, \tau, [\underline{0}]_{\approx_{\mathrm{MB}}}) &= \tfrac{\mu \cdot \gamma}{\mu + \gamma}
\end{aligned}
$$

otherwise for $a = \tau$ we would have:

$$
\begin{aligned}
pbtm(<\tau, \mu>.<\tau, \gamma>.\underline{0} + <a, \lambda>.\underline{0}, [\underline{0}]_{\approx_{\mathrm{MB}}}) &= \{\!|\ \tfrac{\mu}{\mu + \lambda} \cdot \left(\tfrac{1}{\mu + \lambda} + \tfrac{1}{\gamma} \right), \tfrac{\lambda}{\mu + \lambda} \cdot \tfrac{1}{\mu + \lambda}\ |\!\} \\
pbtm(<\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} + <a, \lambda>.\underline{0}, [\underline{0}]_{\approx_{\mathrm{MB}}}) &= \{\!|\ \tfrac{1}{\frac{\mu \cdot \gamma}{\mu + \gamma} + \lambda}\ |\!\}
\end{aligned}
$$

The congruence violation with respect to the alternative composition operator can be prevented by adopting a construction analogous to the one used in [18] for weak bisimilarity over nondeterministic process terms. In other words, we have to apply the exit rate equality check also to fully unstable process terms, with the equivalence classes to consider being the ones with respect to $\approx_{\mathrm{MB}}$.

Definition 5. *Let $P_1, P_2 \in \mathbb{P}_{\rm M}$. We say that P_1 is weakly Markovian bisimulation congruent to P_2, written $P_1 \simeq_{\rm MB} P_2$, iff for all action names $a \in Name$ and equivalence classes $D \in \mathbb{P}_{\rm M}/\approx_{\rm MB}$:*

$$rate(P_1, a, D) \; = \; rate(P_2, a, D)$$ ■

Proposition 3. *$\sim_{\rm MB} \subset \simeq_{\rm MB} \subset \approx_{\rm MB}$, with $\simeq_{\rm MB} = \approx_{\rm MB}$ over $\mathbb{P}_{\rm M,nfu}$.* ■

Proposition 4. *Let $P_1, P_2 \in \mathbb{P}_{\rm M}$ and $<a, \lambda> \in Act_{\rm M}$. Then $<a, \lambda>.P_1 \simeq_{\rm MB} <a, \lambda>.P_2$ iff $P_1 \approx_{\rm MB} P_2$.* ■

It turns out that $\simeq_{\rm MB}$ is the coarsest congruence – with respect to all the operators of MPC as well as recursion – contained in $\approx_{\rm MB}$.

Theorem 1. *Let $P_1, P_2 \in \mathbb{P}_{\rm M}$. Whenever $P_1 \simeq_{\rm MB} P_2$, then:*

1. *$<a, \lambda>.P_1 \simeq_{\rm MB} <a, \lambda>.P_2$ for all $<a, \lambda> \in Act_{\rm M}$.*
2. *$P_1 + P \simeq_{\rm MB} P_2 + P$ and $P + P_1 \simeq_{\rm MB} P + P_2$ for all $P \in \mathbb{P}_{\rm M}$.*
3. *$P_1/H \simeq_{\rm MB} P_2/H$ for all $H \subseteq Name_{\rm v}$.* ■

Theorem 2. *Let $P_1, P_2 \in \mathbb{P}_{\rm M}$. Then $P_1 \simeq_{\rm MB} P_2$ iff $P_1 + P \approx_{\rm MB} P_2 + P$ for all $P \in \mathbb{P}_{\rm M}$.* ■

With regard to recursion, we need to extend $\simeq_{\rm MB}$ to open process terms in the usual way. The congruence proof is based on a notion of weak Markovian bisimulation up to $\approx_{\rm MB}$ inspired by the notion of Markovian bisimulation up to $\sim_{\rm MB}$ of [9]. It differs from its nondeterministic counterpart [18] due to the necessity of working with equivalence classes in this Markovian setting.

Definition 6. *Let $P_1, P_2 \in \mathcal{PL}_{\rm M}$ be process terms containing free occurrences of $k \in \mathbb{N}$ process variables $X_1, \dots, X_k \in Var$ at most. We define $P_1 \simeq_{\rm MB} P_2$ iff $P_1\{Q_i \hookrightarrow X_i \mid 1 \le i \le k\} \simeq_{\rm MB} P_2\{Q_i \hookrightarrow X_i \mid 1 \le i \le k\}$ for all $Q_1, \dots, Q_k \in \mathcal{PL}_{\rm M}$ containing no free occurrences of process variables.* ■

Definition 7. *Let $^+$ denote the operation of transitive closure for relations. A binary relation $\mathcal{B} \subseteq (\mathbb{P}_{\rm M,nfu} \times \mathbb{P}_{\rm M,nfu}) \cup (\mathbb{P}_{\rm M,fu} \times \mathbb{P}_{\rm M,fu})$ is a weak Markovian bisimulation up to $\approx_{\rm MB}$ iff for all $(P_1, P_2) \in \mathcal{B}$:*

- *If $P_1, P_2 \in \mathbb{P}_{\rm M,nfu}$, then for all $a \in Name$ and $D \in \mathbb{P}_{\rm M}/(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$:*
  $$rate(P_1, a, D) \; = \; rate(P_2, a, D)$$
- *If $P_1, P_2 \in \mathbb{P}_{\rm M,fu}$, then for all $D \in \mathbb{P}_{\rm M,nfu}/(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$:*
  $$pbtm(P_1, D) \; = \; pbtm(P_2, D)$$ ■

Proposition 5. *Let $\mathcal{B} \subseteq (\mathbb{P}_{\rm M,nfu} \times \mathbb{P}_{\rm M,nfu}) \cup (\mathbb{P}_{\rm M,fu} \times \mathbb{P}_{\rm M,fu})$. If $\mathcal{B}$ is a weak Markovian bisimulation up to $\approx_{\rm MB}$, then $(P_1, P_2) \in \mathcal{B}$ implies $P_1 \approx_{\rm MB} P_2$ for all $P_1, P_2 \in \mathbb{P}_{\rm M}$. Moreover $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+ = \approx_{\rm MB}$.* ■

Theorem 3. *Let $P_1, P_2 \in \mathcal{PL}_{\rm M}$ be process terms containing free occurrences of $k \in \mathbb{N}$ process variables $X_1, \dots, X_k \in Var$ at most. Whenever $P_1 \simeq_{\rm MB} P_2$, then ${\rm rec}\, X_1 : \ldots : {\rm rec}\, X_k : P_1 \simeq_{\rm MB} {\rm rec}\, X_1 : \ldots : {\rm rec}\, X_k : P_2$.* ■

3.2 Sound and Complete Axiomatization

$\simeq_{\mathrm{MB}}$ has a sound and complete axiomatization over the set $\mathbb{P}_{\mathrm{M,nr}}$ of nonrecursive process terms of $\mathbb{P}_{\mathrm{M}}$, which is shown below. The first four axioms are inherited from $\sim_{\mathrm{MB}}$. They are valid for $\simeq_{\mathrm{MB}}$ too because $\sim_{\mathrm{MB}} \subset \simeq_{\mathrm{MB}}$ as stated by Prop. 3. The fifth axiom characterizes $\simeq_{\mathrm{MB}}$. Its validity comes from Props. 1 and 4. The last four axioms are the usual distributive laws for the hiding operator.

$$
\begin{array}{lrcll}
(\mathcal{A}_{\mathrm{MB},1}) & P_1 + P_2 & = & P_2 + P_1 & \\
(\mathcal{A}_{\mathrm{MB},2}) & (P_1 + P_2) + P_3 & = & P_1 + (P_2 + P_3) & \\
(\mathcal{A}_{\mathrm{MB},3}) & P + \underline{0} & = & P & \\
(\mathcal{A}_{\mathrm{MB},4}) & <a, \lambda_1>.P + <a, \lambda_2>.P & = & <a, \lambda_1 + \lambda_2>.P &
\end{array}
$$

$$
\begin{array}{ll}
(\mathcal{A}_{\mathrm{MB},5}) & <a, \lambda>.\sum\limits_{i \in I} <\tau, \mu_i>.\sum\limits_{j \in J_i} <\tau, \gamma_{i,j}>.P_{i,j} \;= \\
& \qquad <a, \lambda>.\sum\limits_{i \in I}\sum\limits_{j \in J_i} <\tau, \frac{\mu_i}{\mu} \cdot \frac{\gamma_{i,j}}{\gamma} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)^{-1}>.P_{i,j} \\
& \text{if: } I \neq \emptyset \text{ is a finite index set} \\
& \quad J_i \neq \emptyset \text{ is a finite index set for all } i \in I \\
& \quad \mu = \sum\limits_{i \in I} \mu_i \\
& \quad \gamma = \sum\limits_{j \in J_i} \gamma_{i,j} \text{ for all } i \in I
\end{array}
$$

$$
\begin{array}{lrcll}
(\mathcal{A}_{\mathrm{MB},6}) & \underline{0}/H & = & \underline{0} & \\
(\mathcal{A}_{\mathrm{MB},7}) & (<a, \lambda>.P)/H & = & <a, \lambda>.(P/H) & \text{if } a \notin H \\
(\mathcal{A}_{\mathrm{MB},8}) & (<a, \lambda>.P)/H & = & <\tau, \lambda>.(P/H) & \text{if } a \in H \\
(\mathcal{A}_{\mathrm{MB},9}) & (P_1 + P_2)/H & = & P_1/H + P_2/H &
\end{array}
$$

Regarding completeness, we show that every nonrecursive process term can be transformed into a normal form that abstracts from the order of summands (consistent with the first two axioms), rules out all null summands and occurrences of the hiding operator (consistent with the third and the last four axioms), and does not allow for simplifications based on the fourth and fifth axioms.

Unlike the nondeterministic case, we cannot encode any saturation [18] in the normal form, as this would alter the quantitative behavior. In contrast, we elaborate on the result of Prop. 1 so as to discover that pairs of terms related by $\approx_{\mathrm{MB}}$ but not by $\simeq_{\mathrm{MB}}$ enjoy properties concerned with $\mathcal{A}_{\mathrm{MB},4}$ and $\mathcal{A}_{\mathrm{MB},5}$.

Lemma 1. *Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nr}}$. If $P_1 \approx_{\mathrm{MB}} P_2$ but $P_1 \not\simeq_{\mathrm{MB}} P_2$, then P_1 and P_2 are respectively of the form:*

$$\sum_{i \in I_1} <\tau, \mu_{1,i}>.P_{1,i} \; \textit{and} \; \sum_{i \in I_2} <\tau, \mu_{2,i}>.P_{2,i}$$

where $I_1 \neq \emptyset, I_2 \neq \emptyset$ are finite index sets and at least one process term in $\{P_{1,i} \mid i \in I_1\} \cup \{P_{2,i} \mid i \in I_2\}$ is fully unstable. Moreover:

$$\{D \in \mathbb{P}_{\mathrm{M}}/\approx_{\mathrm{MB}} \mid \exists i \in I_1.\, P_{1,i} \in D\} \neq \{D \in \mathbb{P}_{\mathrm{M}}/\approx_{\mathrm{MB}} \mid \exists i \in I_2.\, P_{2,i} \in D\} \quad \blacksquare$$

Proposition 6. *Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nr}}$. If $P_1 \approx_{\mathrm{MB}} P_2$ but $P_1 \not\simeq_{\mathrm{MB}} P_2$, then at least one between P_1 and P_2 is of the form:*

$$\sum_{i \in I} <\tau, \mu_i>.\sum_{j \in J_i} <\tau, \gamma_{i,j}>.P_{i,j}$$

where $I \neq \emptyset$ is a finite index set, $J_i \neq \emptyset$ is a finite index set for all $i \in I$, and one of the following two properties holds:

– $\sum_{j \in J_{i_1}} <\tau, \gamma_{i_1,j}>.P_{i_1,j} \approx_{\rm MB} \sum_{j \in J_{i_2}} <\tau, \gamma_{i_2,j}>.P_{i_2,j}$ *for all* $i_1, i_2 \in I$.
– $\sum_{j \in J_{i_1}} \gamma_{i_1,j} = \sum_{j \in J_{i_2}} \gamma_{i_2,j}$ *for all* $i_1, i_2 \in I$. ■

Definition 8. *We say that* $P \in \mathbb{P}_{\rm M,nr}$ *is in* $\simeq_{\rm MB}$*-normal-form iff either* P *is* $\underline{0}$ *or* P *is of the form* $\sum_{i \in I} <a_i, \lambda_i>.P_i$ *with* I *finite and nonempty,* P *initially minimal with respect to* $\mathcal{A}_{\rm MB,4}$*,* $<a_i, \lambda_i>.P_i$ *initially minimal with respect to* $\mathcal{A}_{\rm MB,5}$ *for all* $i \in I$*, and* P_i *in* $\simeq_{\rm MB}$*-normal-form for all* $i \in I$. ■

In the definition above, by P initially minimal with respect to $\mathcal{A}_{\rm MB,4}$ we mean that P does not contain any two summands like the ones on the left-hand side of $\mathcal{A}_{\rm MB,4}$. Likewise, by $<a_i, \lambda_i>.P_i$ initially minimal with respect to $\mathcal{A}_{\rm MB,5}$ we mean that $<a_i, \lambda_i>.P_i$ does not match the left-hand side of $\mathcal{A}_{\rm MB,5}$.

It is worth noting that by virtue of Prop. 6, whenever it holds $P_1 \approx_{\rm MB} P_2$ but $P_1 \not\simeq_{\rm MB} P_2$, then at least one between P_1 and P_2 is not in $\simeq_{\rm MB}$-normal-form because of a violation of initial minimality with respect to $\mathcal{A}_{\rm MB,4}$ or $\mathcal{A}_{\rm MB,5}$. This fact will be exploited in the proof of the completeness part of Thm. 4 below.

Lemma 2. *For all* $P \in \mathbb{P}_{\rm M,nr}$ *there exists* $Q \in \mathbb{P}_{\rm M,nr}$ *in* $\simeq_{\rm MB}$*-normal-form such that* $\mathcal{A}_{\rm MB} \vdash P = Q$. ■

Theorem 4. *Let* $P_1, P_2 \in \mathbb{P}_{\rm M,nr}$*. Then* $\mathcal{A}_{\rm MB} \vdash P_1 = P_2 \iff P_1 \simeq_{\rm MB} P_2$. ■

3.3 Exactness at Steady State

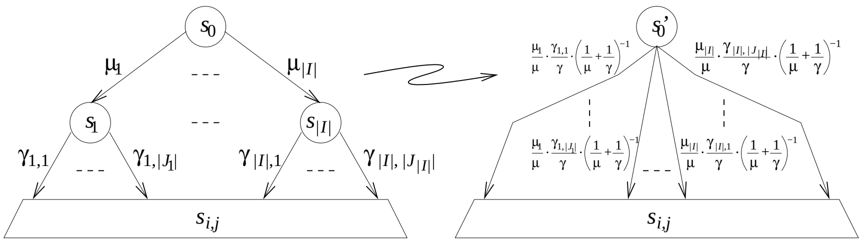
Weak Markovian bisimulation equivalence and the coarsest congruence contained in it are more liberal than Markovian bisimilarity, because they allow every sequence of exponentially timed τ-actions to be considered equivalent to a single exponentially timed τ-action having the same average duration. From a stochastic viewpoint, this amounts to approximating a hypoexponentially or Erlang distributed random variable with an exponentially distributed random variable having the same expected value. From a performance evaluation viewpoint, this can be exploited to assess more quickly properties expressed in terms of the mean time to certain events by working on an aggregated CTMC. [1]

However, it is not necessarily the case that those properties are the only ones preserved by the two weak Markovian behavioral equivalences that we have introduced. This can be investigated by examining the CTMC-level aggregation induced by such equivalences. If it turns out to be an exact CTMC-level aggregation, then the two weak Markovian behavioral equivalences preserve all the performance characteristics. This means that they can be used for reducing the size of models with an underlying CTMC-based semantics without altering the value of any performance measure. [2]

[1] To be precise, since the Markov property of the original CTMC is not preserved but the aggregated stochastic process is still assumed to be a CTMC, it would be more appropriate to call the aggregation a pseudo-aggregation [22].

[2] To be precise, this is true as long as rewards [15, 7] are not associated with fully unstable states and exponentially timed τ-transitions, which is quite reasonable.

Since $\sim_{\rm MB}$ is consistent with ordinary lumpability and the axiomatization of $\simeq_{\rm MB}$ differs from the one of $\sim_{\rm MB}$ only for $\mathcal{A}_{\rm MB,5}$, we can concentrate on this axiom when studying the CTMC-level aggregation induced by $\approx_{\rm MB}$ and $\simeq_{\rm MB}$. If we view $\mathcal{A}_{\rm MB,5}$ without its two $<a, \lambda>$ actions as the following rewriting rule:



then we say that a CTMC is W-lumpable iff a portion of its state space matches the left-hand side of the rewriting rule, in which case it is replaced by the right-hand side where the topmost $1 + |I|$ states have been merged into a single one.

Theorem 5. *W-lumpability is exact at steady state, i.e., the stationary probability of being in a macrostate of a CTMC obtained via W-lumpability is the sum of the stationary probabilities of being in one of the constituent microstates of the CTMC from which the reduced one has been obtained.* ■

Unlike ordinary lumpability and T-lumpability, W-lumpability is not exact at transient state, which means that properties expressed in terms of transient state probabilities may not be preserved. A counterexample is provided by process terms $\bar{P}_1$ and $\bar{P}_2$ of Ex. 1, because the sum of the probabilities of being in one of the first two states of $[\![\bar{P}_1]\!]_{\rm M}$ at time $t \in \mathbb{R}_{>0}$ is different from the probability of being in the first state of $[\![\bar{P}_2]\!]_{\rm M}$ at the same time instant.

In fact, the probability of being in that state of $[\![\bar{P}_2]\!]_{\rm M}$ at that time is the probability that the exponentially distributed duration of its outgoing transition is greater than t, which is $1 - (1 - {\rm e}^{-\frac{\mu \cdot \gamma}{\mu + \gamma} \cdot t}) = {\rm e}^{-\frac{\mu \cdot \gamma}{\mu + \gamma} \cdot t}$ and reduces to ${\rm e}^{-\frac{\mu}{2} \cdot t}$ when $\mu = \gamma$. In contrast, the probability of being in one of those states of $[\![\bar{P}_1]\!]_{\rm M}$ at that time is the probability that the hypoexponentially (for $\mu \neq \gamma$) or Erlang (for $\mu = \gamma$) distributed duration of their two consecutive outgoing transitions is greater than t, which is $1 - (1 - \frac{\gamma}{\gamma - \mu} \cdot {\rm e}^{-\mu \cdot t} + \frac{\mu}{\gamma - \mu} \cdot {\rm e}^{-\gamma \cdot t}) = \frac{\gamma}{\gamma - \mu} \cdot {\rm e}^{-\mu \cdot t} - \frac{\mu}{\gamma - \mu} \cdot {\rm e}^{-\gamma \cdot t}$ or $1 - (1 - (1 + \mu \cdot t) \cdot {\rm e}^{-\mu \cdot t}) = (1 + \mu \cdot t) \cdot {\rm e}^{-\mu \cdot t}$, respectively.

3.4 Decidability in Polynomial Time

In order to check whether $P_1 \approx_{\rm MB} P_2$ or $P_1 \simeq_{\rm MB} P_2$ for any two finite-state processes $P_1, P_2 \in \mathbb{P}_{\rm M}$, similar to other bisimulation equivalences we can employ a partition refinement algorithm based on [20] that:

- Starts with a partition containing one equivalence class for all the non-fully-unstable states of $[\![P_1]\!]_{\rm M}$ and $[\![P_2]\!]_{\rm M}$ and one equivalence class for all the fully unstable states of $[\![P_1]\!]_{\rm M}$ and $[\![P_2]\!]_{\rm M}$.

- Refines the partition until a fixed point is reached, by applying the *rate*-based equality check for splitting the classes of non-fully-unstable states and the *pbtm*-based equality check for splitting the classes of fully unstable states.
- In the case of $\approx_{\rm MB}$, returns yes or no depending on whether P_1 and P_2 belong to the same equivalence class.
- In the case of $\simeq_{\rm MB}$, returns yes or no depending on whether P_1 and P_2 belong to the same equivalence class and satisfy the *rate*-based equality check with respect to all action names and equivalence classes.

Unlike weak bisimulation equivalences for nondeterministic processes and probabilistic processes – which can be decided in polynomial time for all pairs of finite-state processes with analogous partition refinement algorithms [16, 2] – the above algorithm executes in polynomial time only when $[\![P_1]\!]_{\rm M}$ and $[\![P_2]\!]_{\rm M}$ have no cycles of exponentially timed internal transitions.

In fact, while cycles of nondeterministic internal transitions are unimportant from a quantitative viewpoint and cycles of probabilistic internal transitions can be left in the long run with probability 1 (if admitting a way out) or 0 (if connecting an absorbing set of states), cycles of exponentially timed internal transitions cause time to progress and hence cannot be ignored. In particular, their presence causes *pbtm* multisets to be infinite.

For instance, consider $P \equiv <\tau, \mu>. \operatorname{rec} X : (<\tau, \delta>.X + <\tau, \gamma>.Q)$ where $Q \in \mathbb{P}_{\rm M,nfu}$. Due to the presence in $[\![P]\!]_{\rm M}$ of the exponentially timed internal selfloop labeled with $<\tau, \delta>$, we have that $pbtm(P, [Q]_{\approx_{\rm MB}})$ contains infinitely many *probtime* values of the form $(\frac{\delta}{\delta+\gamma})^n \cdot \frac{\gamma}{\delta+\gamma} \cdot (\frac{1}{\mu} + (n+1) \cdot \frac{1}{\delta+\gamma})$ where $n \in \mathbb{N}$. If the selfloop were ignored, then P would erroneously be considered to be weakly Markovian bisimilar to $<\tau, \frac{\mu \cdot \gamma}{\mu+\gamma}>.Q$. Likewise, if only a finite number of *probtime* values were taken into account, then P would erroneously be considered to be weakly Markovian bisimilar to some process such that the average duration of all the reducible computations starting from that process is bounded.

4 Conclusion

In this paper, we have introduced a weak variant $\approx_{\rm MB}$ of Markovian bisimilarity for sequential processes with abstraction, which reduces any sequence of at least two exponentially timed τ-actions to a single exponentially timed τ-action whenever it is possible to preserve the average duration and the execution probability of the sequence. Then, we have characterized the coarsest congruence $\simeq_{\rm MB}$ contained in $\approx_{\rm MB}$ and we have found a sound and complete axiomatization for it, which has been exploited to prove the exactness at steady state of the induced CTMC-level aggregation for all the considered processes. Finally, we have established the decidability in polynomial time of $\approx_{\rm MB}$ and $\simeq_{\rm MB}$ over finite-state processes without cycles of exponentially timed internal actions.

From a different viewpoint, this paper confirms in a Markovian setting the adequacy of the construction used in [18] to single out the coarsest congruence included in a weak bisimulation equivalence for nondeterministic processes that is not closed with respect to alternative composition. It is worth noting that different approaches to the definition of a weak bisimulation equivalence like branching bisimulation [11] and dynamic/progressing bisimulation [19] are no longer suitable in a Markovian setting, as they are too demanding about matching exponentially timed internal actions.

On the stochastic side, we have assumed in this paper that an external observer can see the names of the actions that are performed by the processes as well as the average durations of those actions. Consequently, the external observer is not able to distinguish between an arbitrarily long sequence of exponentially timed τ-actions and a single exponentially timed τ-action having the same average duration. This leads to a state space reduction that preserves steady-state performance measures, but not transient-state performance measures except those that are expressed in terms of mean time to certain events. We point out that considering higher moments – e.g., the variance – of the duration of the actions in addition to its expectation may bring some advantage in terms of transient measure preservation. However, we would end up with a much finer Markovian behavioral equivalence, because the two random variables respectively quantifying the duration of a sequence of exponentially timed τ-actions and the duration of a single exponentially timed τ-action do not necessarily have the same variance when their expected values coincide.

4.1 Related Work

The idea of reducing a sequence of exponentially timed τ-actions to a single exponentially timed τ-action preserving the average duration of the action sequence was originally proposed in [14] through a relation called weak (Markovian) isomorphism. This was shown to be a congruence for both sequential and concurrent processes and to be exact at steady state only for a class of processes satisfying certain constraints on action synchronization. However, unlike $\approx_{\mathrm{MB}}$ and $\simeq_{\mathrm{MB}}$, no axiomatization was provided.

In this paper, we have revisited the idea at the basis of weak (Markovian) isomorphism in the less restrictive bisimulation framework. An important extension with respect to [14] is that we have considered not only individual sequences of exponentially timed τ-actions. In fact, we have addressed trees of exponentially timed τ-actions and we have established the conditions under which such trees can be reduced by preserving both the average duration and the execution probability of their branches. For instance, the pairs of process terms compared in Exs. 2 and 3 are not related by weak (Markovian) isomorphism. A further difference with respect to [14] is that W-lumpability is exact at steady state for all processes even if we consider a Markovian process calculus including parallel composition – as the memoryless property of exponential distributions allows us to take an interleaving view of concurrent process terms – without having to respect any constraint on action synchronization.

Another approach to abstracting from τ-actions in an exponentially timed setting comes from [8], where a variant of Markovian bisimilarity was defined that checks for exit rate equality with respect to all equivalence classes apart from the one including the processes under examination. Congruence and axiomatization results were provided for the proposed equivalence, and a logical characterization based on CSL was illustrated in [3]. However, unlike $\approx_{\mathrm{MB}}$ and $\simeq_{\mathrm{MB}}$, nothing was said about exactness.

4.2 Future Work

A drawback of $\simeq_{\mathrm{MB}}$ is that – unlike weak (Markovian) isomorphism – it is not a congruence with respect to parallel composition, a fact that limits its usefulness for compositional state space reduction purposes. We are currently working on a generalization of $\simeq_{\mathrm{MB}}$ inspired by [14] that exploits context-related information when traversing trees of exponentially timed τ-actions of concurrent processes. The idea is to allow for reductions also in the case of replicas of computations originated from a single process whose local states are fully unstable but are part of global states (due to parallel composition) that are not fully unstable [5].

Having two distinct weak Markovian bisimulation congruences – $\simeq_{\mathrm{MB}}$ and its generalization to concurrent processes – seems to be justified by the tradeoff that exists between achieving compositionality also over concurrent processes and ensuring exactness at steady state for all the considered processes without imposing any constraint.

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Appendix: Proofs of Results of Sect. 3

Proof of Prop. 1. Let us call P_1 and P_2 the two considered process terms and let $\sum_{i \in I} \mu_i \equiv \mu$ and $\sum_{j \in J_{i_1}} \gamma_{i_1,j} = \sum_{j \in J_{i_2}} \gamma_{i_2,j} \equiv \gamma$ for all $i_1, i_2 \in I$. It turns out that $\mathcal{B} = \{(P_1, P_2), (P_2, P_1)\} \cup \{(P, P) \mid P \in \mathbb{P}_{\mathrm{M}}\}$ is a weak Markovian bisimulation. In fact, for all $D \in \mathbb{P}_{\mathrm{M,nfu}}/\mathcal{B}$ there are three nontrivial cases all regarding P_1 and P_2:

– If D does not contain any $P_{i,j}$ and is not reachable via reducible computations from any $P_{i,j}$, then:
$$\mathit{pbtm}(P_1, D) \; = \; \emptyset \; = \; \mathit{pbtm}(P_2, D)$$
– If $D = \{P_{i_0,j_0}\}$ for some $i_0 \in I$ and $j_0 \in J_{i_0}$, then:
$$\begin{aligned}\mathit{pbtm}(P_1, D) \; &= \; \{\!\!| \; \frac{\mu_{i_0}}{\mu} \cdot \frac{\gamma_{i_0,j_0}}{\gamma} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma}\right) \; |\!\!\} \; = \\ &= \; \{\!\!| \; \frac{\frac{\mu_{i_0}}{\mu} \cdot \frac{\gamma_{i_0,j_0}}{\gamma} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)^{-1}}{r} \cdot \frac{1}{r} \; |\!\!\} \; = \; \mathit{pbtm}(P_2, D)\end{aligned}$$
where:
$$r = \sum_{i \in I} \sum_{j \in J_i} \frac{\mu_i}{\mu} \cdot \frac{\gamma_{i,j}}{\gamma} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)^{-1} = \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)^{-1} \cdot \sum_{i \in I} \frac{\mu_i}{\mu} \cdot \sum_{j \in J_i} \frac{\gamma_{i,j}}{\gamma} = \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)^{-1}$$
– If D is reachable via reducible computations from P_{i_0,j_0} for some $i_0 \in I$ and $j_0 \in J_{i_0}$, then for each such reducible computation the *probtime* contribution from P_1 to P_{i_0,j_0} coincides with the *probtime* contribution from P_2 to P_{i_0,j_0}:
$$\frac{\mu_{i_0}}{\mu} \cdot \frac{\gamma_{i_0,j_0}}{\gamma} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma}\right) \; = \; \frac{\frac{\mu_{i_0}}{\mu} \cdot \frac{\gamma_{i_0,j_0}}{\gamma} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)^{-1}}{r} \cdot \frac{1}{r}$$
where r is as in the previous case. Therefore:
$$\mathit{pbtm}(P_1, D) \; = \; \mathit{pbtm}(P_2, D)$$
■

Proof of Prop. 2. Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M}}$ be such that $P_1 \approx_{\mathrm{MB}} P_2$ and let $\mathcal{B}$ be a weak Markovian bisimulation containing the pair (P_1, P_2):

1. Given $<a, \lambda> \in \mathit{Act}_{\mathrm{M}}$, it turns out that the symmetric and transitive closure $\mathcal{B}'$ of the relation $\mathcal{B} \cup \{(<a, \lambda>.P_1, <a, \lambda>.P_2)\}$ is a weak Markovian bisimulation. In fact, there are two nontrivial cases regarding $<a, \lambda>.P_1$ and $<a, \lambda>.P_2$ and the equivalence class D with respect to $\mathcal{B}'$ containing P_1, P_2:
   – If $a \neq \tau$, then $<a, \lambda>.P_1, <a, \lambda>.P_2 \in \mathbb{P}_{\mathrm{M,nfu}}$ and for all $a' \in \mathit{Name}$ and $D' \in \mathbb{P}_{\mathrm{M}}/\mathcal{B}'$ we have that:
   $$\mathit{rate}(<a, \lambda>.P_1, a', D') = \mathit{rate}(<a, \lambda>.P_2, a', D') = \begin{cases} \lambda & \text{if } a' = a \wedge D' = D \\ 0 & \text{if } a' \neq a \vee D' \neq D \end{cases}$$
   – If $a = \tau$, then $<a, \lambda>.P_1, <a, \lambda>.P_2 \in \mathbb{P}_{\mathrm{M,fu}}$ with the only equivalence class reachable in one step by both of them being D. Let $D' \in \mathbb{P}_{\mathrm{M,nfu}}/\mathcal{B}'$. If D' is not reachable from D via reducible computations, then:
   $$\mathit{pbtm}(<a, \lambda>.P_1, D') \; = \; \emptyset \; = \; \mathit{pbtm}(<a, \lambda>.P_2, D')$$
   otherwise for each reducible computation to D' the *probtime* contribution from $<a, \lambda>.P_1$ to D coincides with the *probtime* contribution from $<a, \lambda>.P_2$ to D and hence:
   $$\mathit{pbtm}(<a, \lambda>.P_1, D') \; = \; \mathit{pbtm}(<a, \lambda>.P_2, D')$$

2. Given $H \subseteq \mathit{Name}_{\mathrm{v}}$, it turns out that the transitive closure $\mathcal{B}'$ of the relation $\mathcal{B} \cup \{(P'_1/H, P'_2/H) \mid (P'_1, P'_2) \in \mathcal{B}\}$ is a weak Markovian bisimulation. In fact, there are two nontrivial cases all regarding pairs $(P'_1/H, P'_2/H) \in \mathcal{B}'$ and equivalence classes D of the form $[P'/H]_{\mathcal{B}'} = \{P''/H \in \mathbb{P}_{\mathrm{M}} \mid P'' \in [P']_{\mathcal{B}}\}$:
   - If $P'_1/H, P'_2/H \in \mathbb{P}_{\mathrm{M,nfu}}$, then $\mathit{rate}(P'_1/H, a, D) = \mathit{rate}(P'_2/H, a, D)$ because for $h \in \{1, 2\}$ it holds that $\mathit{rate}(P'_h/H, a, D)$ is equal to:
$$\begin{cases} 0 & \text{if } a \in H \\ \mathit{rate}(P'_h, a, [P']_{\mathcal{B}}) & \text{if } a \notin H \cup \{\tau\} \\ \sum\limits_{b \in H \cup \{\tau\}} \mathit{rate}(P'_h, b, [P']_{\mathcal{B}}) & \text{if } a = \tau \end{cases}$$
   - If $P'_1/H, P'_2/H \in \mathbb{P}_{\mathrm{M,fu}}$, for $D \subseteq \mathbb{P}_{\mathrm{M,nfu}}$ we have that each reducible computation from P'_1 (resp. P'_2) to $[P']_{\mathcal{B}}$ induces a reducible computation from P'_1/H (resp. P'_2/H) to D with the same *probtime* measure. In addition, there might be further reducible computations from P'_1/H (resp. P'_2/H) to D originated from the fact that $\_/H$ has made some intermediate states between P'_1 (resp. P'_2) and $[P']_{\mathcal{B}}$ fully unstable. Since $(P'_1, P'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a weak Markovian bisimulation, those intermediate states have to be pairwise related by $\mathcal{B}$ and hence have to pass the exit rate equality check, which is enough to guarantee that the multiset of additional reducible computations from P'_1/H to D having a certain average duration and the multiset of additional reducible computations from P'_2/H to D having the same average duration have the same sum of *probtime* measures. Therefore $\mathit{pbtm}(P'_1/H, D) = \mathit{pbtm}(P'_2/H, D)$. ■

Proof of Prop. 3. Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M}}$. The proof is divided into five parts:

- Firstly, we prove that $P_1 \sim_{\mathrm{MB}} P_2$ implies $P_1 \approx_{\mathrm{MB}} P_2$. If $P_1 \sim_{\mathrm{MB}} P_2$, then there exists a Markovian bisimulation $\mathcal{B}$ containing the pair (P_1, P_2). It turns out that $\mathcal{B}$ is a weak Markovian bisimulation too. In fact, observed that $\mathcal{B}$ cannot contain any pair composed of a fully unstable process term and a non-fully-unstable process term, the following holds for all $(P'_1, P'_2) \in \mathcal{B}$:
  - If $P'_1, P'_2 \in \mathbb{P}_{\mathrm{M,nfu}}$, then for all $a \in \mathit{Name}$ and $D \in \mathbb{P}_{\mathrm{M}}/\mathcal{B}$:
$$\mathit{rate}(P'_1, a, D) = \mathit{rate}(P'_2, a, D)$$
  The reason is that $(P'_1, P'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a Markovian bisimulation.
  - If $P'_1, P'_2 \in \mathbb{P}_{\mathrm{M,fu}}$, then for all $D \subseteq \mathbb{P}_{\mathrm{M,nfu}}/\mathcal{B}$:
$$\mathit{pbtm}(P'_1, D) = \mathit{pbtm}(P'_2, D)$$
  The reason is that, since $(P'_1, P'_2) \in \mathcal{B}$ and $\mathcal{B}$ is a Markovian bisimulation, for each maximal multiset of reducible computations from P'_1 (resp. P'_2) to D whose corresponding traversed states form pairs contained in $\mathcal{B}$, there exists a maximal multiset of reducible computations from P'_2 (resp. P'_1) to D whose corresponding traversed states form pairs contained in $\mathcal{B}$, such that all corresponding states traversed by the reducible computations in the two multisets form pairs contained in $\mathcal{B}$. Therefore, the two multisets contribute to *pbtm* with the same sum of *probtime* measures.

– Secondly, we demonstrate that $P_1 \sim_{\rm MB} P_2$ implies $P_1 \simeq_{\rm MB} P_2$. Since we have proved that $\sim_{\rm MB} \subseteq \approx_{\rm MB}$, the equivalence classes of $\approx_{\rm MB}$ are unions of equivalence classes of $\sim_{\rm MB}$. Thus, if $P_1 \sim_{\rm MB} P_2$ and we take $a \in \mathit{Name}$ and $D \in \mathbb{P}_{\rm M}/\approx_{\rm MB}$ with $D = \cup_{i \in I} D_i$ and $D_i \in \mathbb{P}_{\rm M}/\sim_{\rm MB}$ for all $i \in I$, we have:

$$\mathit{rate}(P_1, a, D) \; = \; \sum_{i \in I} \mathit{rate}(P_1, a, D_i) \; = \; \sum_{i \in I} \mathit{rate}(P_2, a, D_i) \; = \; \mathit{rate}(P_2, a, D)$$

which means that $P_1 \simeq_{\rm MB} P_2$.

– Thirdly, we show that $P_1 \simeq_{\rm MB} P_2$ implies $P_1 \approx_{\rm MB} P_2$. Whenever $P_1 \simeq_{\rm MB} P_2$, then $P_1 \approx_{\rm MB} P_2$ because:
  - If $P_1, P_2 \in \mathbb{P}_{\rm M,nfu}$, then for all $a \in \mathit{Name}$ and $D \in \mathbb{P}_{\rm M}/\approx_{\rm MB}$:
  $$\mathit{rate}(P_1, a, D) \; = \; \mathit{rate}(P_2, a, D)$$
  The reason is that $P_1 \simeq_{\rm MB} P_2$.
  - If $P_1, P_2 \in \mathbb{P}_{\rm M,fu}$, then for all $D \subseteq \mathbb{P}_{\rm M,nfu}/\approx_{\rm MB}$:
  $$\mathit{pbtm}(P_1, D) \; = \; \mathit{pbtm}(P_2, D)$$
  The reason is that, since $P_1 \simeq_{\rm MB} P_2$, both P_1 and P_2 reach in one step the same equivalence classes at the same rates and hence the first step towards D contributes to *pbtm* in the same way for P_1 and P_2. At that point, it is enough to consider among those equivalence classes reached in one step by P_1 and P_2 both D itself (if reachable in one step) and the ones from which it is possible to arrive at D via reducible computations.

– Fourthly, we prove that the inclusions are strict. For example, we have:

$$<a, \lambda>.<\tau, \mu>.<\tau, \gamma>.\underline{0} \;\not\sim_{\rm MB}\; <a, \lambda>.<\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0}$$
$$<a, \lambda>.<\tau, \mu>.<\tau, \gamma>.\underline{0} \;\simeq_{\rm MB}\; <a, \lambda>.<\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0}$$

and:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} \;\not\simeq_{\rm MB}\; <\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0}$$
$$<\tau, \mu>.<\tau, \gamma>.\underline{0} \;\approx_{\rm MB}\; <\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0}$$

– Finally, the fact that $P_1 \simeq_{\rm MB} P_2$ iff $P_1 \approx_{\rm MB} P_2$ when $P_1, P_2 \in \mathbb{P}_{\rm M,nfu}$ stems immediately from the definitions of $\simeq_{\rm MB}$ and $\approx_{\rm MB}$. ■

Proof of Prop. 4. A straightforward consequence of the definition of $\simeq_{\rm MB}$. ■

Proof of Thm. 1. Let $P_1, P_2 \in \mathbb{P}_{\rm M}$ be such that $P_1 \simeq_{\rm MB} P_2$:

1. By virtue of Prop. 3, from $P_1 \simeq_{\rm MB} P_2$ it follows that $P_1 \approx_{\rm MB} P_2$ and hence P_1 and P_2 belong to the same equivalence class D with respect to $\approx_{\rm MB}$. Given $<a, \lambda> \in \mathit{Act}_{\rm M}$, for all $a' \in \mathit{Name}$ and $D' \in \mathbb{P}_{\rm M}/\approx_{\rm MB}$ we have that:
$$\mathit{rate}(<a, \lambda>.P_1, a', D') \; = \; \mathit{rate}(<a, \lambda>.P_2, a', D') \; = \; \begin{cases} \lambda & \text{if } a' = a \wedge D' = D \\ 0 & \text{if } a' \neq a \vee D' \neq D \end{cases}$$
Therefore $<a, \lambda>.P_1 \simeq_{\rm MB} <a, \lambda>.P_2$.
2. Given $P \in \mathbb{P}_{\rm M}$, for all $a \in \mathit{Name}$ and $D \in \mathbb{P}_{\rm M}/\approx_{\rm MB}$ we have that:
$$\begin{aligned} \mathit{rate}(P_1 + P, a, D) \; &= \; \mathit{rate}(P_1, a, D) + \mathit{rate}(P, a, D) \; = \\ &= \; \mathit{rate}(P_2, a, D) + \mathit{rate}(P, a, D) \; = \; \mathit{rate}(P_2 + P, a, D) \end{aligned}$$
because $P_1 \simeq_{\rm MB} P_2$. Therefore $P_1 + P \simeq_{\rm MB} P_2 + P$ and $P + P_1 \simeq_{\rm MB} P + P_2$.
3. Given $H \subseteq \mathit{Name}_{\rm v}$, for all $a \in \mathit{Name}$ and $D \in \mathbb{P}_{\rm M}/\approx_{\rm MB}$ there are two cases:
   – If D does not contain any term of the form P/H, then:
   $$\mathit{rate}(P_1/H, a, D) \; = \; 0 \; = \; \mathit{rate}(P_2/H, a, D)$$

– If $D = [P / H]_{\approx_{\rm MB}}$, then we can exploit the congruence property of $\approx_{\rm MB}$ with respect to the hiding operator as established by Prop. 2 in order to express D as $\bigcup_{P' \in D'} [P']_{\approx_{\rm MB}} / H$, where D' is a maximal set including P of process terms that are pairwise not related by $\approx_{\rm MB}$, such that $P' / H \approx_{\rm MB} P / H$ for all $P' \in D'$. As a consequence:

$$\mathit{rate}(P_1 / H, a, D) \; = \; \mathit{rate}(P_2 / H, a, D)$$

because for $h \in \{1, 2\}$ it holds that $\mathit{rate}(P_h / H, a, D)$ is equal to:

$$\begin{cases} 0 & \text{if } a \in H \\ \sum\limits_{P' \in D'} \mathit{rate}(P_h, a, [P']_{\approx_{\rm MB}}) & \text{if } a \notin H \cup \{\tau\} \\ \sum\limits_{P' \in D'} \sum\limits_{b \in H \cup \{\tau\}} \mathit{rate}(P_h, b, [P']_{\approx_{\rm MB}}) & \text{if } a = \tau \end{cases}$$

and $P_1 \simeq_{\rm MB} P_2$.

Therefore $P_1 / H \simeq_{\rm MB} P_2 / H$. ■

Proof of Thm. 2. Let $P_1, P_2 \in \mathbb{P}_{\rm M}$. The proof is divided into two parts:

⇒ If $P_1 \simeq_{\rm MB} P_2$, then by virtue of Thm. 1 it follows that $P_1 + P \simeq_{\rm MB} P_2 + P$ for all $P \in \mathbb{P}_{\rm M}$. Due to Prop. 3, this implies that $P_1 + P \approx_{\rm MB} P_2 + P$ for all $P \in \mathbb{P}_{\rm M}$.

⇐ Suppose that $P_1 + P \approx_{\rm MB} P_2 + P$ for all $P \in \mathbb{P}_{\rm M}$. Since it is possible to find $\bar{P} \in \mathbb{P}_{\rm M}$ such that neither $P_1 + \bar{P}$ nor $P_2 + \bar{P}$ is fully unstable, from $P_1 + \bar{P} \approx_{\rm MB} P_2 + \bar{P}$ it follows that $P_1 + \bar{P} \simeq_{\rm MB} P_2 + \bar{P}$ because $\simeq_{\rm MB}$ and $\approx_{\rm MB}$ coincide over $\mathbb{P}_{\rm M,nfu}$ as established by Prop. 3. Since for all $a \in \mathit{Name}$ and $D \in \mathbb{P}_{\rm M} / \approx_{\rm MB}$ it then holds that:

$$\begin{aligned} \mathit{rate}(P_1, a, D) \; &= \; \mathit{rate}(P_1 + \bar{P}, a, D) - \mathit{rate}(\bar{P}, a, D) \; = \\ &= \; \mathit{rate}(P_2 + \bar{P}, a, D) - \mathit{rate}(\bar{P}, a, D) \; = \; \mathit{rate}(P_2, a, D) \end{aligned}$$

we have that $P_1 \simeq_{\rm MB} P_2$. ■

Proof of Prop. 5. Let $\mathcal{B}$ be a weak Markovian bisimulation up to $\approx_{\rm MB}$. We first show that $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$ is a weak Markovian bisimulation by proving by induction on $n \in \mathbb{N}_{>0}$ that for all $(P_1, P_2) \in (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^n$:

– If $P_1, P_2 \in \mathbb{P}_{\rm M,nfu}$, then for all $a \in \mathit{Name}$ and $D \in \mathbb{P}_{\rm M} / (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$:

$$\mathit{rate}(P_1, a, D) \; = \; \mathit{rate}(P_2, a, D)$$

– If $P_1, P_2 \in \mathbb{P}_{\rm M,fu}$, then for all $D \in \mathbb{P}_{\rm M,nfu} / (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$:

$$\mathit{pbtm}(P_1, D) \; = \; \mathit{pbtm}(P_2, D)$$

Let $(P_1, P_2) \in (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^n$:

– If $n = 1$, then $(P_1, P_2) \in \mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB}$. There are two cases:
  - If $(P_1, P_2) \in \mathcal{B} \cup \mathcal{B}^{-1}$, then the result immediately follows from the fact that $\mathcal{B}$ is a weak Markovian bisimulation up to $\approx_{\rm MB}$.
  - If $(P_1, P_2) \in \approx_{\rm MB}$, then the result stems from the fact that $\approx_{\rm MB} \subseteq (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$ and hence each equivalence class of $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$ is the union of some equivalence classes of $\approx_{\rm MB}$.

– Let $n > 1$ and the result hold for all $(Q_1, Q_2) \in (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^{n-1}$. From $(P_1, P_2) \in (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^n$, we derive that there exists $P \in \mathbb{P}_{\rm M}$ such that $(P_1, P) \in (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^{n-1}$ and $(P, P_2) \in \mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB}$. Then the result holds both for the pair (P_1, P) – by the induction hypothesis – and for the pair (P, P_2) – by reasoning like in the case $n = 1$. As a consequence, the three process terms P_1, P_2, and P all belong either to $\mathbb{P}_{\rm M,nfu}$ or to $\mathbb{P}_{\rm M,fu}$ and hence the result follows for the pair (P_1, P_2) by transitivity of *rate* equality or *pbtm* equality, respectively.

Since we have proved that $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$ is a weak Markovian bisimulation, $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+ \subseteq \approx_{\rm MB}$. On the other hand, $\mathcal{B} \subseteq (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$. Therefore, $\mathcal{B} \subseteq \approx_{\rm MB}$ by transitivity of set inclusion, i.e., $(P_1, P_2) \in \mathcal{B}$ implies $P_1 \approx_{\rm MB} P_2$ for all $P_1, P_2 \in \mathbb{P}_{\rm M}$. We also note that $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+ = \approx_{\rm MB}$. ■

Proof of Thm. 3. Without loss of generality, we assume for simplicity that the two process terms $P_1, P_2 \in \mathcal{PL}_{\rm M}$ such that $P_1 \simeq_{\rm MB} P_2$ contain free occurrences of a single process variable $X \in \mathit{Var}$. Consider the binary relation:

$$\mathcal{B} \;=\; \{(P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, P\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}) \mid P \in \mathcal{PL}_{\rm M} \text{ containing free occurrences of } X \text{ at most}\}$$

which is a subset of $(\mathcal{PL}_{\rm M,nfu} \times \mathcal{PL}_{\rm M,nfu}) \cup (\mathcal{PL}_{\rm M,fu} \times \mathcal{PL}_{\rm M,fu})$. In fact, e.g., the case $P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\} \in \mathcal{PL}_{\rm M,nfu}$ and $P\{\mathrm{rec}\, X : P_2 \hookrightarrow X\} \in \mathcal{PL}_{\rm M,fu}$ is not possible because:

– If P is not a process variable, then the actions enabled by $P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ and the actions enabled by $P\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}$ coincide with the actions enabled by P.
– If P is a process variable, which must be X, then $P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ is equal to $\mathrm{rec}\, X : P_1$ and $P\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}$ is equal to $\mathrm{rec}\, X : P_2$. The two resulting process terms are isomorphic to $P_1\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ and $P_2\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}$, respectively, with $P_1\{\mathrm{rec}\, X : P_1 \hookrightarrow X\} \simeq_{\rm MB} P_2\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}$ because $P_1 \simeq_{\rm MB} P_2$.

Similar to [18], we show that $\mathcal{B}$ has a property stronger than being a weak Markovian bisimulation up to $\approx_{\rm MB}$: for each $P \in \mathcal{PL}_{\rm M}$ containing free occurrences of X at most, it holds that for all action names $a \in \mathit{Name}$ and equivalence classes $D \in \mathcal{PL}_{\rm M} / (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB})^+$:

$$\mathit{rate}(P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) \;\leq\; \mathit{rate}(P\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D)$$

(like in [9], $\geq$ can be established between the two *rate* values with a symmetric argument, from which it can be concluded that the two *rate* values coincide).

If $\mathit{rate}(P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) = 0$, then the property trivially holds, otherwise we proceed by induction on the maximum depth $d \in \mathbb{N}_{>0}$ of the inferences of the transitions from $P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ to D labeled with a:

– If $d = 1$, then only the semantic rule for the action prefix operator has been applied and hence P must be of the form $<a, \lambda>.P'$ (notice that it cannot be P equal to X because in that case $P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ would be equal to $\mathrm{rec}\, X : P_1$, which would contradict $d = 1$). Thus, for $i \in \{1, 2\}$ we have that $P\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$ is of the form $<a, \lambda>.(P'\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$. Since P' contains free occurrences of X at most, $(P'\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, P'\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}) \in \mathcal{B}$ and hence both process terms belong to D. Thus $\mathit{rate}(P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) = \lambda = \mathit{rate}(P\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D)$.

- Let $d > 1$ and suppose that the property holds for all triples composed of a pair of process terms in $\mathcal{B}$, an equivalence class D', and an action name a' such that there are transitions from the first process term of the pair to D' labeled with a' and the maximum depth of their inferences is at most $d - 1$. There are four cases:
  - If P is of the form $P' + P''$, then for $i \in \{1, 2\}$ we have that $P\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$ is of the form $P'\{\mathrm{rec}\, X : P_i \hookrightarrow X\} + P''\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$ and hence $\mathit{rate}(P\{\mathrm{rec}\, X : P_i \hookrightarrow X\}, a, D) = \mathit{rate}(P'\{\mathrm{rec}\, X : P_i \hookrightarrow X\}, a, D) + \mathit{rate}(P''\{\mathrm{rec}\, X : P_i \hookrightarrow X\}, a, D)$. In this case, the semantic rules for the alternative composition operator are applied first and hence the transitions from $P'\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ and $P''\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ to D labeled with a are considered (their inferences have maximum depth $d - 1$). If there are no such transitions from $P'\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$, then $\mathit{rate}(P'\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) = 0$, otherwise – since P' contains free occurrences of X at most – from the induction hypothesis it follows that $\mathit{rate}(P'\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) \leq \mathit{rate}(P'\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D)$. Using a similar argument, we have that $\mathit{rate}(P''\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) = 0$ or by the induction hypothesis $\mathit{rate}(P''\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) \leq \mathit{rate}(P''\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D)$. Thus $\mathit{rate}(P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) \leq \mathit{rate}(P\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D)$.
  - If P is a process variable, which must be X, then for $i \in \{1, 2\}$ we have that $P\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$ is equal to $\mathrm{rec}\, X : P_i$, which in turn is isomorphic to $P_i\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$ and hence $\mathit{rate}(\mathrm{rec}\, X : P_i, a, D) = \mathit{rate}(P_i\{\mathrm{rec}\, X : P_i \hookrightarrow X\}, a, D)$. In this case, the semantic rule for recursion is applied first and hence the transitions from $P_1\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ to D labeled with a are considered (their inferences have maximum depth $d - 1$). Since P_1 contains free occurrences of X at most, from the induction hypothesis it follows that $\mathit{rate}(P_1\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) \leq \mathit{rate}(P_1\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D)$, with $\mathit{rate}(P_1\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D) = \mathit{rate}(P_2\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D)$ because $P_1 \simeq_{\mathrm{MB}} P_2$. Thus $\mathit{rate}(\mathrm{rec}\, X : P_1, a, D) \leq \mathit{rate}(\mathrm{rec}\, X : P_2, a, D)$.
  - If P is of the form $\mathrm{rec}\, Y : P'$, then there are two subcases:
    * If $Y = X$, then P contains no free occurrences of X. Therefore, for $i \in \{1, 2\}$ we have that $P\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$ is equal to P and hence $\mathit{rate}(P\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}, a, D) = \mathit{rate}(P\{\mathrm{rec}\, X : P_2 \hookrightarrow X\}, a, D)$.
    * If $Y \neq X$, then for $i \in \{1, 2\}$ we have that $P\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$ is isomorphic to $P'\{\mathrm{rec}\, Y : P' \hookrightarrow Y\}\{\mathrm{rec}\, X : P_i \hookrightarrow X\}$ and hence $\mathit{rate}(P\{\mathrm{rec}\, X : P_i \hookrightarrow X\}, a, D) = \mathit{rate}(P'\{\mathrm{rec}\, Y : P' \hookrightarrow Y\}\{\mathrm{rec}\, X : P_i \hookrightarrow X\}, a, D)$. In this case, the semantic rule for recursion is applied first and hence the transitions from $P'\{\mathrm{rec}\, Y : P' \hookrightarrow Y\}\{\mathrm{rec}\, X : P_1 \hookrightarrow X\}$ to D labeled with a are considered (their inferences have maximum depth $d - 1$). Since

$P'\{\text{rec}\, Y : P' \hookrightarrow Y\}$ contains free occurrences of X at most, from the induction hypothesis it follows that $\mathit{rate}(P'\{\text{rec}\, Y : P' \hookrightarrow Y\} \{\text{rec}\, X : P_1 \hookrightarrow X\}, a, D) \leq \mathit{rate}(P'\{\text{rec}\, Y : P' \hookrightarrow Y\} \{\text{rec}\, X : P_2 \hookrightarrow X\}, a, D)$. Thus $\mathit{rate}(P\{\text{rec}\, X : P_1 \hookrightarrow X\}, a, D) \leq \mathit{rate}(P\{\text{rec}\, X : P_2 \hookrightarrow X\}, a, D)$.

- If P is of the form P'/H, then for $i \in \{1, 2\}$ we have that $P\{\text{rec}\, X : P_i \hookrightarrow X\}$ is of the form $(P'\{\text{rec}\, X : P_i \hookrightarrow X\})/H$. In this case, the semantic rules for the hiding operator are applied first and hence the transitions from $P'\{\text{rec}\, X : P_1 \hookrightarrow X\}$ to $[Q']_{(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\text{MB}})^+}$ labeled with a' are considered (their inferences have maximum depth $d - 1$), where $Q'/H \in D$ and $a' = a$ for $a \notin H \cup \{\tau\}$, $a' \in H \cup \{\tau\}$ for $a = \tau$. Since $\approx_{\text{MB}}$ is a congruence with respect to the hiding operator by virtue of Prop. 2, and $\mathcal{B}$ can be easily shown to be a congruence with respect to the hiding operator, the equivalence class D can be expressed as $\bigcup_{Q'' \in D'} [Q'']_{(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\text{MB}})^+}/H$, where D' is a maximal set including Q' of process terms that are pairwise not related by $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\text{MB}})^+$, such that $(Q''/H, Q'/H) \in (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\text{MB}})^+$ for all $Q'' \in D'$. As a consequence, for $i \in \{1, 2\}$ we have that $\mathit{rate}((P'\{\text{rec}\, X : P_i \hookrightarrow X\})/H, a, D)$ is equal to:
  - ∗ 0 if $a \in H$.
  - ∗ $\sum_{Q'' \in D'} \mathit{rate}(P'\{\text{rec}\, X{:}P_i \hookrightarrow X\}, a, [Q'']_{(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\text{MB}})^+})$ if $a \notin H \cup \{\tau\}$.
  - ∗ $\sum_{Q'' \in D'} \sum_{a' \in H \cup \{\tau\}} \mathit{rate}(P'\{\text{rec}\, X{:}P_i \hookrightarrow X\}, a', [Q'']_{(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\text{MB}})^+})$ if $a = \tau$.

  For each of the *rate* summands giving rise to the P_1-related value $\mathit{rate}((P'\{\text{rec}\, X : P_1 \hookrightarrow X\})/H, a, D)$, if there are no transitions contributing to the *rate* summand, then the summand is equal to 0, otherwise – since P' contains free occurrences of X at most – from the induction hypothesis it follows that the summand is not greater than the corresponding *rate* summand for P_2. Therefore, we conclude that $\mathit{rate}((P'\{\text{rec}X{:}P_1 \hookrightarrow X\})/H, a, D) \leq \mathit{rate}((P'\{\text{rec}X{:}P_2 \hookrightarrow X\})/H, a, D)$.

From the property of $\mathcal{B}$ that we have proved (and the symmetrical property), it follows that $\mathcal{B}$ is a weak Markovian bisimulation up to $\approx_{\text{MB}}$. In fact, if $P\{\text{rec}\, X : P_1 \hookrightarrow X\}, P\{\text{rec}\, X : P_2 \hookrightarrow X\} \in \mathcal{PL}_{\text{M,fu}}$, then for all $D \in \mathcal{PL}_{\text{M,nfu}}/(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\text{MB}})^+$ it holds that $\mathit{pbtm}(P\{\text{rec}\, X : P_1 \hookrightarrow X\}, D) = \mathit{pbtm}(P\{\text{rec}\, X : P_2 \hookrightarrow X\}, D)$. The reason is that both $P\{\text{rec}\, X : P_1 \hookrightarrow X\}$ and $P\{\text{rec}\, X : P_2 \hookrightarrow X\}$ reach in one step the same equivalence classes at the same rates and hence the first step towards D contributes to *pbtm* in the same way for $P\{\text{rec}\, X : P_1 \hookrightarrow X\}$ and $P\{\text{rec}\, X : P_2 \hookrightarrow X\}$.

Therefore, by virtue of Prop. 5 we have that $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\text{MB}})^+ = \approx_{\text{MB}}$ and hence what we have proved is that, for each $P \in \mathcal{PL}_{\text{M}}$ containing free occurrences of X at most, it holds that for all $a \in \mathit{Name}$ and $D \in \mathcal{PL}_{\text{M}}/\approx_{\text{MB}}$:

$$\mathit{rate}(P\{\text{rec}\, X : P_1 \hookrightarrow X\}, a, D) \; = \; \mathit{rate}(P\{\text{rec}\, X : P_2 \hookrightarrow X\}, a, D)$$

This means that $P\{\text{rec}\, X : P_1 \hookrightarrow X\} \simeq_{\text{MB}} P\{\text{rec}\, X : P_2 \hookrightarrow X\}$ for all $P \in \mathcal{PL}_{\text{M}}$ containing free occurrences of X at most. We finally derive $\text{rec}\, X : P_1 \simeq_{\text{MB}} \text{rec}\, X : P_2$ by taking P equal to X. ∎

Proof of Lemma 1. Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nr}}$ be such that $P_1 \approx_{\mathrm{MB}} P_2$ but $P_1 \not\simeq_{\mathrm{MB}} P_2$. The proof is divided into three parts:

- Since $\approx_{\mathrm{MB}}$ and $\simeq_{\mathrm{MB}}$ coincide over $\mathbb{P}_{\mathrm{M,nfu}}$ as established by Prop. 3, both P_1 and P_2 must be fully unstable. Since P_1 and P_2 are nonrecursive, no rec binder can occur in them and hence both of them must start with one or more alternative exponentially timed τ-actions, i.e., $P_1 \equiv \sum_{i \in I_1} <\tau, \mu_{1,i}>.P_{1,i}$ and $P_2 \equiv \sum_{i \in I_2} <\tau, \mu_{2,i}>.P_{2,i}$ where $I_1 \neq \emptyset, I_2 \neq \emptyset$ are finite index sets.
- If all the derivative process terms $P_{1,i}$, $i \in I_1$, of P_1 and $P_{2,i}$, $i \in I_2$, of P_2 were not fully unstable, then for $k \in \{1, 2\}$ and $D \in \mathbb{P}_{\mathrm{M,nfu}} / \approx_{\mathrm{MB}}$ we would have:
$$pbtm(P_k, D) \; = \; \{\!| \, \frac{rate(P_k, \tau, D)}{rate_{\mathrm{t}}(P_k)} \cdot \frac{1}{rate_{\mathrm{t}}(P_k)} \, |\!\}$$
If we let:
$$v(P_k, D) \; = \; \frac{rate(P_k, \tau, D)}{rate_{\mathrm{t}}(P_k)} \cdot \frac{1}{rate_{\mathrm{t}}(P_k)}$$
we would derive:
$$rate(P_k, \tau, D) \; = \; v(P_k, D) \cdot rate_{\mathrm{t}}(P_k) \cdot rate_{\mathrm{t}}(P_k)$$
and also:
$$\sum_{D' \in \mathbb{P}_{\mathrm{M,nfu}} / \approx_{\mathrm{MB}}} v(P_k, D') \; = \; \frac{1}{rate_{\mathrm{t}}(P_k)} \cdot \sum_{D' \in \mathbb{P}_{\mathrm{M,nfu}} / \approx_{\mathrm{MB}}} \frac{rate(P_k, \tau, D')}{rate_{\mathrm{t}}(P_k)} \; = \; \frac{1}{rate_{\mathrm{t}}(P_k)}$$
or equivalently:
$$rate_{\mathrm{t}}(P_k) \; = \; 1 / \sum_{D' \in \mathbb{P}_{\mathrm{M,nfu}} / \approx_{\mathrm{MB}}} v(P_k, D')$$
so that:
$$rate(P_k, \tau, D) \; = \; v(P_k, D) / \left(\sum_{D' \in \mathbb{P}_{\mathrm{M,nfu}} / \approx_{\mathrm{MB}}} v(P_k, D') \right)^2$$
From $P_1 \approx_{\mathrm{MB}} P_2$ and the fact that both P_1 and P_2 are fully unstable, for all $D \in \mathbb{P}_{\mathrm{M,nfu}} / \approx_{\mathrm{MB}}$ it would then follow:
$$rate(P_1, \tau, D) \; = \; v(P_1, D) / \left(\sum_{D' \in \mathbb{P}_{\mathrm{M,nfu}} / \approx_{\mathrm{MB}}} v(P_1, D') \right)^2 \; = \; v(P_2, D) / \left(\sum_{D' \in \mathbb{P}_{\mathrm{M,nfu}} / \approx_{\mathrm{MB}}} v(P_2, D') \right)^2 \; = \; rate(P_2, \tau, D)$$
while for $a \neq \tau$ or $D'' \in \mathbb{P}_{\mathrm{M,fu}} / \approx_{\mathrm{MB}}$ we would have:
$$rate(P_1, a, D'') \; = \; 0 \; = \; rate(P_2, a, D'')$$
In conclusion, $P_1 \not\simeq_{\mathrm{MB}} P_2$ would be violated. Therefore, at least one process term belonging to $\{P_{1,i} \mid i \in I_1\} \cup \{P_{2,i} \mid i \in I_2\}$ must be fully unstable.
- If the two sets of equivalence classes with respect to $\approx_{\mathrm{MB}}$ reachable in one step by P_1 and P_2 were the same, say $\{D_1, D_2, \dots, D_n\}$ with $n \in \mathbb{N}_{>0}$, from $P_1 \approx_{\mathrm{MB}} P_2$ we would derive that for all $1 \leq i \leq n$:
$$\frac{rate(P_1, \tau, D_i)}{rate_{\mathrm{t}}(P_1)} \cdot \frac{1}{rate_{\mathrm{t}}(P_1)} \; = \; \frac{rate(P_2, \tau, D_i)}{rate_{\mathrm{t}}(P_2)} \cdot \frac{1}{rate_{\mathrm{t}}(P_2)}$$
and hence:
$$\sum_{i=1}^{n} \frac{rate(P_1, \tau, D_i)}{rate_{\mathrm{t}}(P_1)} \cdot \frac{1}{rate_{\mathrm{t}}(P_1)} \; = \; \sum_{i=1}^{n} \frac{rate(P_2, \tau, D_i)}{rate_{\mathrm{t}}(P_2)} \cdot \frac{1}{rate_{\mathrm{t}}(P_2)}$$
or equivalently:
$$\frac{1}{rate_{\mathrm{t}}(P_1)} \; = \; \frac{1}{rate_{\mathrm{t}}(P_2)}$$
As a consequence, for all $1 \leq i \leq n$ we would have:

$$rate(P_1, \tau, D_i) \; = \; rate(P_2, \tau, D_i)$$

while for $a \neq \tau$ or $D' \notin \{D_1, D_2, \dots, D_n\}$ we would have:

$$rate(P_1, a, D') \; = \; 0 \; = \; rate(P_2, a, D')$$

In conclusion, $P_1 \not\simeq_{\rm MB} P_2$ would be violated. Therefore, it must be $\{D \in \mathbb{P}_{\rm M}/\approx_{\rm MB} \mid \exists i \in I_1. P_{1,i} \in D\} \neq \{D \in \mathbb{P}_{\rm M}/\approx_{\rm MB} \mid \exists i \in I_2. P_{2,i} \in D\}$. ■

Proof of Prop. 6. Let $P_1, P_2 \in \mathbb{P}_{\rm M,nr}$ be such that $P_1 \approx_{\rm MB} P_2$ but $P_1 \not\simeq_{\rm MB} P_2$. By virtue of Lemma 1, it turns out that P_1 is of the form $\sum_{i \in I_1} {<}\tau, \mu_{1,i}{>}.P_{1,i}$ and P_2 is of the form $\sum_{i \in I_2} {<}\tau, \mu_{2,i}{>}.P_{2,i}$ where $I_1 \neq \emptyset, I_2 \neq \emptyset$ are finite index sets and at least one process term in $\{P_{1,i} \mid i \in I_1\} \cup \{P_{2,i} \mid i \in I_2\}$ is fully unstable. The proof is divided into two parts:

- Firstly, we demonstrate that at least one between P_1 and P_2 is of the form $\sum_{i \in I} {<}\tau, \mu_i{>}. \sum_{j \in J_i} {<}\tau, \gamma_{i,j}{>}.P_{i,j}$ where $I \neq \emptyset$ is a finite index set and $J_i \neq \emptyset$ is a finite index set for all $i \in I$. There are two cases:
  - Suppose that at least one between P_1 and P_2 can reach in one step only a single equivalence class $D \in \mathbb{P}_{\rm M}/\approx_{\rm MB}$. Assuming that it is P_1, there are two subcases:
    * If $D \subseteq \mathbb{P}_{\rm M,fu}$, then we immediately derive that P_1 is of the considered form, i.e., $P_1 \equiv \sum_{i \in I_1} {<}\tau, \mu_{1,i}{>}. \sum_{j \in J_i} {<}\tau, \gamma_{1,i,j}{>}.P_{1,i,j}$ where $J_i \neq \emptyset$ is a finite index set for all $i \in I_1$.
    * If $D \subseteq \mathbb{P}_{\rm M,nfu}$, then by virtue of Lemma 1 it cannot be the only equivalence class with respect to $\approx_{\rm MB}$ reachable in one step by P_2. Since $D \subseteq \mathbb{P}_{\rm M,nfu}$ and D is the only equivalence class with respect to $\approx_{\rm MB}$ reachable in one step by P_1, from $P_1 \approx_{\rm MB} P_2$ it follows that all the other classes with respect to $\approx_{\rm MB}$ reachable in one step by P_2 must be subset of $\mathbb{P}_{\rm M,fu}$ and lead only to D. Furthermore, P_2 cannot reach D in one step, because otherwise $pbtm(P_2, D)$ would contain at least two values whereas $pbtm(P_1, D)$ contains only $\frac{1}{rate_{\rm t}(P_1)}$, thus violating $P_1 \approx_{\rm MB} P_2$. Hence $P_2 \equiv \sum_{i \in I_2} {<}\tau, \mu_{2,i}{>}. \sum_{j \in J_i} {<}\tau, \gamma_{2,i,j}{>}.P_{2,i,j}$ where $J_i \neq \emptyset$ is a finite index set for all $i \in I_2$.
  - Suppose that both P_1 and P_2 can reach in one step several equivalence classes with respect to $\approx_{\rm MB}$. The two behavioral equivalences $\approx_{\rm MB}$ and $\simeq_{\rm MB}$ differ only for the treatment of fully unstable process terms. In fact, the former equivalence applies a *pbtm*-based equality check to reducible computations, whereas the latter equivalence applies a *rate*-based equality check to initial transitions. As a consequence, from $P_1 \approx_{\rm MB} P_2$ but $P_1 \not\simeq_{\rm MB} P_2$ it follows that some reducible computations of one between P_1 and P_2 must necessarily occur reduced in the other one, with the reductions taking place at the beginning of those computations and preserving their execution probability and their average duration (reductions preserving those quantities can occur at any stage of the considered computations as $P_1 \approx_{\rm MB} P_2$, but only the absence of such reductions taking place right at the beginning of the computations violates $P_1 \not\simeq_{\rm MB} P_2$).

To be precise, in addition to initial reductions, there is another reason – initial permutations – for which some reducible computations of one between P_1 and P_2 may differ at the beginning in the other one. In fact, since *pbtm* abstracts not only from the length of reducible computations but also from the order of the exponentially timed τ-transitions forming those computations, some reducible computations of P_1 (resp. P_2) may occur in P_2 (resp. P_1) with the actions labeling their first transitions exchanged with the actions labeling their second transitions. However, since both P_1 and P_2 can reach in one step several equivalence classes with respect to $\approx_{\mathrm{MB}}$, any such permutation would lead either to the same process term – if the rates of the involved actions are the same – or to a process term not $\approx_{\mathrm{MB}}$-equivalent to the original one because of the alteration of initial action execution probabilities or of the average sojourn time of the original process term and its derivatives – if the rates of the involved actions are different.

In general, when a reduction takes place at the beginning of a fully unstable process term P, the reduction cannot be concerned with a single reducible computation, but must involve all the reducible computations of P. The reason is that the reduction must preserve the execution probability and the average duration of all the computations of P. In fact, since it takes place at the beginning of P, the reduction produces another fully unstable process term P' whose initial actions are slower than the initial actions of P. Therefore, the average sojourn time of P' is necessarily greater than the average sojourn time of P:

$$\frac{1}{rate_{\mathrm{t}}(P')} > \frac{1}{rate_{\mathrm{t}}(P)}$$

Now, if P had a computation that cannot be reduced because it contains a single exponentially timed τ-transition – say of rate $\mu \in \mathbb{R}_{>0}$ – ending up in a non-fully-unstable state – say belonging to $D \in \mathbb{P}_{\mathrm{M}}/\approx_{\mathrm{MB}}$ – then that computation would have an execution probability in P' greater than its execution probability in P:

$$\frac{\mu}{rate_{\mathrm{t}}(P')} > \frac{\mu}{rate_{\mathrm{t}}(P)}$$

In order to avoid this alteration of the execution probability of the considered computation, in P' we should change the rate of the corresponding initial action from μ to $\frac{\mu}{rate_{\mathrm{t}}(P)}$ multiplied by the reciprocal of the average duration of the other initial actions of P', but then we would increase the average duration of the considered computation with respect to P. Therefore, in any case $pbtm(P', D)$ and $pbtm(P, D)$ would be different and hence it would turn out $P' \not\approx_{\mathrm{MB}} P$.

As a consequence of the fact that all reducible computations of one between P_1 and P_2 must necessarily occur reduced at the beginning in the other one, at least one between P_1 and P_2 must be of the form $\sum_{i \in I} {<}\tau, \mu_i{>}. \sum_{j \in J_i} {<}\tau, \gamma_{i,j}{>}.P_{i,j}$ where $I \neq \emptyset$ is a finite index set and $J_i \neq \emptyset$ is a finite index set for all $i \in I$.

- Secondly, we demonstrate that the term between P_1 and P_2 of the form $\sum_{i \in I} {<}\tau, \mu_i{>}. \sum_{j \in J_i} {<}\tau, \gamma_{i,j}{>}.P_{i,j}$ satisfies one of the two properties mentioned at the end of the proposition. For simplicity, we assume that only one between P_1 and P_2 is of that form. There are two cases:

- If that term can reach in one step only a single equivalence class with respect to $\approx_{\rm MB}$, then we immediately derive that $\sum_{j \in J_{i_1}} <\tau, \gamma_{i_1,j}>.P_{i_1,j} \approx_{\rm MB} \sum_{j \in J_{i_2}} <\tau, \gamma_{i_2,j}>.P_{i_2,j}$ for all $i_1, i_2 \in I$.
- If that term can reach in one step several equivalence classes with respect to $\approx_{\rm MB}$, then the property satisfied in the previous case cannot hold. However, as shown in the first part of the proof (see the second subcase of the first case for P_2 reaching several classes, as well as the second case), all the reducible computations of that term must necessarily occur reduced at the beginning of the other term because $P_1 \approx_{\rm MB} P_2$ but $P_1 \not\simeq_{\rm MB} P_2$. This is possible iff $\sum_{j \in J_{i_1}} \gamma_{i_1,j} = \sum_{j \in J_{i_2}} \gamma_{i_2,j}$ for all $i_1, i_2 \in I$. In fact:
  - ∗ If the property above is satisfied, then by virtue of Prop. 1 all the computations of that term can be reduced at the beginning.
  - ∗ Suppose that all the computations of that term can be reduced at the beginning but the property above is not satisfied. Therefore, there exist $i_1, i_2 \in I$ such that $\sum_{j \in J_{i_1}} \gamma_{i_1,j} \neq \sum_{j \in J_{i_2}} \gamma_{i_2,j}$ and hence the derivative of $<\tau, \mu_{i_1}>$ and the derivative of $<\tau, \mu_{i_2}>$ have average sojourn times different from each other. If we let $\mu = \sum_{i \in I} \mu_i$, $\gamma_1 = \sum_{j \in J_{i_1}} \gamma_{i_1,j}$, and $\gamma_2 = \sum_{j \in J_{i_2}} \gamma_{i_2,j}$, then it holds that $\gamma_1 \neq \gamma_2$ and $\frac{1}{\mu} + \frac{1}{\gamma_1} \neq \frac{1}{\mu} + \frac{1}{\gamma_2}$.

    Since all computations must be reduced at the beginning in a way that preserves their execution probability and their average duration, these pieces of information must necessarily be part of the rates of the new initial exponentially timed τ-actions resulting from the reduction. In particular, the reduction of $<\tau, \mu_{i_1}>$ with $<\tau, \gamma_{i_1,j}>$, $j \in J_{i_1}$, gives rise to an exponentially timed τ-action whose rate is $\frac{\mu_{i_1}}{\mu} \cdot \frac{\gamma_{i_1,j}}{\gamma_1} \cdot (\frac{1}{\mu} + \frac{1}{\gamma_1})^{-1}$, whereas the reduction of $<\tau, \mu_{i_2}>$ with $<\tau, \gamma_{i_2,j}>$, $j \in J_{i_2}$, gives rise to an exponentially timed τ-action whose rate is $\frac{\mu_{i_2}}{\mu} \cdot \frac{\gamma_{i_2,j}}{\gamma_2} \cdot (\frac{1}{\mu} + \frac{1}{\gamma_2})^{-1}$.

    However, the resulting process term is not $\approx_{\rm MB}$-equivalent to the original one, which contradicts $P_1 \approx_{\rm MB} P_2$. In fact, while in the original process term the *probtime* of reaching $\{P_{i_1,j} \mid j \in J_{i_1}\}$ is $\frac{\mu_{i_1}}{\mu} \cdot (\frac{1}{\mu} + \frac{1}{\gamma_1})^{-1}$, in the process term resulting from the reduction it is the previous value divided by the square of the sum of the rates of the new initial exponentially timed τ-actions (see Ex. 4). This sum is not equal to $(\frac{1}{\mu} + \frac{1}{\gamma_1})^{-1}$, because each of its summands is given by a fraction of μ multiplied by $(\frac{1}{\mu} + \frac{1}{\gamma})^{-1}$ for some $\gamma \in \mathbb{R}_{>0}$, with two of these γ values – γ_1 and γ_2 – being different from each other. ■

Proof of Lemma 2. We proceed by induction on the syntactical structure of $P \in \mathbb{P}_{\rm M,nr}$:

– If P is $\underline{0}$, then the result follows by taking $Q \equiv \underline{0}$ (which is in $\simeq_{\rm MB}$-normal-form) and using reflexivity.

– If P is of the form $<a, \lambda>.P'$, then by the induction hypothesis there exists $Q' \in \mathbb{P}_{\mathrm{M,nr}}$ in $\simeq_{\mathrm{MB}}$-normal-form such that $\mathcal{A}_{\mathrm{MB}} \vdash P' = Q'$. From substitutivity with respect to action prefix, we obtain that $\mathcal{A}_{\mathrm{MB}} \vdash <a, \lambda>.P' = <a, \lambda>.Q'$. There are two cases:
  - If $<a, \lambda>.Q'$ is in $\simeq_{\mathrm{MB}}$-normal-form, then we are done.
  - If $<a, \lambda>.Q'$ is not in $\simeq_{\mathrm{MB}}$-normal-form, then the result follows after applying $\mathcal{A}_{\mathrm{MB},5}$ by virtue of transitivity.

– If P is of the form $P_1 + P_2$, then by the induction hypothesis there exist $Q_1, Q_2 \in \mathbb{P}_{\mathrm{M,nr}}$ in $\simeq_{\mathrm{MB}}$-normal-form such that $\mathcal{A}_{\mathrm{MB}} \vdash P_1 = Q_1$ and $\mathcal{A}_{\mathrm{MB}} \vdash P_2 = Q_2$. From substitutivity with respect to alternative composition, we obtain that $\mathcal{A}_{\mathrm{MB}} \vdash P_1 + P_2 = Q_1 + Q_2$. There are two cases:
  - If $Q_1 + Q_2$ is in $\simeq_{\mathrm{MB}}$-normal-form, then we are done.
  - If $Q_1 + Q_2$ is not in $\simeq_{\mathrm{MB}}$-normal-form, then the result follows after as many applications of $\mathcal{A}_{\mathrm{MB},3}$ and $\mathcal{A}_{\mathrm{MB},4}$ as needed – possibly preceded by applications of $\mathcal{A}_{\mathrm{MB},1}$ and $\mathcal{A}_{\mathrm{MB},2}$ – by virtue of substitutivity with respect to alternative composition as well as transitivity.

– If P is of the form P'/H, then by the induction hypothesis there exists $Q' \in \mathbb{P}_{\mathrm{M,nr}}$ in $\simeq_{\mathrm{MB}}$-normal-form such that $\mathcal{A}_{\mathrm{MB}} \vdash P' = Q'$. From substitutivity with respect to hiding, we obtain that $\mathcal{A}_{\mathrm{MB}} \vdash P'/H = Q'/H$. The result then follows after as many applications of $\mathcal{A}_{\mathrm{MB},6}$ to $\mathcal{A}_{\mathrm{MB},9}$ as needed – possibly followed by applications of $\mathcal{A}_{\mathrm{MB},4}$ and $\mathcal{A}_{\mathrm{MB},5}$ in turn preceded by applications of $\mathcal{A}_{\mathrm{MB},1}$ and $\mathcal{A}_{\mathrm{MB},2}$ – by virtue of substitutivity with respect to action prefix and alternative composition as well as transitivity. ■

Proof of Thm. 4. The proof is divided into two parts:

⇒ The soundness part of the result comes from the following remarks:
  - Since $\simeq_{\mathrm{MB}}$ is an equivalence relation and a congruence with respect to action prefix, alternative composition, and hiding by virtue of Thm. 1, in any deduction based on $\mathcal{A}_{\mathrm{MB}}$ it is correct to use reflexivity, symmetry, transitivity, and substitutivity with respect to action prefix, alternative composition, and hiding.
  - The validity of axioms $\mathcal{A}_{\mathrm{MB},1}$ to $\mathcal{A}_{\mathrm{MB},4}$ – which are sound for $\sim_{\mathrm{MB}}$ – is ensured by $\sim_{\mathrm{MB}} \subset \simeq_{\mathrm{MB}}$ as established by Prop. 3.
  - The validity of axiom $\mathcal{A}_{\mathrm{MB},5}$ stems from Props. 1 and 4.
  - The validity of axioms $\mathcal{A}_{\mathrm{MB},6}$ to $\mathcal{A}_{\mathrm{MB},9}$ stems from the operational semantic rules of MPC.

⇐ Given $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nr}}$ such that $P_1 \simeq_{\mathrm{MB}} P_2$, we prove that $\mathcal{A}_{\mathrm{MB}} \vdash P_1 = P_2$ by assuming without loss of generality that both P_1 and P_2 are in $\simeq_{\mathrm{MB}}$-normal-form. In fact, if this were not the case, by virtue of Lemma 2 we could derive $Q_1, Q_2 \in \mathbb{P}_{\mathrm{M,nr}}$ in $\simeq_{\mathrm{MB}}$-normal-form such that $\mathcal{A}_{\mathrm{MB}} \vdash P_1 = Q_1$ and $\mathcal{A}_{\mathrm{MB}} \vdash P_2 = Q_2$ (hence $P_1 \simeq_{\mathrm{MB}} Q_1$ and $P_2 \simeq_{\mathrm{MB}} Q_2$ due to the soundness of the axioms with respect to $\simeq_{\mathrm{MB}}$), with $Q_1 \simeq_{\mathrm{MB}} Q_2$ (because it also holds $P_1 \simeq_{\mathrm{MB}} P_2$ and $\simeq_{\mathrm{MB}}$ is a transitive relation). So, if we proved $\mathcal{A}_{\mathrm{MB}} \vdash Q_1 = Q_2$ from $Q_1 \simeq_{\mathrm{MB}} Q_2$, it would then follow $\mathcal{A}_{\mathrm{MB}} \vdash P_1 = P_2$

by transitivity.
Let us proceed by induction on the syntactical structure of $P_1 \in \mathbb{P}_{\mathrm{M,nr}}$ in $\simeq_{\mathrm{MB}}$-normal-form:

- If P_1 is $\underline{0}$, then from $P_1 \simeq_{\mathrm{MB}} P_2$ and P_2 in $\simeq_{\mathrm{MB}}$-normal-form it follows that P_2 is $\underline{0}$ too, hence the result by reflexivity.
- If P_1 is of the form $\sum_{i \in I_1} <a_i, \lambda_i>.P_{1,i}$ with I_1 finite and nonempty, then from $P_1 \simeq_{\mathrm{MB}} P_2$ and P_2 in $\simeq_{\mathrm{MB}}$-normal-form it follows that P_2 is of the form $\sum_{j \in I_2} <b_j, \mu_j>.P_{2,j}$ with I_2 finite and nonempty.
  Moreover, for all $i, i' \in I_1$ such that $i \neq i'$ (resp. $j, j' \in I_2$ such that $j \neq j'$) it must hold that $a_i \neq a_{i'}$ or $P_{1,i} \not\simeq_{\mathrm{MB}} P_{1,i'}$ (resp. $b_j \neq b_{j'}$ or $P_{2,j} \not\simeq_{\mathrm{MB}} P_{2,j'}$). In fact, if it were $a_h = a_{h'}$ and $P_{1,h} \simeq_{\mathrm{MB}} P_{1,h'}$ for some $h, h' \in I_1$ such that $h \neq h'$, then by the induction hypothesis we would have $\mathcal{A}_{\mathrm{MB}} \vdash P_{1,h} = P_{1,h'}$ and hence $\mathcal{A}_{\mathrm{MB}} \vdash <a_h, \lambda_h>.P_{1,h} + <a_{h'}, \lambda_{h'}>.P_{1,h'} = <a_h, \lambda_h>.P_{1,h} + <a_h, \lambda_{h'}>.P_{1,h}$ by substitutivity, which would contradict the initial minimality of P_1 with respect to $\mathcal{A}_{\mathrm{MB},4}$.
  In addition, for all $i, i' \in I_1$ such that $i \neq i'$ (resp. $j, j' \in I_2$ such that $j \neq j'$) it must hold that $a_i \neq a_{i'}$ or $P_{1,i} \not\approx_{\mathrm{MB}} P_{1,i'}$ (resp. $b_j \neq b_{j'}$ or $P_{2,j} \not\approx_{\mathrm{MB}} P_{2,j'}$). In fact, if it were $a_h = a_{h'}$ and $P_{1,h} \approx_{\mathrm{MB}} P_{1,h'}$ for some $h, h' \in I_1$ such that $h \neq h'$, then $P_{1,h} \not\simeq_{\mathrm{MB}} P_{1,h'}$ and $P_{1,h} \approx_{\mathrm{MB}} P_{1,h'}$ would contradict the initial minimality of P_1 summand $<a_h, \lambda_h>.P_{1,h}$ or $<a_{h'}, \lambda_{h'}>.P_{1,h'}$ with respect to $\mathcal{A}_{\mathrm{MB},5}$ by virtue of Prop. 6.
  As a consequence, since $P_1 \simeq_{\mathrm{MB}} P_2$ and hence for all $a \in \mathit{Name}$ and $D \in \mathbb{P}_{\mathrm{M}} / \approx_{\mathrm{MB}}$ we have that $\mathit{rate}(P_1, a, D) = \mathit{rate}(P_2, a, D)$, a bijective correspondence can be established between the set of summands of P_1 and the set of summands of P_2. For each summand $<a_i, \lambda_i>.P_{1,i}$ there exists exactly one summand $<b_j, \mu_j>.P_{2,j}$ such that $a_i = b_j$, $\lambda_i = \mu_j$, and $P_{1,i} \approx_{\mathrm{MB}} P_{2,j}$ – and hence $P_{1,i} \simeq_{\mathrm{MB}} P_{2,j}$ otherwise $<a_i, \lambda_i>.P_{1,i}$ or $<b_j, \mu_j>.P_{2,j}$ would not be initially minimal with respect to $\mathcal{A}_{\mathrm{MB},5}$ by virtue of Prop. 6 – and vice versa. For each pair of corresponding summands $<a_i, \lambda_i>.P_{1,i}$ and $<b_j, \mu_j>.P_{2,j}$, from $P_{1,i} \simeq_{\mathrm{MB}} P_{2,j}$ and the induction hypothesis it follows that $\mathcal{A}_{\mathrm{MB}} \vdash P_{1,i} = P_{2,j}$ and hence $\mathcal{A}_{\mathrm{MB}} \vdash <a_i, \lambda_i>.P_{1,i} = <b_j, \mu_j>.P_{2,j}$ by substitutivity with respect to action prefix ($a_i = b_j$ and $\lambda_i = \mu_j$). Due to the bijectivity of the correspondence, we have $\mathcal{A}_{\mathrm{MB}} \vdash \sum_{i \in I_1} <a_i, \lambda_i>.P_{1,i} = \sum_{j \in I_2} <b_j, \mu_j>.P_{2,j}$ by substitutivity with respect to alternative composition. ■

Proof of Thm. 5. Let us preliminarily define the backward reachability set of $P \in \mathbb{P}_{\mathrm{M}}$ as follows:

$$\mathit{brs}(P) \; = \; \{P' \in \mathbb{P}_{\mathrm{M}} \mid \exists a \in \mathit{Name}, \lambda \in \mathbb{R}_{>0}. \, P' \xrightarrow{a,\lambda}_{\mathrm{M}} P\}$$

and the backward rate of $P \in \mathbb{P}_{\mathrm{M}}$ with respect to $P' \in \mathit{brs}(P)$ as follows:

$$\mathit{rate}_{\mathrm{b}}(P', P) \; = \; \sum \{\!| \, \lambda \in \mathbb{R}_{>0} \mid \exists a \in \mathit{Name}. \, P' \xrightarrow{a,\lambda}_{\mathrm{M}} P \, |\!\}$$

In the following, we use subscript l (resp. r) to denote quantities related to the original (resp. aggregated) CTMC on the left (resp. right) of the rewriting rule depicted just before the theorem.

Whenever the two stationary state probability vectors $\boldsymbol{\pi}_{\mathrm{l}}$ and $\boldsymbol{\pi}_{\mathrm{r}}$ exist, we have that $\boldsymbol{\pi}_{\mathrm{l}}$ satisfies the following linear system of global balance equations for the original CTMC on the left:

$$\begin{array}{rcll} \pi_{\mathrm{l}}[s_i] \cdot \gamma & = & \pi_{\mathrm{l}}[s_0] \cdot \mu_i & i \in I \\ \pi_{\mathrm{l}}[s_{i,j}] \cdot rate_{\mathrm{t}}(s_{i,j}) & = & \pi_{\mathrm{l}}[s_i] \cdot \gamma_{i,j} & i \in I, j \in J_i \\ \pi_{\mathrm{l}}[s] \cdot rate_{\mathrm{t}}(s) & = & \sum\limits_{s' \in brs(s)} \pi_{\mathrm{l}}[s'] \cdot rate_{\mathrm{b}}(s', s) & \text{any other state } s \end{array}$$

while for the aggregated CTMC on the right we have that $\boldsymbol{\pi}_{\mathrm{r}}$ satisfies the following linear system of global balance equations:

$$\begin{array}{rcll} \pi_{\mathrm{r}}[s_{i,j}] \cdot rate_{\mathrm{t}}(s_{i,j}) & = & \pi_{\mathrm{r}}[s'_0] \cdot \frac{\mu_i}{\mu} \cdot \frac{\gamma_{i,j}}{\gamma} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma}\right)^{-1} & i \in I, j \in J_i \\ \pi_{\mathrm{r}}[s] \cdot rate_{\mathrm{t}}(s) & = & \sum\limits_{s' \in brs(s)} \pi_{\mathrm{r}}[s'] \cdot rate_{\mathrm{b}}(s', s) & \text{any other state } s \end{array}$$

with both the $\pi_{\mathrm{l}}[.]$'s and the $\pi_{\mathrm{r}}[.]$'s summing up to 1. Since $\frac{1}{\mu} + \frac{1}{\gamma} = \frac{\mu + \gamma}{\mu \cdot \gamma}$, the second linear system is equivalent to:

$$\begin{array}{rcll} \pi_{\mathrm{r}}[s_{i,j}] \cdot rate_{\mathrm{t}}(s_{i,j}) & = & \pi_{\mathrm{r}}[s'_0] \cdot \frac{\mu_i \cdot \gamma_{i,j}}{\mu + \gamma} & i \in I, j \in J_i \\ \pi_{\mathrm{r}}[s] \cdot rate_{\mathrm{t}}(s) & = & \sum\limits_{s' \in brs(s)} \pi_{\mathrm{r}}[s'] \cdot rate_{\mathrm{b}}(s', s) & \text{any other state } s \end{array}$$

We show that, through the introduction of a new variable y replacing the set of $|I| + 1$ variables $\{\pi_{\mathrm{l}}[s_i] \mid i \in I \cup \{0\}\}$, the system of global balance equations for the original CTMC on the left can be transformed into a linear system having the same number of variables and equations as well as the same coefficient matrix as the linear system equivalent to the system of global balance equations for the aggregated CTMC on the right.

By summing up over all $i \in I$ the first group of equations in the linear system for the original CTMC, we derive the following equation:

$$\sum_{i \in I} \pi_{\mathrm{l}}[s_i] \cdot \gamma \; = \; \pi_{\mathrm{l}}[s_0] \cdot \mu$$

If we let:

$$y \; = \; \pi_{\mathrm{l}}[s_0] + \sum_{i \in I} \pi_{\mathrm{l}}[s_i]$$

or equivalently:

$$\sum_{i \in I} \pi_{\mathrm{l}}[s_i] \; = \; y - \pi_{\mathrm{l}}[s_0]$$

then the last derived equation can be rewritten as follows:

$$y \cdot \gamma - \pi_{\mathrm{l}}[s_0] \cdot \gamma \; = \; \pi_{\mathrm{l}}[s_0] \cdot \mu$$

and hence:

$$\pi_{\mathrm{l}}[s_0] \; = \; y \cdot \frac{\gamma}{\mu + \gamma}$$

Since for all $i \in I$ it holds that:

$$\pi_{\mathrm{l}}[s_i] \; = \; \pi_{\mathrm{l}}[s_0] \cdot \frac{\mu_i}{\gamma} \; = \; y \cdot \frac{\mu_i}{\mu + \gamma}$$

the second group of equations in the linear system for the original CTMC can be rewritten as follows:

$$\pi_{\mathrm{l}}[s_{i,j}] \cdot rate_{\mathrm{t}}(s_{i,j}) \; = \; y \cdot \frac{\mu_i \cdot \gamma_{i,j}}{\mu + \gamma} \qquad i \in I, j \in J_i$$

In conclusion, the introduction of variable y causes the system of global balance equations for the original CTMC to be equivalent to the following one:

$$\begin{array}{rcll} \pi_{\mathrm{l}}[s_{i,j}] \cdot rate_{\mathrm{t}}(s_{i,j}) & = & y \cdot \frac{\mu_i \cdot \gamma_{i,j}}{\mu + \gamma} & i \in I, j \in J_i \\ \pi_{\mathrm{l}}[s] \cdot rate_{\mathrm{t}}(s) & = & \sum\limits_{s' \in brs(s)} \pi_{\mathrm{l}}[s'] \cdot rate_{\mathrm{b}}(s', s) & \text{any other state } s \end{array}$$

with all the occurring $\pi_{\mathrm{l}}[.]$'s plus y summing up to 1, which has the same form as the linear system equivalent to the system of global balance equations for the aggregated CTMC. As a consequence:

$$
\begin{array}{rcll}
y & = & \pi_{\mathrm{r}}[s'_0] & \\
\pi_{\mathrm{l}}[s_{i,j}] & = & \pi_{\mathrm{r}}[s_{i,j}] & i \in I, j \in J_i \\
\pi_{\mathrm{l}}[s] & = & \pi_{\mathrm{r}}[s] & \text{any other state } s
\end{array}
$$

from which exactness at steady state follows because $\pi_{\mathrm{r}}[s'_0] = \pi_{\mathrm{l}}[s_0] + \sum_{i \in I} \pi_{\mathrm{l}}[s_i]$.

■