Abstracting from Exponentially Timed Internal Actions:
Weak Markovian Bisimulation Congruences and Exact CTMC-Level Aggregations

Marco Bernardo

University of Urbino – Italy

Markovian Behavioral Equivalences

- Tools for relating and manipulating formal models with an underlying continuous-time Markov chain (CTMC) semantics.
- Markovian bisimilarity: two processes are equivalent whenever they are able to mimic each other's functional and performance behavior step by step [Hil1994, Buc1994, HerRet1994].
- Markovian testing equivalence: two processes are equivalent whenever an external observer is not able to distinguish between them from a functional or performance viewpoint by interacting with them by means of tests and comparing their reactions [BerCle2000, Ber2007].
- Markovian trace equivalence: two processes are equivalent whenever they are able to perform computations with the same functional and performance characteristics [WolBaiMaj2005, Ber2007].
- Strong vs. weak Markovian behavioral equivalences.

Abstracting from Internal Actions

- When comparing nondeterministic processes, internal actions can be abstracted away via **weak behavioral equivalences**: $a.\tau.b.\underline{0} \approx a.b.\underline{0}$.

- Abstraction not always possible when comparing Markovian processes:

  △ <u>*Immediate internal actions*</u> are invisible and take no time hence they can be safely left out [Her1995, Ret1995, MarTrc2006, BerAld2007].

  ▽ <u>*Exponentially timed internal actions*</u> are invisible but take time.

- $<a, \lambda>.<\tau, \gamma>.<b, \mu>.\underline{0}$ is not equivalent to $<a, \lambda>.<b, \mu>.\underline{0}$ because we can observe a nonzero delay between a and b in the first case.

- But $<a, \lambda>.<\tau, \gamma_1>.<\tau, \gamma_2>.<b, \mu>.\underline{0} \approx_{\rm M} <a, \lambda>.<\tau, \gamma>.<b, \mu>.\underline{0}$ if the average duration of the sequence of τ-actions on the left ($\frac{1}{\gamma_1} + \frac{1}{\gamma_2}$) is equal to the average duration of the τ-action on the right ($\frac{1}{\gamma}$), i.e., if $\gamma = \frac{\gamma_1 \cdot \gamma_2}{\gamma_1 + \gamma_2}$.

Open Problems

- *To what extent can we abstract from exp. timed internal actions? Only from sequences of at least two* [Hil1994] *or also from branches?*
- *How to define a Markovian behavioral equivalence abstracting from sequences/branches of exponentially timed internal actions?*
- *Will it be a congruence with respect to typical process operators?*
- *Will it have a sound and complete axiomatization?*
- *Will it be decidable in polynomial time?*
- *Will it induce an exact CTMC-level aggregation?*
- *Any tradeoff among all of these properties?*

Achievements

- *Yes, we can abstract also from branches of exp. timed internal actions.*
- *Defined a weak Markovian bisimulation equivalence ($\approx_{\rm MB}$) that is coarser than Markovian bisimilarity and weak Markovian isomorphism [Hil1994].*
- *Shown that Milner's construction makes it a congruence with respect to alternative composition ($\simeq_{\rm MB}$).*
- *Shown that a more elaborated definition inspired by [Hil1994] makes it a congruence with respect to parallel composition ($\approx_{\rm MB,g}$, $\simeq_{\rm MB,g}$).*
- *Sound and complete axiomatization of $\simeq_{\rm MB}$ over nonrecursive sequential processes with abstraction.*
- *Polynomial-time decidability of $\approx_{\rm MB}$ / $\simeq_{\rm MB}$ over finite-state processes having no cycles of exponentially timed internal actions.*
- $\approx_{\rm MB}$ / $\simeq_{\rm MB}$ *exact only at steady state and this holds for all processes, while for $\approx_{\rm MB,g}$ / $\simeq_{\rm MB,g}$ it holds only for certain classes of processes.*

Markovian Process Calculus

- Conduct the study in a process algebraic framework.
- Durational actions each formed by its name and its rate.
- Dynamic operators and recursion for representing all the CTMCs.
- Parallel composition for representing the system structure.
- Hiding operator for abstraction purposes.
- $Name_{\mathrm{v}}$: set of visible action names.
- $Name = Name_{\mathrm{v}} \cup \{\tau\}$: set of all action names.
- $Act_{\mathrm{M}} = Name \times \mathbb{R}_{>0}$: set of exponentially timed actions.
- Var: set of process variables.

- Process term syntax for process language $\mathcal{PL}_{\rm M}$:

P	$::=$	$\underline{0}$	inactive process	
	$\mid$	$<a, \lambda>.P$	exp. timed action prefix	$(a \in Name, \lambda \in \mathbb{R}_{>0})$
	$\mid$	$P + P$	alternative composition	
	$\mid$	$P \parallel_S P$	parallel composition	$(S \subseteq Name_{\rm v})$
	$\mid$	$P \,/\, H$	hiding	$(H \subseteq Name_{\rm v})$
	$\mid$	X	process variable	$(X \in Var)$
	$\mid$	$\text{rec}\, X : P$	recursion	$(X \in Var)$

- $\mathbb{P}_{\rm M}$: set of closed and guarded process terms.
- Race policy: whenever several exponentially timed actions are enabled, the action that is executed is the one sampling the least duration.

- Every $P \in \mathbb{P}_{\rm M}$ is mapped to a labeled multitransition system $[\![P]\!]_{\rm M}$:
  - ⊙ Each state corresponds to a process term into which P can evolve.
  - ⊙ The initial state corresponds to P.
  - ⊙ Each transition from a source state to a target state is labeled with the action that determines the corresponding state change.
- State transition graph expressing all *computations* and *branching points* and accounting for *transition multiplicity* ($<a, \lambda>.\underline{0} + <a, \lambda>.\underline{0}$ vs. $<a, \lambda>.\underline{0}$).
- Every $P \in \mathbb{P}_{\rm M}$ is mapped to a CTMC:
  - ⊙ Dropping action names from all transitions of $[\![P]\!]_{\rm M}$.
  - ⊙ Collapsing all the transitions between any two states of $[\![P]\!]_{\rm M}$ into a single transition by summing up the rates of the original transitions.

- Operational semantic rules for dynamic operators and recursion:

$$(\textsc{Pre}_{\mathrm{M}}) \quad \frac{}{<a,\lambda>.P \xrightarrow{a,\lambda}_{\mathrm{M}} P}$$

$$(\textsc{Alt}_{\mathrm{M},1}) \quad \frac{P_1 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{P_1 + P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'} \qquad (\textsc{Alt}_{\mathrm{M},2}) \quad \frac{P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{P_1 + P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'}$$

$$(\textsc{Rec}_{\mathrm{M}}) \quad \frac{P\{\mathrm{rec}\, X : P \hookrightarrow X\} \xrightarrow{a,\lambda}_{\mathrm{M}} P'}{\mathrm{rec}\, X : P \xrightarrow{a,\lambda}_{\mathrm{M}} P'}$$

- Operational semantic rules for static operators:

$$(\textsc{Par}_{\mathrm{M},1})\quad \frac{P_1 \xrightarrow{a,\lambda}_{\mathrm{M}} P'_1 \quad a \notin S}{P_1 \parallel_S P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'_1 \parallel_S P_2} \qquad (\textsc{Par}_{\mathrm{M},2})\quad \frac{P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P'_2 \quad a \notin S}{P_1 \parallel_S P_2 \xrightarrow{a,\lambda}_{\mathrm{M}} P_1 \parallel_S P'_2}$$

$$(\textsc{Syn}_{\mathrm{M}})\quad \frac{P_1 \xrightarrow{a,\lambda_1}_{\mathrm{M}} P'_1 \quad P_2 \xrightarrow{a,\lambda_2}_{\mathrm{M}} P'_2 \quad a \in S}{P_1 \parallel_S P_2 \xrightarrow{a,\lambda_1 \otimes \lambda_2}_{\mathrm{M}} P'_1 \parallel_S P'_2}$$

$$(\textsc{Hid}_{\mathrm{M},1})\quad \frac{P \xrightarrow{a,\lambda}_{\mathrm{M}} P' \quad a \notin H}{P/H \xrightarrow{a,\lambda}_{\mathrm{M}} P'/H} \qquad (\textsc{Hid}_{\mathrm{M},2})\quad \frac{P \xrightarrow{a,\lambda}_{\mathrm{M}} P' \quad a \in H}{P/H \xrightarrow{\tau,\lambda}_{\mathrm{M}} P'/H}$$

(Strong) Markovian Bisimilarity

- Comparing the exit rates of process terms [Hil1994, Buc1994, HerRet1994].

- The exit rate of a process term $P \in \mathbb{P}_{\rm M}$ is the rate at which P can execute actions of a certain name $a \in Name$ that lead to a certain destination $D \subseteq \mathbb{P}_{\rm M}$:

$$rate(P, a, D) \; = \; \sum \{\!|\, \lambda \in \mathbb{R}_{>0} \mid \exists P' \in D.\, P \xrightarrow{a,\lambda}_{\rm M} P' \,|\!\}$$

- Summation stems from the adoption of the race policy.

- The total exit rate of P is the reciprocal of the average sojourn time in the state associated with P:

$$rate_{\rm t}(P) \; = \; \sum_{a \in Name} rate(P, a, \mathbb{P}_{\rm M})$$

- An equivalence relation $\mathcal{B}$ over $\mathbb{P}_{\mathrm{M}}$ is a Markovian bisimulation iff, whenever $(P_1, P_2) \in \mathcal{B}$, then for all action names $a \in \mathit{Name}$ and equivalence classes $D \in \mathbb{P}_{\mathrm{M}}/\mathcal{B}$:

$$\mathit{rate}(P_1, a, D) \;=\; \mathit{rate}(P_2, a, D)$$

- Markovian bisimilarity, denoted $\sim_{\mathrm{MB}}$, is the union of all the Markovian bisimulations.

- $\sim_{\mathrm{MB}}$ can be proved to be a congruence with respect to all the operators of MPC.

- $\sim_{\mathrm{MB}}$ is a congruence with respect to recursion as well.

- $\sim_{\mathrm{MB}}$ can be decided in polynomial time over finite-state processes.

- Sound and complete axiomatization over the set of nonrecursive process terms of MPC.

- Basic laws for dynamic operators:

$$
\begin{array}{lrcl}
(\mathcal{A}_{\mathrm{MB},1}) & P_1 + P_2 & = & P_2 + P_1 \\
(\mathcal{A}_{\mathrm{MB},2}) & (P_1 + P_2) + P_3 & = & P_1 + (P_2 + P_3) \\
(\mathcal{A}_{\mathrm{MB},3}) & P + \underline{0} & = & P \\
(\mathcal{A}_{\mathrm{MB},4}) & <a, \lambda_1>.P + <a, \lambda_2>.P & = & <a, \lambda_1 + \lambda_2>.P
\end{array}
$$

- Usual expansion law for the parallel composition operator.

- Usual distribution laws for the other static operators.

- $\sim_{\mathrm{MB}}$ induces a CTMC-level aggregation that is consistent with ordinary lumping and hence it is exact both at steady state and at transient state.

Defining a Weak Markovian Bisimilarity

- Basic idea: weaken the distinguishing power of $\sim_{\mathrm{MB}}$ by viewing every sequence of exp. timed τ-actions as a single exp. timed τ-action with the same average duration and execution probability as the sequence.

- $\mathbb{P}_{\mathrm{M,s}}$: set of stable process terms, which can perform no exponentially timed τ-actions ($P \in \mathbb{P}_{\mathrm{M,s}}$ iff $P \xrightarrow{\tau,\lambda}\!\!\!\!\!\!/\;\;_{\mathrm{M}} P'$ for all λ and P').

- $\mathbb{P}_{\mathrm{M,u}}$: set of unstable process terms, which can perform at least one exponentially timed τ-action ($\mathbb{P}_{\mathrm{M,u}} = \mathbb{P}_{\mathrm{M}} \setminus \mathbb{P}_{\mathrm{M,s}}$).

- $\mathbb{P}_{\mathrm{M,fu}}$: set of fully unstable process terms, which can perform only exponentially timed τ-actions (most natural candidates for abstraction).

- A computation c having the form $P_1 \xrightarrow{\tau,\lambda_1}_{\rm M} P_2 \xrightarrow{\tau,\lambda_2}_{\rm M} \dots \xrightarrow{\tau,\lambda_n}_{\rm M} P_{n+1}$ is reducible iff $P_i \in \mathbb{P}_{\rm M,fu}$ for all $i = 1, \dots, n$.

- Length-abstracting measure of a reducible computation c:

$$\boxed{probtime(c) \; = \; \left(\prod_{i=1}^{n} \frac{\lambda_i}{rate(P_i, \tau, \mathbb{P}_{\rm M})} \right) \cdot \left(\sum_{i=1}^{n} \frac{1}{rate(P_i, \tau, \mathbb{P}_{\rm M})} \right)}$$

  based on the product of the execution probabilities of the transitions of c and the sum of the average sojourn times of the states traversed by c.

- Finite-length reducible computations are enough to distinguish between fully unstable process terms that must be told apart ($\lambda_1 \neq \lambda_2$ and $a \in Name_{\rm v}$):

$$<\tau, \lambda_1>.<a, \lambda>.P \quad \text{vs.} \quad <\tau, \lambda_2>.<a, \lambda>.P$$

$$\text{rec}\, X : <\tau, \lambda_1>.X \quad \text{vs.} \quad \text{rec}\, X : <\tau, \lambda_2>.X$$

- The weak variant of $\sim_{\rm MB}$ should work like $\sim_{\rm MB}$ over $\mathbb{P}_{\rm M,nfu}$ and abstract from the length of reducible computations while preserving their execution probability and average duration over $\mathbb{P}_{\rm M,fu}$.

- Need to lift measure *probtime* from a single reducible computation to a multiset of reducible computations with the same origin and destination:

$$pbtm(P, D) \; = \; \{\!| \sum_{c \in reducomp(P, D, t)} probtime(c) \mid t \in \mathbb{R}_{>0} \land reducomp(P, D, t) \neq \emptyset \,|\!\}$$

  where $reducomp(P, D, t)$ is the multiset of reducible computations from $P \in \mathbb{P}_{\rm M}$ to $D \subseteq \mathbb{P}_{\rm M}$ whose average duration is $t \in \mathbb{R}_{>0}$.

- Measures must be summed up (otherwise $<\tau, \lambda_1>.\underline{0} + <\tau, \lambda_2>.\underline{0}$ would not be weakly equivalent to $<\tau, \lambda_1 + \lambda_2>.\underline{0}$) but only in case of equal average durations.

- An equivalence relation $\mathcal{B} \subseteq (\mathbb{P}_{\mathrm{M,nfu}} \times \mathbb{P}_{\mathrm{M,nfu}}) \cup (\mathbb{P}_{\mathrm{M,fu}} \times \mathbb{P}_{\mathrm{M,fu}})$ is a weak Markovian bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$:
  - If $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nfu}}$, then for all action name $a \in \mathit{Name}$ and equivalence classes $D \in \mathbb{P}_{\mathrm{M}}/\mathcal{B}$:
  $$\mathit{rate}(P_1, a, D) \;=\; \mathit{rate}(P_2, a, D)$$
  - If $P_1, P_2 \in \mathbb{P}_{\mathrm{M,fu}}$, then for all equivalence classes $D \in \mathbb{P}_{\mathrm{M,nfu}}/\mathcal{B}$:
  $$\mathit{pbtm}(P_1, D) \;=\; \mathit{pbtm}(P_2, D)$$
- Weak Markovian bisimilarity, denoted $\approx_{\mathrm{MB}}$, is the union of all the weak Markovian bisimulations.

- **Example 1** – Consider the two process terms:

$$\bar{P}_1 \equiv <\tau, \mu>.<\tau, \gamma>.Q \qquad (\equiv <\tau, \gamma>.<\tau, \mu>.Q)$$

$$\bar{P}_2 \equiv <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.Q$$

with $Q \in \mathbb{P}_{\mathrm{M,nfu}}$.

- Then $\bar{P}_1 \approx_{\mathrm{MB}} \bar{P}_2$ because:

$$\begin{aligned} pbtm(\bar{P}_1, [Q]_{\approx_{\mathrm{MB}}}) &= \{\!| (1 \cdot 1) \cdot (\tfrac{1}{\mu} + \tfrac{1}{\gamma}) |\!\} = \\ &= \{\!| 1 \cdot \tfrac{\mu + \gamma}{\mu \cdot \gamma} |\!\} = pbtm(\bar{P}_2, [Q]_{\approx_{\mathrm{MB}}}) \end{aligned}$$

- In general, for $l \in \mathbb{N}_{>0}$ we have that:

$$<\tau, \mu>.<\tau, \gamma_1>. \dots .<\tau, \gamma_l>.Q \approx_{\mathrm{MB}} <\tau, \left(\frac{1}{\mu} + \frac{1}{\gamma_1} + \dots + \frac{1}{\gamma_l}\right)^{-1}>.Q$$

- **Example 2** – Consider the two process terms:

$$\bar{P}_3 \equiv <\tau, \mu>.(<\tau, \gamma_1>.Q_1 + <\tau, \gamma_2>.Q_2)$$

$$\bar{P}_4 \equiv <\tau, \frac{\gamma_1}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right)^{-1}>.Q_1 + <\tau, \frac{\gamma_2}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right)^{-1}>.Q_2$$

  with $Q_1, Q_2 \in \mathbb{P}_{\mathrm{M,nfu}}$ and $Q_1 \not\approx_{\mathrm{MB}} Q_2$.

- Then $\bar{P}_3 \approx_{\mathrm{MB}} \bar{P}_4$ because:

$$pbtm(\bar{P}_3, [Q_1]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\gamma_1}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right) |\!\} = pbtm(\bar{P}_4, [Q_1]_{\approx_{\mathrm{MB}}})$$

$$pbtm(\bar{P}_3, [Q_2]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\gamma_2}{\gamma_1+\gamma_2} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\gamma_2}\right) |\!\} = pbtm(\bar{P}_4, [Q_2]_{\approx_{\mathrm{MB}}})$$

- In general, for $n \in \mathbb{N}_{>0}$ we have that:

$$<\tau, \mu>.(<\tau, \gamma_1>.Q_1 + \ldots + <\tau, \gamma_n>.Q_n) \approx_{\mathrm{MB}} <\tau, \frac{\gamma_1}{\gamma_1+\ldots+\gamma_n} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\ldots+\gamma_n}\right)^{-1}>.Q_1 + \ldots + <\tau, \frac{\gamma_n}{\gamma_1+\ldots+\gamma_n} \cdot \left(\frac{1}{\mu} + \frac{1}{\gamma_1+\ldots+\gamma_n}\right)^{-1}>.Q_n$$

- **Example 3** – Consider the two process terms:

$$\bar{P}_5 \equiv \,<\!\tau, \mu_1\!>.<\!\tau, \gamma\!>.Q_1 + <\!\tau, \mu_2\!>.<\!\tau, \gamma\!>.Q_2$$

$$\bar{P}_6 \equiv \,<\!\tau, \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right)^{-1}\!>.Q_1 + <\!\tau, \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right)^{-1}\!>.Q_2$$

  with $Q_1, Q_2 \in \mathbb{P}_{\mathrm{M,nfu}}$ and $Q_1 \not\approx_{\mathrm{MB}} Q_2$.

- Then $\bar{P}_5 \approx_{\mathrm{MB}} \bar{P}_6$ because:

$$pbtm(\bar{P}_5, [Q_1]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right) |\!\} = pbtm(\bar{P}_6, [Q_1]_{\approx_{\mathrm{MB}}})$$

$$pbtm(\bar{P}_5, [Q_2]_{\approx_{\mathrm{MB}}}) = \{\!| \frac{\mu_2}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right) |\!\} = pbtm(\bar{P}_6, [Q_2]_{\approx_{\mathrm{MB}}})$$

- In general, for $n \in \mathbb{N}_{>0}$ we have that:

$$<\!\tau, \mu_1\!>.<\!\tau, \gamma\!>.Q_1 + \ldots + <\!\tau, \mu_n\!>.<\!\tau, \gamma\!>.Q_n \approx_{\mathrm{MB}} \,<\!\tau, \frac{\mu_1}{\mu_1+\ldots+\mu_n} \cdot \left(\frac{1}{\mu_1+\ldots+\mu_n} + \frac{1}{\gamma}\right)^{-1}\!>.Q_1 + \ldots + <\!\tau, \frac{\mu_n}{\mu_1+\ldots+\mu_n} \cdot \left(\frac{1}{\mu_1+\ldots+\mu_n} + \frac{1}{\gamma}\right)^{-1}\!>.Q_n$$

- **Example 4** – No variant of $<\tau, \mu_1>.<\tau, \gamma>.Q_1 + <\tau, \mu_2>.<\tau, \gamma>.Q_2$ related to actions $<\tau, \gamma>$ leads to a reduction.

- If we consider:

$$\begin{aligned} \bar{P}_7 &\equiv <\tau, \mu_1>.<\tau, \gamma_1>.Q_1 + <\tau, \mu_2>.<\tau, \gamma_2>.Q_2 \\ \bar{P}_8 &\equiv <\tau, \tfrac{\mu_1}{\mu_1+\mu_2} \cdot \left(\tfrac{1}{\mu_1+\mu_2} + \tfrac{1}{\gamma_1}\right)^{-1}>.Q_1 + <\tau, \tfrac{\mu_2}{\mu_1+\mu_2} \cdot \left(\tfrac{1}{\mu_1+\mu_2} + \tfrac{1}{\gamma_2}\right)^{-1}>.Q_2 \end{aligned}$$

  with $\gamma_1 \neq \gamma_2$, then $\bar{P}_7 \not\approx_{\rm MB} \bar{P}_8$.

- If we consider:

$$\begin{aligned} \bar{P}_9 &\equiv <\tau, \mu_1>.<\tau, \gamma>.Q_1 + <\tau, \mu_2>.Q_2 \\ \bar{P}_{10} &\equiv <\tau, \frac{\mu_1}{\mu_1+\mu_2} \cdot \left(\frac{1}{\mu_1+\mu_2} + \frac{1}{\gamma}\right)^{-1}>.Q_1 + <\tau, \mu_2>.Q_2 \end{aligned}$$

  then $\bar{P}_9 \not\approx_{\rm MB} \bar{P}_{10}$.

- Let:
  - $I \neq \emptyset$ be a finite index set.
  - $\mu_i \in \mathbb{R}_{>0}$ for all $i \in I$.
  - $J_i \neq \emptyset$ be a finite index set for all $i \in I$.
  - $\gamma_{i,j} \in \mathbb{R}_{>0}$ and $P_{i,j} \in \mathbb{P}_{\mathrm{M}}$ for all $i \in I$ and $j \in J_i$.

  Then:

$$\sum_{i \in I} <\tau, \mu_i>. \sum_{j \in J_i} <\tau, \gamma_{i,j}>.P_{i,j} \;\approx_{\mathrm{MB}}$$

$$\sum_{i \in I} \sum_{j \in J_i} <\tau, \frac{\mu_i}{\sum\limits_{k \in I} \mu_k} \cdot \frac{\gamma_{i,j}}{\sum\limits_{h \in J_i} \gamma_{i,h}} \cdot \left(\frac{1}{\sum\limits_{k \in I} \mu_k} + \frac{1}{\sum\limits_{h \in J_i} \gamma_{i,h}} \right)^{-1} >.P_{i,j}$$

  whenever:

$$\textstyle\sum_{j \in J_1} \gamma_{i_1,j} \;=\; \sum_{j \in J_2} \gamma_{i_2,j} \text{ for all } i_1, i_2 \in I$$

Congruence Property

- $\approx_{\rm MB}$ is a congruence with respect to action prefix and hiding.
- Let $P_1, P_2 \in \mathbb{P}_{\rm M}$. Whenever $P_1 \approx_{\rm MB} P_2$, then:
  - $<a, \lambda>.P_1 \approx_{\rm MB} <a, \lambda>.P_2$ for all $<a, \lambda> \in Act_{\rm M}$.
  - $P_1 / H \approx_{\rm MB} P_2 / H$ for all $H \subseteq Name_{\rm v}$.
- $\approx_{\rm MB}$ is *not* a congruence with respect to alternative composition due to fully unstable process terms.
- For instance:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} \ \approx_{\rm MB} \ <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0}$$

  but:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} + <a, \lambda>.\underline{0} \ \not\approx_{\rm MB} \ <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} + <a, \lambda>.\underline{0}$$

  both for $a \neq \tau$ and for $a = \tau$.

- Adapt Milner's construction for getting a weak bisimulation congruence.
- Apply the exit rate equality check also to fully unstable process terms, with the equivalence classes to consider being the ones w.r.t. $\approx_{\mathrm{MB}}$.
- We say that P_1 is weakly Markovian bisimulation congruent to P_2, written $P_1 \simeq_{\mathrm{MB}} P_2$, iff for all action names $a \in \mathit{Name}$ and equivalence classes $D \in \mathbb{P}_{\mathrm{M}} / \approx_{\mathrm{MB}}$:
$$\mathit{rate}(P_1, a, D) \; = \; \mathit{rate}(P_2, a, D)$$
- $\sim_{\mathrm{MB}} \subset \simeq_{\mathrm{MB}} \subset \approx_{\mathrm{MB}}$.
- $\simeq_{\mathrm{MB}}$ and $\approx_{\mathrm{MB}}$ coincide over $\mathbb{P}_{\mathrm{M,nfu}}$.
- $<a, \lambda>.P_1 \simeq_{\mathrm{MB}} <a, \lambda>.P_2$ iff $P_1 \approx_{\mathrm{MB}} P_2$.

- $\simeq_{\rm MB}$ is a congruence over the set $\mathbb{P}_{\rm M,seq}$ of process terms of $\mathbb{P}_{\rm M}$ that do not contain any occurrence of the parallel composition operator.
- Let $P_1, P_2 \in \mathbb{P}_{\rm M}$. Whenever $P_1 \simeq_{\rm MB} P_2$, then:
  - $<a, \lambda>.P_1 \simeq_{\rm MB} <a, \lambda>.P_2$ for all $<a, \lambda> \in Act_{\rm M}$.
  - $P_1 + P \simeq_{\rm MB} P_2 + P$ and $P + P_1 \simeq_{\rm MB} P + P_2$ for all $P \in \mathbb{P}_{\rm M}$.
  - $P_1 / H \simeq_{\rm MB} P_2 / H$ for all $H \subseteq Name_{\rm v}$.
- $\simeq_{\rm MB}$ is the coarsest congruence contained in $\approx_{\rm MB}$ over $\mathbb{P}_{\rm M,seq}$.
- Let $P_1, P_2 \in \mathbb{P}_{\rm M}$. Then $P_1 \simeq_{\rm MB} P_2$ iff $P_1 + P \approx_{\rm MB} P_2 + P$ for all $P \in \mathbb{P}_{\rm M}$.
- $\simeq_{\rm MB}$ is a congruence with respect to recursion (up to, open terms).

- A binary relation $\mathcal{B} \subseteq (\mathbb{P}_{\mathrm{M,nfu}} \times \mathbb{P}_{\mathrm{M,nfu}}) \cup (\mathbb{P}_{\mathrm{M,fu}} \times \mathbb{P}_{\mathrm{M,fu}})$ is a weak Markovian bisimulation up to $\approx_{\mathrm{MB}}$ iff for all $(P_1, P_2) \in \mathcal{B}$:
  - If $P_1, P_2 \in \mathbb{P}_{\mathrm{M,nfu}}$, then for all action names $a \in \mathit{Name}$ and equivalence classes $D \in \mathbb{P}_{\mathrm{M}}/(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\mathrm{MB}})^+$:
  $$\mathit{rate}(P_1, a, D) = \mathit{rate}(P_2, a, D)$$
  - If $P_1, P_2 \in \mathbb{P}_{\mathrm{M,fu}}$, then for all equivalence classes $D \in \mathbb{P}_{\mathrm{M,nfu}}/(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\mathrm{MB}})^+$:
  $$\mathit{pbtm}(P_1, D) = \mathit{pbtm}(P_2, D)$$
- If $\mathcal{B} \subseteq (\mathbb{P}_{\mathrm{M,nfu}} \times \mathbb{P}_{\mathrm{M,nfu}}) \cup (\mathbb{P}_{\mathrm{M,fu}} \times \mathbb{P}_{\mathrm{M,fu}})$ is a weak Markovian bisimulation up to $\approx_{\mathrm{MB}}$, then $(P_1, P_2) \in \mathcal{B}$ implies $P_1 \approx_{\mathrm{MB}} P_2$. Moreover $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\mathrm{MB}})^+$ coincides with $\approx_{\mathrm{MB}}$.

- We extend $\simeq_{\mathrm{MB}}$ to open process terms by replacing all variables freely occurring outside rec binders with every closed process term.
- Let $P_1, P_2 \in \mathcal{PL}_{\mathrm{M}}$ be guarded process terms containing free occurrences of $k \in \mathbb{N}$ process variables $X_1, \dots, X_k \in \mathit{Var}$ at most.
- We define $P_1 \simeq_{\mathrm{MB}} P_2$ iff:
$$P_1\{Q_i \hookrightarrow X_i \mid 1 \leq i \leq k\} \simeq_{\mathrm{MB}} P_2\{Q_i \hookrightarrow X_i \mid 1 \leq i \leq k\}$$
for all $Q_1, \dots, Q_k \in \mathcal{PL}_{\mathrm{M}}$ containing no free occurrences of process variables.
- Whenever $P_1 \simeq_{\mathrm{MB}} P_2$, then:
$$\mathrm{rec}\, X_1 : \dots : \mathrm{rec}\, X_k : P_1 \simeq_{\mathrm{MB}} \mathrm{rec}\, X_1 : \dots : \mathrm{rec}\, X_k : P_2$$

Sound and Complete Axiomatization

- $\simeq_{\rm MB}$ has a sound and complete axiomatization over the set $\mathbb{P}_{\rm M,seq,nr}$ of nonrecursive process terms of $\mathbb{P}_{\rm M,seq}$.

- The basic axioms of $\simeq_{\rm MB}$ coincide with those of $\sim_{\rm MB}$:

$$
\begin{array}{lrcl}
(\mathcal{A}_{\rm MB,1}) & P_1 + P_2 & = & P_2 + P_1 \\
(\mathcal{A}_{\rm MB,2}) & (P_1 + P_2) + P_3 & = & P_1 + (P_2 + P_3) \\
(\mathcal{A}_{\rm MB,3}) & P + \underline{0} & = & P \\
(\mathcal{A}_{\rm MB,4}) & <a, \lambda_1>.P + <a, \lambda_2>.P & = & <a, \lambda_1 + \lambda_2>.P
\end{array}
$$

- Axiom characterizing $\simeq_{\rm MB}$:

$$(\mathcal{A}_{\rm MB,5}) \quad <a,\lambda>.\sum_{i\in I}<\tau,\mu_i>.\sum_{j\in J_i}<\tau,\gamma_{i,j}>.P_{i,j} \;=\; <a,\lambda>.\sum_{i\in I}\sum_{j\in J_i}<\tau,\frac{\mu_i}{\mu}\cdot\frac{\gamma_{i,j}}{\gamma}\cdot\left(\frac{1}{\mu}+\frac{1}{\gamma}\right)^{-1}>.P_{i,j}$$

if: $I \neq \emptyset$ is a finite index set

$J_i \neq \emptyset$ is a finite index set for all $i \in I$

$\mu = \sum_{i\in I}\mu_i$

$\gamma = \sum_{j\in J_i}\gamma_{i,j}$ for all $i \in I$

- Same distributive axioms for hiding as $\sim_{\rm MB}$:

$(\mathcal{A}_{\rm MB,6})$	$\underline{0}/H$	$=$	$\underline{0}$	
$(\mathcal{A}_{\rm MB,7})$	$(<a,\lambda>.P)/H$	$=$	$<a,\lambda>.(P/H)$	if $a \notin H$
$(\mathcal{A}_{\rm MB,8})$	$(<a,\lambda>.P)/H$	$=$	$<\tau,\lambda>.(P/H)$	if $a \in H$
$(\mathcal{A}_{\rm MB,9})$	$(P_1+P_2)/H$	$=$	$P_1/H+P_2/H$	

- For proving completeness, we cannot resort to normal form saturation as this would alter the quantitative behavior.
- Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M,seq,nr}}$. If $P_1 \approx_{\mathrm{MB}} P_2$ but $P_1 \not\simeq_{\mathrm{MB}} P_2$, then at least one between P_1 and P_2 (both of which must be fully unstable) is of the form:

$$\sum_{i \in I} <\tau, \mu_i>. \sum_{j \in J_i} <\tau, \gamma_{i,j}>.P_{i,j}$$

  where $I \neq \emptyset$ is a finite index set, $J_i \neq \emptyset$ is a finite index set for all $i \in I$, and one of the following two properties holds:
  - $\sum_{j \in J_{i_1}} <\tau, \gamma_{i_1,j}>.P_{i_1,j} \approx_{\mathrm{MB}} \sum_{j \in J_{i_2}} <\tau, \gamma_{i_2,j}>.P_{i_2,j}$ for all $i_1, i_2 \in I$.
  - $\sum_{j \in J_{i_1}} \gamma_{i_1,j} = \sum_{j \in J_{i_2}} \gamma_{i_2,j}$ for all $i_1, i_2 \in I$.
- Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M,seq,nr}}$. Then $\mathcal{A}_{\mathrm{MB},1..9} \vdash P_1 = P_2 \iff P_1 \simeq_{\mathrm{MB}} P_2$.

Polynomial-Time Decidability

- Partition refinement algorithm like for other bisimulation equivalences.
- Steps for checking whether $P_1 \approx_{\mathrm{MB}} P_2$ or $P_1 \simeq_{\mathrm{MB}} P_2$ (finite-state processes):
  1. Build a partition with one class for all the non-fully-unstable states of $[\![P_1]\!]_{\mathrm{M}}$ and $[\![P_2]\!]_{\mathrm{M}}$ and one class for all the fully unstable states of $[\![P_1]\!]_{\mathrm{M}}$ and $[\![P_2]\!]_{\mathrm{M}}$.
  2. Refine the partition by applying the *rate*-based check to the classes of non-fully-unstable states and the *pbtm*-based check to the classes of fully unstable states, until a fixed point is reached.
  3. $\approx_{\mathrm{MB}}$: return yes or no depending on whether P_1 and P_2 belong to the same class.
  4. $\simeq_{\mathrm{MB}}$: return yes or no depending on whether P_1 and P_2 belong to the same class and satisfy the *rate*-based check w.r.t. all classes.

- The algorithm executes in polynomial time only when $[\![P_1]\!]_{\mathrm{M}}$ and $[\![P_2]\!]_{\mathrm{M}}$ have no cycles of exponentially timed internal transitions.
- Cycles of nondeterministic internal transitions are unimportant from a quantitative viewpoint.
- Cycles of probabilistic internal transitions can be left in the long run with probability 1 (a way out exists) or 0 (connecting an absorbing set of states).
- Cycles of exponentially timed internal transitions cause time to progress and hence cannot be ignored: *pbtm* multisets become infinite.
- Consider $P \equiv \,<\tau, \mu>. \, \mathrm{rec} \, X : (<\tau, \delta>.X + <\tau, \gamma>.Q)$ with $Q \in \mathbb{P}_{\mathrm{M,nfu}}$.
- Due to the exponentially timed internal selfloop labeled with $<\tau, \delta>$, we have that $pbtm(P, [Q]_{\approx_{\mathrm{MB}}})$ contains infinitely many *probtime* values of the form $(\frac{\delta}{\delta+\gamma})^n \cdot \frac{\gamma}{\delta+\gamma} \cdot (\frac{1}{\mu} + (n+1) \cdot \frac{1}{\delta+\gamma})$ where $n \in \mathbb{N}$.

Exactness of CTMC-Level Aggregation

- $\approx_{\mathrm{MB}}$ and $\simeq_{\mathrm{MB}}$ allow every sequence of exponentially timed τ-actions to be considered equivalent to a single exponentially timed τ-action having the same average duration (and execution probability).

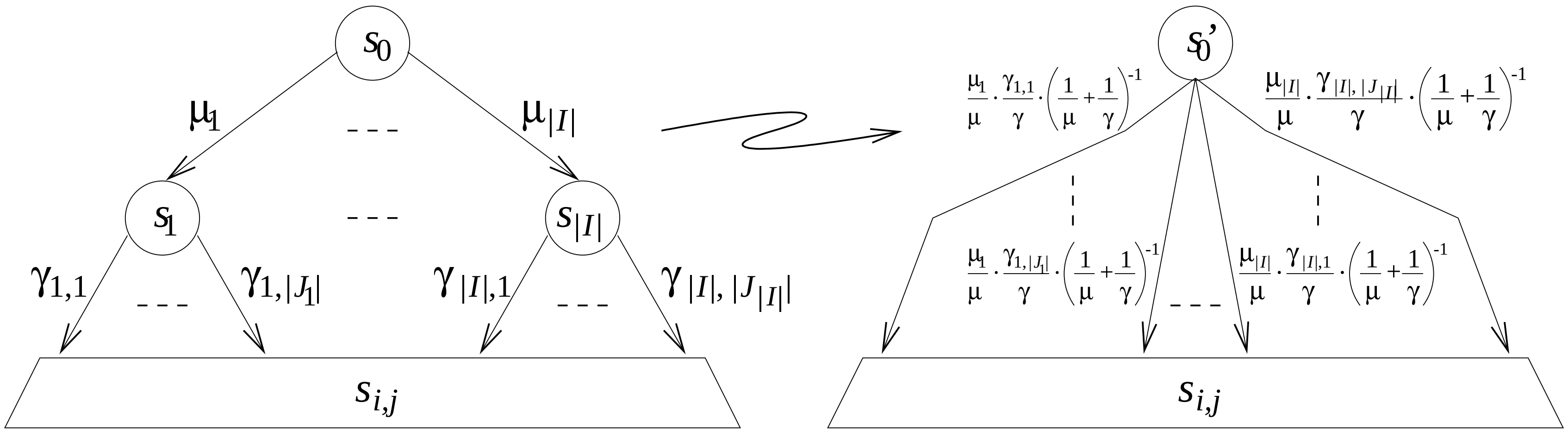
- This amounts to approximating a hypoexponentially or Erlang distributed random variable with an exponentially distributed random variable having the same expected value.

- This can be exploited to assess more quickly properties expressed in terms of the mean time to certain events by working on a CTMC obtained from the original one via pseudo-aggregation.

- *Is there any other performance property that is preserved?*

- Concentrate on $\mathcal{A}_{\mathrm{MB},5}$ because this is the only new axiom w.r.t. $\sim_{\mathrm{MB}}$ and $\sim_{\mathrm{MB}}$ is consistent with ordinary lumpability.

- The induced CTMC-level aggregation, called W-lumpability, eliminates $|I|$ states and $|I|$ transitions by merging the first $1 + |I|$ states into a single one:



- W-lumpability is exact *at steady state*, i.e., the stationary probability of being in a macrostate of a CTMC obtained via W-lumpability is the sum of the stationary probabilities of being in one of the constituent microstates of the CTMC from which the reduced one has been derived.

- Unlike ordinary lumpability and T-lumpability, W-lumpability may *not* preserve properties expressed in terms of transient state probabilities.

- Reconsider $\bar{P}_1 \equiv <\tau, \mu>.<\tau, \gamma>.Q$ and $\bar{P}_2 \equiv <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.Q$.

- The probability of being in the first state of $[\![\bar{P}_2]\!]_{\mathrm{M}}$ at time $t \in \mathbb{R}_{>0}$ is $1 - (1 - \mathrm{e}^{-\frac{\mu \cdot \gamma}{\mu + \gamma} \cdot t}) = \mathrm{e}^{-\frac{\mu \cdot \gamma}{\mu + \gamma} \cdot t}$, which reduces to $\mathrm{e}^{-\frac{\mu}{2} \cdot t}$ when $\mu = \gamma$.

- The sum of the probabilities of being in one of the first two states of $[\![\bar{P}_1]\!]_{\mathrm{M}}$ at the same time instant is $\frac{\gamma}{\gamma - \mu} \cdot \mathrm{e}^{-\mu \cdot t} - \frac{\mu}{\gamma - \mu} \cdot \mathrm{e}^{-\gamma \cdot t}$ for $\mu \neq \gamma$ or $(1 + \mu \cdot t) \cdot \mathrm{e}^{-\mu \cdot t}$ for $\mu = \gamma$.

Compositionality/Exactness Tradeoff for Concurrent Processes

- $\approx_{\mathrm{MB}}$ and $\simeq_{\mathrm{MB}}$ are *not* congruences with respect to parallel composition due to fully unstable process terms (different from nondeterministic processes).

- For instance:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} \approx_{\mathrm{MB}} <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0}$$

  but:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0} \not\approx_{\mathrm{MB}} <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}$$

  both for $a \neq \tau$ and for $a = \tau$.

- Not being a congruence with respect to parallel composition significantly reduces the usefulness of $\simeq_{\mathrm{MB}}$ for compositional state space reduction.

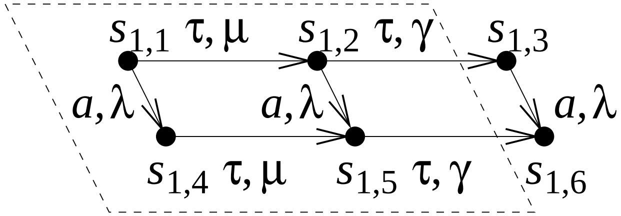
- We can retrieve full compositionality by enhancing the abstraction capability of $\simeq_{\mathrm{MB}}$ in the case of concurrent computations ...
- ... but exactness will hold at steady state only for certain processes (sequential processes with abstraction and concurrent processes with limited synchronization).
- So far considered only computations traversing fully unstable states.
- Revise the notion of reducible computation by admitting the traversal of unstable states (that are not fully unstable) satisfying certain conditions.
- Focus on local computations that traverse fully unstable local states that may be part of global states that are not fully unstable.
- Replicated (trees of) reducible computations due to interleaving.

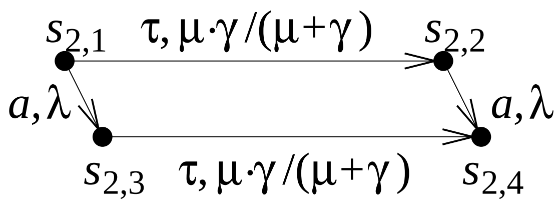
- The process terms:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}$$

$$<\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}$$

  should be considered weakly Markovian bisimulation equivalent and should give rise to the following CTMC-level aggregation:





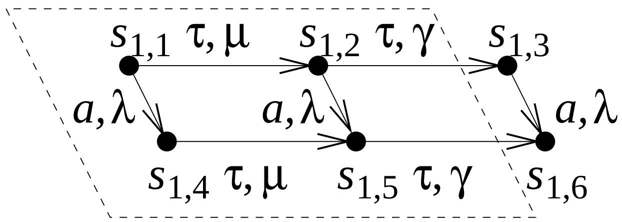
- Due to interleaving, a sequence of exponentially timed τ-actions may be replicated in the sense that it may label several replicas of the same computation that traverses fully unstable local states.

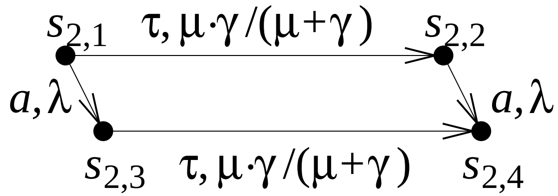
- Recognize – and take into account at once – all the replicas of that computation and pinpoint their initial and final states.

- Consider again:

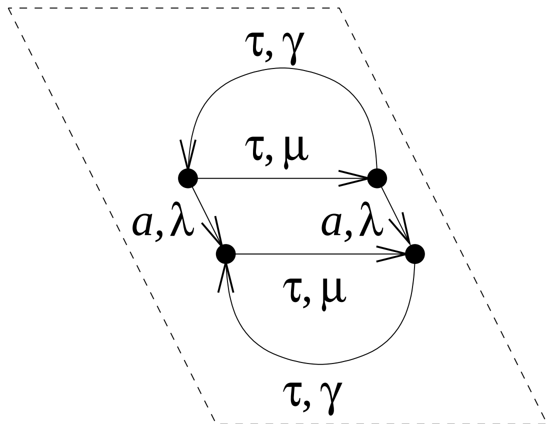
- The initial states are $s_{1,1}$ and $s_{1,4}$.
- The final states are $s_{1,3}$ and $s_{1,6}$.
- A one-to-one correspondence can be established between the states traversed by any two replicas: $(s_{1,1}, s_{1,4})$, $(s_{1,2}, s_{1,5})$, $(s_{1,3}, s_{1,6})$.
- *When moving vertically, the current stage of the replicas is preserved.*
- Any two states traversed by the same replica can only possess transitions that are pairwise identically labeled.
- The context of a replica is the set of transitions not belonging to the replica that depart from a state traversed by the replica: $\{<a, \lambda>\}, \emptyset$.
- *When moving horizontally, the context of each replica is preserved.*

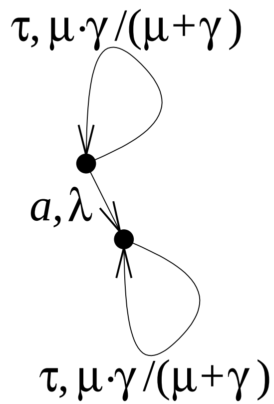
- Identification of the boundary of the replicas of a reducible computation:
  - The final states have no exponentially timed τ-transition:





  - Each of the final states has an exponentially timed τ-transition back to one of the preceding states of the same replica (local recursion):





- The notion of replicated reducible computation must be accompanied by an adjustment of measure *probtime* (now) and multiset *pbtm* (later).

- Given a computation c of the form $P_1 \xrightarrow{\tau,\lambda_1}_{\mathrm{M}} P_2 \xrightarrow{\tau,\lambda_2}_{\mathrm{M}} \dots \xrightarrow{\tau,\lambda_n}_{\mathrm{M}} P_{n+1}$ that is reducible in the old sense, we have that:

$$\boxed{\mathit{probtime}(c) \;=\; \left(\prod_{i=1}^{n} \frac{\lambda_i}{\mathit{rate}(P_i,\tau,\mathbb{P}_{\mathrm{M}})}\right) \cdot \left(\sum_{i=1}^{n} \frac{1}{\mathit{rate}(P_i,\tau,\mathbb{P}_{\mathrm{M}})}\right)}$$

- The denominators can indifferently be of the form $\mathit{rate}(P_i, \tau, \mathbb{P}_{\mathrm{M}})$ or $\mathit{rate}_{\mathrm{t}}(P_i)$ because $P_i \in \mathbb{P}_{\mathrm{M,fu}}$ for all $i = 1, \dots, n$.

- In the case of a replicated reducible computation, each replica has a possibly different context and hence we need to take values of the form $\mathit{rate}(P_i, \tau, \mathbb{P}_{\mathrm{M}})$ as denominators, so as to focus on τ-transitions.

- Since τ-transitions can be also in the context, denominators will be of the form $\mathit{rate}(P_i, \tau, \mathcal{P})$ with $\mathcal{P}$ containing *only the states traversed by the replicas*, so as to achieve context independence.

- Consider $m \in \mathbb{N}_{>0}$ process terms $P_1, P_2, \dots, P_m \in \mathbb{P}_{\mathrm{M}}$ different from each other, which can be reached from the same process term.
- Let C_k^τ be the set of all the finite-length computations starting from P_k such that each of them:
  - Is labeled with a sequence of exponentially timed τ-actions.
  - Traverses different states except at most the final state and one of the preceding states.
  - Shares no transitions with computations in $C_{k'}^\tau$ for all $k' \neq k$.
- Assume that those sets are not empty and that their computations can be partitioned into $n \in \mathbb{N}_{>0}$ groups of replicas each consisting of m computations from all the m sets, such that all the computations in the same group have the same length and are labeled with the same sequence of exponentially timed τ-actions.
- $C_k^\tau = \{c_{k,i} \equiv P_{k,i,1} \xrightarrow{\tau,\lambda_{i,1}}_{\mathrm{M}} P_{k,i,2} \xrightarrow{\tau,\lambda_{i,2}}_{\mathrm{M}} \dots \xrightarrow{\tau,\lambda_{i,l_i}}_{\mathrm{M}} P_{k,i,l_i+1} \mid 1 \leq i \leq n\}$ where $P_{k,i,1}$ is P_k and $l_i \in \mathbb{N}_{>0}$ is the computation length.

- The family of computations $\mathcal{C}^\tau = \{C_1^\tau, C_2^\tau, \dots, C_m^\tau\}$ is g-reducible iff either $m = 1$ and for all $i = 1, \dots, n$:
  - $P_{1,i,j} \in \mathbb{P}_{\mathrm{M,fu}}$ for all $j = 1, \dots, l_i$;
  - $P_{1,i,l_i+1} \in \mathbb{P}_{\mathrm{M,nfu}}$ or P_{1,i,l_i+1} is $P_{1,i,j}$ for some $j = 1, \dots, l_i$;

  or $m \geq 1$, $P_{1,i,j} \in \mathbb{P}_{\mathrm{M,nfu}}$ for every $i = 1, \dots, n$ and $j = 1, \dots, l_i$ when $m = 1$, and for all $i = 1, \dots, n$:
  - For all $k = 1, \dots, m$, $j = 1, \dots, l_i$, and $<a, \lambda> \in \mathit{Act}_{\mathrm{M}}$:
    1. Whenever $P_{k,i,j} \xrightarrow{a,\lambda}_{\mathrm{M}} P'$ with P' different from $P_{k,i,j+1}$, then:
       a) either P' is $P_{k',i,j}$ for some $k' = 1, \dots, m$;
       b) or P' is $P_{k,i',j'}$ with $a = \tau$ and $\lambda = \lambda_{i',j'-1}$ for some $i' = 1, \dots, n$ such that $i' \neq i$ and some $j' = 2, \dots, l_{i'+1}$.

2. For all $k' = 1, \ldots, m$, it holds that $P_{k,i,j} \xrightarrow{a,\lambda}_{\mathrm{M}} P_{k',i,j}$ iff $P_{k,i,j'} \xrightarrow{a,\lambda}_{\mathrm{M}} P_{k',i,j'}$ for all $j' = 1, \ldots, l_i$.
3. For all $i' = 1, \ldots, n$ such that $i' \neq i$ and $j' = 2, \ldots, l_{i'+1}$, it holds that $P_{k,i,j} \xrightarrow{a,\lambda}_{\mathrm{M}} P_{k,i',j'}$ iff $P_{k',i,j} \xrightarrow{a,\lambda}_{\mathrm{M}} P_{k',i',j'}$ for all $k' = 1, \ldots, m$.

– One of the following holds:

$\overline{4}$. If there exists $\lambda_{i,l_i+1} \in \mathbb{R}_{>0}$ such that $P_{k,i,l_i+1} \xrightarrow{\tau,\lambda_{i,l_i+1}}_{\mathrm{M}} P_{k,i,l_i+2}$ for all $k = 1, \ldots, m$, then at least one of conditions 1, 2, and 3 above is not satisfied by P_{k',i,l_i+1} for some $k' = 1, \ldots, m$.

$\widetilde{4}$. There is no $\lambda_{i,l_i+1} \in \mathbb{R}_{>0}$ such that $P_{k,i,l_i+1} \xrightarrow{\tau,\lambda_{i,l_i+1}}_{\mathrm{M}} P_{k,i,l_i+2}$ for all $k = 1, \ldots, m$.

$\widehat{4}$. P_{k,i,l_i+1} is $P_{k,i,j}$ for all $k = 1, \ldots, m$ and some $j = 1, \ldots, l_i$.

- In the case $m = 1$ with all the traversed states being fully unstable, the definition of g-reducible family of computations coincides with the old definition of reducible computation (the former directly considers a tree).

- In the case $m = 1$ with every $P_{1,i,j} \in \mathbb{P}_{\mathrm{M,nfu}}$, all the sequential process terms in parallel with the one originating the considered tree of reducible computations repeatedly execute a single action, thus causing no replica of the tree to be formed (observable selfloops).

- Condition 1 establishes that each transition that deviates from the considered reducible computation of $\mathcal{C}^{\tau}$ (maximality of $\mathcal{C}^{\tau}$):
  - either is a vertical transition that preserves the current stage of the replicas and hence causes the passage to the corresponding state of another replica when $k' \neq k$ or to the same state of the same replica when $k' = k$ (subcase a);
  - or is a transition belonging to some other computation in $\mathcal{C}^{\tau}$ starting from the same process term P_k (subcase b).

- Condition 2 is related to condition 1.a and ensures that the context of a replica is preserved along each state traversed by the replica.
- Condition 3 is related to condition 1.b and ensures that any transition belonging neither to the considered computation nor to its context is present at the same stage of each replica of the considered computation.
- The three variants of condition 4 establish the boundary of the replicas of the considered computation in a way that guarantees the maximality of the length of the replicas themselves under:
  - Conditions 1, 2, 3 (variant $\overline{4}$).
  - The constraint that all the transitions of the replicas be labeled with exponentially timed τ-actions (variant $\widetilde{4}$).
  - The constraint that all the traversed states be different except at most the final state and one of the preceding states (variant $\widehat{4}$).

- $source(\mathcal{C}^\tau) = \{P_k \mid 1 \leq k \leq m\}$: set of initial states of $\mathcal{C}^\tau$.
- $target(\mathcal{C}^\tau) = \{P_{k,i,l_i+1} \mid 1 \leq k \leq m, 1 \leq i \leq n\}$: set of final states of $\mathcal{C}^\tau$.
- In order to avoid interferences between computations in $C_1^\tau, C_2^\tau, \ldots, C_m^\tau$ and transitions belonging to the context of those computations, let:

$$\boxed{probtime_{\mathrm{cf}}(c_{k,i}) \;=\; \left(\prod_{j=1}^{l_i} \frac{\lambda_{i,j}}{rate(P_{k,i,j},\tau,\mathcal{P}_k)}\right) \cdot \left(\sum_{j=1}^{l_i} \frac{1}{rate(P_{k,i,j},\tau,\mathcal{P}_k)}\right)}$$

  where $\mathcal{P}_k = \{P_{k,i',j'} \mid 1 \leq i' \leq n, 2 \leq j' \leq l_{i'+1}\}$.

- We redefine $reducomp(P_k, D, t)$ as the multiset of computations *identical to those in* C_k^τ that go from P_k to D and have average duration t.
- Whenever $\mathcal{C}^\tau$ is g-reducible, then $probtime_{\mathrm{cf}}(c_{k,i}) = probtime_{\mathrm{cf}}(c_{k',i})$ and $pbtm_{\mathrm{cf}}(P_k, target(\mathcal{C}^\tau)) = pbtm_{\mathrm{cf}}(P_{k'}, target(\mathcal{C}^\tau))$.

- An equivalence relation $\mathcal{B}$ over $\mathbb{P}_{\mathrm{M}}$ is a g-weak Markovian bisimulation iff, whenever $(P_1, P_2) \in \mathcal{B}$, then:
  - For all visible action names $a \in \mathit{Name}_{\mathrm{v}}$ and equivalence classes $D \in \mathbb{P}_{\mathrm{M}} / \mathcal{B}$:
  $$\mathit{rate}(P_1, a, D) \ = \ \mathit{rate}(P_2, a, D)$$
  - If P_1 is not an initial state of any g-reducible family of computations, then P_2 is not an initial state of any g-reducible family of computations either, and for all equivalence classes $D \in \mathbb{P}_{\mathrm{M}} / \mathcal{B}$:
  $$\mathit{rate}(P_1, \tau, D) \ = \ \mathit{rate}(P_2, \tau, D)$$
  - If P_1 is an initial state of some g-reducible family of computations, then P_2 is an initial state of some g-reducible family of computations too, and for all g-reducible families of computations $\mathcal{C}_1^\tau$ with $P_1 \in \mathit{source}(\mathcal{C}_1^\tau)$ there exists a g-reducible family of computations $\mathcal{C}_2^\tau$ with $P_2 \in \mathit{source}(\mathcal{C}_2^\tau)$ such that for all equivalence classes $D \in \mathbb{P}_{\mathrm{M}} / \mathcal{B}$:
  $$\mathit{pbtm}_{\mathrm{cf}}(P_1, D \cap \mathit{target}(\mathcal{C}_1^\tau)) \ = \ \mathit{pbtm}_{\mathrm{cf}}(P_2, D \cap \mathit{target}(\mathcal{C}_2^\tau))$$
- G-weak Markovian bisimilarity, denoted $\approx_{\mathrm{MB,g}}$, is the union of all the g-weak Markovian bisimulations.

- All the examples that we have seen before for $\approx_{\rm MB}$ are valid for $\approx_{\rm MB,g}$, because a tree of computations reducible in the sense of the original definition forms a g-reducible family of computations.

- **Example 5** – It holds that:

$$<\tau, \mu>.<\tau, \gamma>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0} \approx_{\rm MB,g} <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}$$

For $a \neq \tau$ and D containing the final state $\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}$:

$$\begin{array}{rcl} pbtm_{\rm cf}(<\tau, \mu>.<\tau, \gamma>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}, D \cap target(\mathcal{C}_1^{\tau})) & = & \{\!|\, (1 \cdot 1) \cdot (\frac{1}{\mu} + \frac{1}{\gamma}) \,|\!\} \\ pbtm_{\rm cf}(<\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}, D \cap target(\mathcal{C}_2^{\tau})) & = & \{\!|\, 1 \cdot \frac{\mu + \gamma}{\mu \cdot \gamma} \,|\!\} \end{array}$$

For $a = \tau$ and D containing the final states $<\tau, \mu>.<\tau, \gamma>.\underline{0} \parallel_{\emptyset} \underline{0}$ and $<\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} \parallel_{\emptyset} \underline{0}$:

$$\begin{array}{rcl} pbtm_{\rm cf}(<\tau, \mu>.<\tau, \gamma>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}, D \cap target(\mathcal{C}_1'^{\tau})) & = & \{\!|\, 1 \cdot \frac{1}{\lambda} \,|\!\} \\ pbtm_{\rm cf}(<\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.\underline{0} \parallel_{\emptyset} <a, \lambda>.\underline{0}, D \cap target(\mathcal{C}_2'^{\tau})) & = & \{\!|\, 1 \cdot \frac{1}{\lambda} \,|\!\} \end{array}$$

Congruence Property

- Unlike $\approx_{\rm MB}$, it turns out that $\approx_{\rm MB,g}$ is a congruence with respect to parallel composition too.
- Let $P_1, P_2 \in \mathbb{P}_{\rm M}$. Whenever $P_1 \approx_{\rm MB,g} P_2$, then:
  - $<a, \lambda>.P_1 \approx_{\rm MB,g} <a, \lambda>.P_2$ for all $<a, \lambda> \in Act_{\rm M}$.
  - $P_1 / H \approx_{\rm MB,g} P_2 / H$ for all $H \subseteq Name_{\rm v}$.
  - $P_1 \,\|_S\, P \approx_{\rm MB,g} P_2 \,\|_S\, P$ and $P \,\|_S\, P_1 \approx_{\rm MB,g} P \,\|_S\, P_2$ for all $S \subseteq Name_{\rm v}$ and $P \in \mathbb{P}_{\rm M}$.
- Like $\approx_{\rm MB}$, we have that $\approx_{\rm MB,g}$ is not a congruence with respect to alternative composition either.
- We say that P_1 is g-weakly Markovian bisimulation congruent to P_2, written $P_1 \simeq_{\rm MB,g} P_2$, iff for all action names $a \in Name$ and equivalence classes $D \in \mathbb{P}_{\rm M} / \approx_{\rm MB,g}$:

$$rate(P_1, a, D) \;=\; rate(P_2, a, D)$$

- $\sim_{\mathrm{MB}} \subset \simeq_{\mathrm{MB,g}} \subset \approx_{\mathrm{MB,g}}$.
- $\simeq_{\mathrm{MB,g}}$ and $\approx_{\mathrm{MB,g}}$ coincide over the set of process terms of $\mathbb{P}_{\mathrm{M}}$ that are not initial states of any g-reducible family of computations.
- $<a, \lambda>.P_1 \simeq_{\mathrm{MB,g}} <a, \lambda>.P_2$ iff $P_1 \approx_{\mathrm{MB,g}} P_2$.
- Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M}}$. Whenever $P_1 \simeq_{\mathrm{MB,g}} P_2$, then:
  - $<a, \lambda>.P_1 \simeq_{\mathrm{MB,g}} <a, \lambda>.P_2$ for all $<a, \lambda> \in \mathit{Act}_{\mathrm{M}}$.
  - $P_1 + P \simeq_{\mathrm{MB,g}} P_2 + P$ and $P + P_1 \simeq_{\mathrm{MB,g}} P + P_2$ for all $P \in \mathbb{P}_{\mathrm{M}}$.
  - $P_1 / H \simeq_{\mathrm{MB,g}} P_2 / H$ for all $H \subseteq \mathit{Name}_{\mathrm{v}}$.
  - $P_1 \,\|_S\, P \simeq_{\mathrm{MB,g}} P_2 \,\|_S\, P$ and $P \,\|_S\, P_1 \simeq_{\mathrm{MB,g}} P \,\|_S\, P_2$ for all $S \subseteq \mathit{Name}_{\mathrm{v}}$ and $P \in \mathbb{P}_{\mathrm{M}}$.
- Let $P_1, P_2 \in \mathbb{P}_{\mathrm{M}}$. Then $P_1 \simeq_{\mathrm{MB,g}} P_2$ iff $P_1 + P \approx_{\mathrm{MB,g}} P_2 + P$ for all $P \in \mathbb{P}_{\mathrm{M}}$.

- A binary relation $\mathcal{B}$ over $\mathbb{P}_{\rm M}$ is a g-weak Markovian bisimulation up to $\approx_{\rm MB,g}$ iff, whenever $(P_1, P_2) \in \mathcal{B}$, then:
  - For all visible action names $a \in Name_{\rm v}$ and equivalence classes $D \in \mathbb{P}_{\rm M} / (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB,g})^+$:
  $$rate(P_1, a, D) \; = \; rate(P_2, a, D)$$
  - If P_1 is not an initial state of any g-reducible family of computations, then P_2 is not an initial state of any g-reducible family of computations either, and for all equivalence classes $D \in \mathbb{P}_{\rm M} / (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB,g})^+$:
  $$rate(P_1, \tau, D) \; = \; rate(P_2, \tau, D)$$
  - If P_1 is an initial state of some g-reducible family of computations, then P_2 is an initial state of some g-reducible family of computations too, and for all g-reducible families of computations $\mathcal{C}_1^\tau$ with $P_1 \in source(\mathcal{C}_1^\tau)$ there exists a g-reducible family of computations $\mathcal{C}_2^\tau$ with $P_2 \in source(\mathcal{C}_2^\tau)$ such that for all equivalence classes $D \in \mathbb{P}_{\rm M} / (\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB,g})^+$:
  $$pbtm_{\rm cf}(P_1, D \cap target(\mathcal{C}_1^\tau)) \; = \; pbtm_{\rm cf}(P_2, D \cap target(\mathcal{C}_2^\tau))$$
- If $\mathcal{B} \subseteq \mathbb{P}_{\rm M} \times \mathbb{P}_{\rm M}$ is a g-weak Markovian bisimulation up to $\approx_{\rm MB,g}$, then $(P_1, P_2) \in \mathcal{B}$ implies $P_1 \approx_{\rm MB,g} P_2$. Moreover $(\mathcal{B} \cup \mathcal{B}^{-1} \cup \approx_{\rm MB,g})^+$ coincides with $\approx_{\rm MB,g}$.

- We extend $\simeq_{\mathrm{MB,g}}$ to open process terms by replacing all variables freely occurring outside rec binders with every closed process term.
- Let $P_1, P_2 \in \mathcal{PL}_{\mathrm{M}}$ be guarded process terms containing free occurrences of $k \in \mathbb{N}$ process variables $X_1, \ldots, X_k \in \mathit{Var}$ at most.
- We define $P_1 \simeq_{\mathrm{MB,g}} P_2$ iff:

  $$P_1\{Q_i \hookrightarrow X_i \mid 1 \leq i \leq k\} \simeq_{\mathrm{MB,g}} P_2\{Q_i \hookrightarrow X_i \mid 1 \leq i \leq k\}$$

  for all $Q_1, \ldots, Q_k \in \mathcal{PL}_{\mathrm{M}}$ containing no free occurrences of process variables.
- Whenever $P_1 \simeq_{\mathrm{MB,g}} P_2$, then:

  $$\mathrm{rec}\, X_1 : \ldots : \mathrm{rec}\, X_k : P_1 \simeq_{\mathrm{MB,g}} \mathrm{rec}\, X_1 : \ldots : \mathrm{rec}\, X_k : P_2$$

Exactness of CTMC-Level Aggregation

- The CTMC-level aggregation induced by $\approx_{\rm MB,g}$ and $\simeq_{\rm MB,g}$ is exact at steady-state only for those process terms with a restricted use of synchronization (in addition to sequential process terms with abstraction).

- This limitation stems from insensitivity conditions for GSMPs (with GSMPs coming into play due to the reduction of sequences of exponentially timed τ-transitions) and emphasizes a tradeoff between achieving compositionality over concurrent processes and preserving steady-state exactness everywhere.

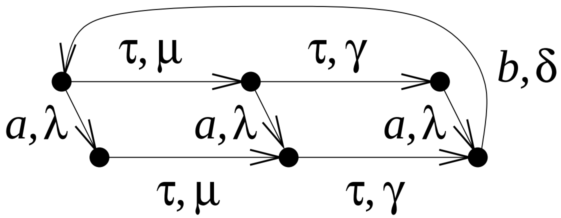
- GW-lumpability is exact at steady state over each process term $P \in \mathbb{P}_{\rm M}$ such that, for all g-reducible families of computations $\mathcal{C}^{\tau}$ in $[\![P]\!]_{\rm M}$ with size $m \geq 2$ or size $m = 1$ and all the traversed states being not fully unstable, no transition in $[\![P]\!]_{\rm M}$ arising from action synchronization has an element of $source(\mathcal{C}^{\tau})$ as its target state.

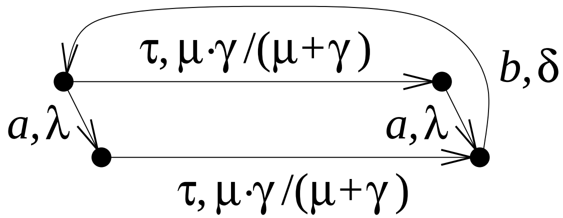
- **Example 6** – Consider the two process terms *with synchronization*:

$$\text{rec}\, X : <\tau, \mu>.<\tau, \gamma>.<b, \delta>.X \parallel_{\{b\}} \text{rec}\, Y : <a, \lambda>.<b, \delta>.Y$$

$$\text{rec}\, X : <\tau, \frac{\mu \cdot \gamma}{\mu + \gamma}>.<b, \delta>.X \parallel_{\{b\}} \text{rec}\, Y : <a, \lambda>.<b, \delta>.Y$$

- Resulting CTMC-level aggregation:





- Not exact due to the following steady-state probabilities $(\mu = \gamma = \lambda = \delta = 1)$:

2/13 1/13 1/13 2/10 1/10

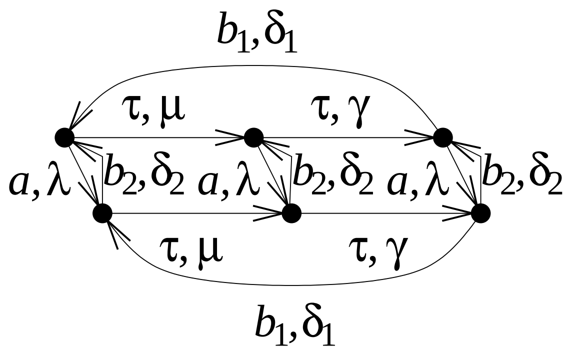
2/13 3/13 4/13 4/10 3/10

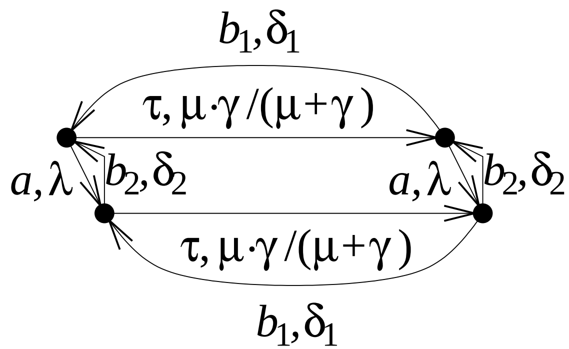
- **Example 7** – Consider the two process terms *without synchronization*:

$$\text{rec}\, X : <\tau, \mu>.<\tau, \gamma>.<b_1, \delta_1>.X \;\|_{\emptyset}\; \text{rec}\, Y : <a, \lambda>.<b_2, \delta_2>.Y$$

$$\text{rec}\, X : <\tau, \tfrac{\mu \cdot \gamma}{\mu + \gamma}>.<b_1, \delta_1>.X \;\|_{\emptyset}\; \text{rec}\, Y : <a, \lambda>.<b_2, \delta_2>.Y$$

- Resulting CTMC-level aggregation:





- Exact due to the following steady-state probabilities $(\mu = \gamma = \lambda = \delta_1 = \delta_2 = 1)$:

1/6 1/6 1/6 2/6 1/6

1/6 1/6 1/6 2/6 1/6

Related and Future Work

- Problem originally addressed in [Hil1994] through weak isomorphism: preserving average duration, notion of context.

- Congruence and steady-state exactness of weak isomorphism have been investigated, but no axiomatization is known.

- In [Bra2002] a variant of Markovian bisimilarity is defined that checks for exit rate equality with respect to all equivalence classes apart from the one including the process terms under examination.

- Congruence and axiomatization results have been provided, but nothing is said about exactness.

- We have considered a framework more useful than isomorphism, addressed not only sequences but also branches of exp. timed τ-actions, and investigated congruence-axiomatization-decidability-exactness.

- There exists a tradeoff between achieving compositionality also over concurrent processes and ensuring exactness at steady state for all the considered processes.

- Milner's weak bisimulation framework smoothly applies (up to completeness).

- We could have considered branching or dynamic bisimulation instead, but they are too demanding about matching exp. timed τ-transitions.

- Open issues:
  - Axiomatization of $\simeq_{\mathrm{MB,g}}$?
  - Algorithm for $\approx_{\mathrm{MB,g}}$ and $\simeq_{\mathrm{MB,g}}$?
  - Is $\approx_{\mathrm{MB,g}}$ (resp. $\simeq_{\mathrm{MB,g}}$) the coarsest congruence with respect to parallel composition contained in $\approx_{\mathrm{MB}}$ (resp. $\simeq_{\mathrm{MB}}$)?
  - Modal logic characterizations?