A Process Algebraic Theory of Reversible Concurrent Systems

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Concurrency: Nondeterminism vs. Irreversibility

- Systems composed of several interconnected computing parts that communicate by exchanging information or simply synchronizing.
- Models: shared memory, message passing, web services, . . .
- Types: centralized/distributed/decentralized, static/dynamic/mobile.
- Aspects: functionality, security, reliability, performance, . . .

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- Different advancing speeds, scheduling policies, ...
- What if the output does not uniquely define the input?
- Irreversibility: typical of functions that are *not invertible*.
- Example 1: conjunctions/disjunctions are irreversible.
- Example 2: negation is reversible.

Reversible Computing

- What does (ir)reversibility mean in computing?
- Well established concept in mathematics, physics, chemistry, biology: inverse relation/function/operation, formula/law/reaction, ...
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- Landauer principle states that any manipulation of information that is *irreversible* – i.e., causes information loss – such as:
  - erasure/overwriting of bits
  - merging of computation paths

  must be accompanied by a corresponding *entropy increase*.
- Minimal *heat generation* due to *extra work* for standardizing signals and making them independent of their history, so that it becomes *impossible to determine the input from the output*.

- Due to Landauer principle, the logical irreversibility of a function implies the physical irreversibility of computing that function and the consequent dissipative effects.
- Experimentally verified by Bérut et al in 2012 and revisited in terms of its physical foundations by Frank in 2018.
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- Every reversible computation, where no information is lost instead, may be potentially carried out without dissipating further heat.
- Lower energy consumption could therefore be achieved by resorting to reversible computing.
- There are many other applications of reversible computing:
  - Biochemical reaction modeling (nature).
  - Parallel discrete-event simulation (speedup).
  - Fault tolerant computing systems (rollback).
  - Robotics and control theory (backtrack).
  - Concurrent program debugging (reproducibility).
  - Distributed algorithms (deadlock, consensus).

- Two directions of computation characterize every reversible system:
  - Forward: coincides with the normal way of computing.
  - Backward: the effects of the forward one are undone (when needed).
- How to proceed backward? Same path as the forward direction?
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- Different notions of reversibility developed in different settings:
  - Causal reversibility is the capability of going back to a past state *consistently with the computational history* (easy for sequential ones, not trivial for concurrent and distributed systems) [DanosKrivine04].
  - Time reversibility refers to the conditions under which the stochastic behavior remains the same when the *direction of time* is reversed (quantitative models, efficient performance evaluation) [Kelly79].
  - Only recently the relationships between the two have been investigated (the former implies the latter in models based on Markov chains in certain circumstances).

Reversibility in Process Algebra

- There are no inverse process algebraic operators!
- The dynamic approach of [DanosKrivine04] yielding RCCS uses explicit stack-based memories attached to processes to record all executed actions and all discarded subprocesses.
- A single transition relation is defined, while actions are divided into forward and backward resulting in forward and backward transitions.

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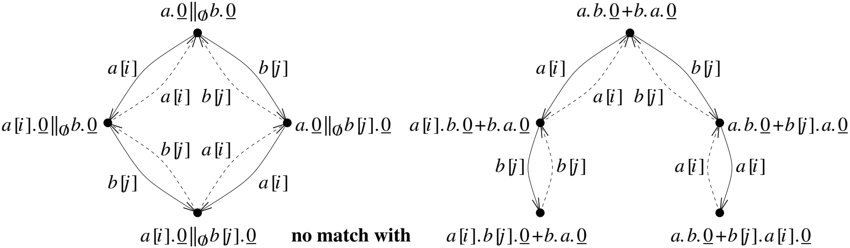
- The static approach of [PhillipsUlidowski07a] yielding CCSK is a method to reverse calculi by retaining within process syntax:
  - all executed actions, which are suitably decorated;
  - all dynamic operators, which are therefore treated as static.
- A forward transition relation and a backward transition relation are separately defined, labeled with communication keys so as to know who synchronized with whom when building backward transitions.

- In [PU07a] forward-reverse bisimilarity has been introduced too, which is truly concurrent as it does not satisfy the expansion law of parallel composition into a choice among all possible action sequencings ($a \neq b$):



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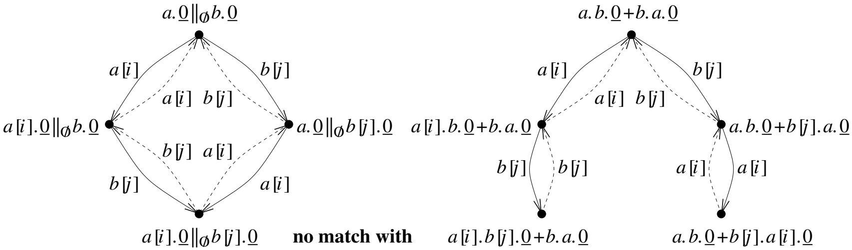

- With back-and-forth bisimilarity [DeNicolaMontanariVaandrager90] the interleaving view can be restored as this bisimilarity is defined on computations instead of states to preserve both causality and history (one transition relation, viewed as bidirectional, outgoing/incoming).

- What are the properties of bisimilarity over reversible processes?
- Minimal process calculus tailored for reversible processes to *comparatively* study congruence, axioms, and logics for:
  - Forward-reverse bisimilarity.
  - Forward-only bisimilarity.
  - Reverse-only bisimilarity.

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- Two different kinds of bisimilarities:
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  - Weak bisimilarities (abstraction from unobservable actions).
- Initially only sequential processes (i.e., no parallel composition) to be neutral with respect to interleaving view vs. true concurrency.
- Then add parallel composition and investigate expansion laws (relate sequential *specifications* to concurrent *implementations*).

Reversible Sequential Processes

- Usually only the future behavior of processes is described.
- We store the past behavior in the syntax like in [PU07a]:

$$P \ ::= \ \underline{0} \mid a \,.\, P \mid a^{\dagger} .\, P \mid P + P$$

- Countable set A of actions, including the unobservable action τ.
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- $a^{\dagger}.\,P$ executed action a, its forward continuation is inside P, and can undo a after all executed actions within P have been undone.
- A single transition relation like in [DMV90] labeled just with actions.
- Therefore there is no need of communication keys [PU07a], which allows for uniform action decorations like in [BoudolCastellani94].
- No need to distinguish between forward and backward actions or resort to stack-based memories [DK04].

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- Work with the set $\mathbb{P}$ of reachable processes:

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\mathit{reachable}(\underline{0}) & & \\
\mathit{reachable}(a\,.\,P) & \Longleftarrow & \mathit{initial}(P) \\
\mathit{reachable}(a^{\dagger}.\,P) & \Longleftarrow & \mathit{reachable}(P) \\
\mathit{reachable}(P_1 + P_2) & \Longleftarrow & (\mathit{reachable}(P_1) \wedge \mathit{initial}(P_2)) \vee \\
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\end{array}$$

- In $P_1 + P_2$ both subprocesses can be initial (at least one must be).
- Every initial or final process is reachable too ($\underline{0}$ is both).
- $\mathbb{P}$ also contains processes that are neither initial nor final: $a^{\dagger}.\,b\,.\,\underline{0}$.
- Past actions can never follow future actions: $b\,.\,a^{\dagger}.\,\underline{0} \notin \mathbb{P}$.

- Since all information needed to enable reversibility is in the syntax, action prefix and choice are made static by the semantics [PU07a].
- Semantics defined according to the structural operational approach: labeled transition system $(\mathbb{P}, A, \longrightarrow)$ where $\longrightarrow \subseteq \mathbb{P} \times A \times \mathbb{P}$.
- Single transition relation viewed as symmetric to meet loop property: *executed actions can be undone and undone actions can be redone* (necessary condition for any reasonable notion of reversibility).

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- Outgoing/incoming transitions for forward/reverse bisimilarity.
- Like in [DMV90] a transition $P \xrightarrow{a} P'$ goes:
  - forward if it is viewed as an outgoing transition of P, in which case action a is done;
  - backward if it is viewed as an incoming transition of P', in which case action a is undone.

- Semantic rules for action prefix:

$$\frac{\mathit{initial}(P)}{a \,.\, P \xrightarrow{a} a^{\dagger} \,.\, P} \qquad\qquad \frac{P \xrightarrow{b} P'}{a^{\dagger} \,.\, P \xrightarrow{b} a^{\dagger} \,.\, P'}$$

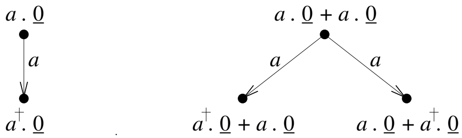
- The prefix related to the executed action is *not discarded*.
- It becomes a †-decorated part of the target process, necessary to offer again that action after rolling back.
- Additional rule for performing unexecuted actions that are preceded by already executed actions (direct consequence of making prefix static).
- This rule propagates actions executed by initial subprocesses.
- Can we view $a^{\dagger} \,.\, \_$ as the inverse operator of $a \,.\, \_$?

- Semantic rules for alternative composition:

$$\frac{P_1 \xrightarrow{a} P_1' \quad \mathit{initial}(P_2)}{P_1 + P_2 \xrightarrow{a} P_1' + P_2} \qquad \frac{P_2 \xrightarrow{a} P_2' \quad \mathit{initial}(P_1)}{P_1 + P_2 \xrightarrow{a} P_1 + P_2'}$$

- The subprocess not involved in the executed action is *not discarded* but cannot proceed further (only the non-initial subprocess can).
- It becomes part of the target process, which is necessary for offering again the original choice after undoing all the executed actions.
- If both subprocesses are initial, both rules apply (nondet. choice).
- If not, should operator $+$ become something like $+^{\dagger}$?
Not needed due to action decorations within either subprocess.

- The labeled transition system underlying an initial process is a *tree*, whose branching points correspond to occurrences of $+$:
  - Every non-final process has at least one outgoing transition.
  - Every non-initial process has exactly one incoming transition due to decorations associated with executed actions.
- Consider the two initial processes $a \,.\, \underline{0}$ and $a \,.\, \underline{0} + a \,.\, \underline{0}$:



- Single a-transition on the right in a forward-only process calculus.
- These two distinct processes should be considered equivalent though.

Bisimilarities for Reversible Nondeterministic Processes

- Bisimulation game: *outgoing* transitions for forward direction and *incoming* transitions for backward direction [DMV90].
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a:
  - Forward bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
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  - Reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
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    - for each $P'_1 \xrightarrow{a} P_1$ there exists $P'_2 \xrightarrow{a} P_2$ such that $(P'_1, P'_2) \in \mathcal{B}$.
- Largest such relations: $\sim_{\rm FB}$, $\sim_{\rm RB}$, $\sim_{\rm FRB}$.
- In order for $P_1, P_2 \in \mathbb{P}$ to be identified by $\sim_{\rm FB}/\sim_{\rm RB}$, the sets of actions labeling their outgoing/incoming transitions must coincide (forward/backward ready sets).

Discriminating Power

- $\sim_{\rm FRB} \subsetneq \sim_{\rm FB} \cap \sim_{\rm RB}$:
  - The inclusion is strict because the final processes $a^{\dagger} . \underline{0}$ and $a^{\dagger} . \underline{0} + c . \underline{0}$ are identified by $\sim_{\rm FB}$ and $\sim_{\rm RB}$, but distinguished by $\sim_{\rm FRB}$.
  - $\sim_{\rm FB}$ and $\sim_{\rm RB}$ are incomparable because $a^{\dagger} . \underline{0} \sim_{\rm FB} \underline{0}$ but $a^{\dagger} . \underline{0} \not\sim_{\rm RB} \underline{0}$ while $a . \underline{0} \sim_{\rm RB} \underline{0}$ but $a . \underline{0} \not\sim_{\rm FB} \underline{0}$.

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- First comparative remark ($\sim_{\mathrm{FB}}$ vs. $\sim_{\mathrm{RB}}$):
  - $\sim_{\mathrm{FRB}} = \sim_{\mathrm{FB}}$ over initial processes, with $\sim_{\mathrm{RB}}$ strictly coarser.
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- $a \, . \, \underline{0}$ and $a \, . \, \underline{0} + a \, . \, \underline{0}$ are identified by all three bisimilarities as witnessed by any bisimulation containing the pairs $(a \, . \, \underline{0}, a \, . \, \underline{0} + a \, . \, \underline{0})$, $(a^{\dagger} \, . \, \underline{0}, a^{\dagger} \, . \, \underline{0} + a \, . \, \underline{0})$, $(a^{\dagger} \, . \, \underline{0}, a \, . \, \underline{0} + a^{\dagger} \, . \, \underline{0})$.

Compositionality Properties

- $\sim_{\mathrm{FB}}$ equates processes with different past: $a_1^{\dagger} . \underline{0} \sim_{\mathrm{FB}} a_2^{\dagger} . \underline{0} \sim_{\mathrm{FB}} \underline{0}$.
- $\sim_{\mathrm{RB}}$ equates processes with different future: $a_1 . \underline{0} \sim_{\mathrm{RB}} a_2 . \underline{0} \sim_{\mathrm{RB}} \underline{0}$.

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  - $a^{\dagger} . b \,.\, \underline{0} \;\not\sim_{\rm RB}\; b \,.\, \underline{0}$ hence no such compositionality violation for $\sim_{\rm RB}$.
- $\sim_{\rm RB}$ and $\sim_{\rm FRB}$ never identify an initial process with a non-initial one, hence $\sim_{\rm FB}$ has to be made sensitive to the *presence of the past*.
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a past-sensitive forward bisimulation iff it is a forward bisimulation in which $\mathit{initial}(P_1) \iff \mathit{initial}(P_2)$ for all $(P_1, P_2) \in \mathcal{B}$. Largest such relation: $\sim_{\rm FB:ps}$.
- $a_1^{\dagger} \,.\, \underline{0} \sim_{\rm FB:ps} a_2^{\dagger} \,.\, \underline{0}$, but $a^{\dagger} \,.\, \underline{0} \not\sim_{\rm FB:ps} \underline{0}$ and $a^{\dagger} . b \,.\, \underline{0} \not\sim_{\rm FB:ps} b \,.\, \underline{0}$.

- Let $P_1, P_2 \in \mathbb{P}$ be s.t. $P_1 \sim P_2$ and take arbitrary $a \in A$ and $P \in \mathbb{P}$.
- All the considered bisimilarities are congruences w.r.t. action prefix:
  - $a\,.\,P_1 \sim a\,.\,P_2$ provided that *initial*$(P_1) \wedge$ *initial*(P_2).
  - $a^{\dagger}\,.\,P_1 \sim a^{\dagger}\,.\,P_2$.
- $\sim_{\mathrm{FB:ps}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$ are congruences w.r.t. alternative composition:
  - $P_1 + P \sim P_2 + P$ and $P + P_1 \sim P + P_2$ provided that *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\sim_{\mathrm{FB:ps}}$ is the coarsest congruence w.r.t. $+$ contained in $\sim_{\mathrm{FB}}$:
  - $P_1 \sim_{\mathrm{FB:ps}} P_2$ iff $P_1 + P \sim_{\mathrm{FB}} P_2 + P$ for all $P \in \mathbb{P}$ s.t. *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.

Equational Characterizations

- Deduction system $\vdash$ based on these axioms and inference rules on $\mathbb{P}$:
  - Reflexivity: $P = P$.
  - Symmetry: $\dfrac{P_1 = P_2}{P_2 = P_1}$.
  - Transitivity: $\dfrac{P_1 = P_2 \quad P_2 = P_3}{P_1 = P_3}$.
  - .-Substitutivity: $\dfrac{P_1 = P_2 \quad \textit{initial}(P_1) \wedge \textit{initial}(P_2)}{a \,.\, P_1 = a \,.\, P_2}$, $\dfrac{P_1 = P_2}{a^\dagger \,.\, P_1 = a^\dagger \,.\, P_2}$.
  - +-Substitutivity: $\dfrac{P_1 = P_2 \quad \textit{initial}(P) \vee (\textit{initial}(P_1) \wedge \textit{initial}(P_2))}{P_1 + P = P_2 + P \quad P + P_1 = P + P_2}$.
- Correspond to $\sim_{\mathrm{FB:ps}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$ being equivalence relations as well as congruences w.r.t. action prefix and alternative composition.
- $\vdash$ is sound and complete w.r.t. $\sim$ when $\vdash P_1 = P_2$ iff $P_1 \sim P_2$.

- Operator-specific axioms:

$(\mathcal{A}_1)$		$(P + Q) + R$	$=$	$P + (Q + R)$	
$(\mathcal{A}_2)$		$P + Q$	$=$	$Q + P$	
$(\mathcal{A}_3)$		$P + \underline{0}$	$=$	P	
$(\mathcal{A}_4)$	$[\sim_{\mathrm{FB:ps}}]$	$a^{\dagger} . P$	$=$	P	if $\neg initial(P)$
$(\mathcal{A}_5)$	$[\sim_{\mathrm{FB:ps}}]$	$a^{\dagger} . P$	$=$	$b^{\dagger} . P$	if $initial(P)$
$(\mathcal{A}_6)$	$[\sim_{\mathrm{FB:ps}}]$	$P + Q$	$=$	P	if $\neg initial(P)$, where $initial(Q)$
$(\mathcal{A}_7)$	$[\sim_{\mathrm{RB}}]$	$a . P$	$=$	P	where $initial(P)$
$(\mathcal{A}_8)$	$[\sim_{\mathrm{RB}}]$	$P + Q$	$=$	P	if $initial(Q)$
$(\mathcal{A}_9)$	$[\sim_{\mathrm{FB:ps}}]$	$P + P$	$=$	P	where $initial(P)$
$(\mathcal{A}_{10})$	$[\sim_{\mathrm{FRB}}]$	$P + Q$	$=$	P	if $initial(Q) \wedge to\_initial(P) = Q$

- $\mathcal{A}_8$ subsumes $\mathcal{A}_3$ (with $Q = \underline{0}$) and $\mathcal{A}_9$ (with $Q = P$).
- $\mathcal{A}_9$ and $\mathcal{A}_6$ apply in two different cases (P initial or not).
- $\mathcal{A}_{10}$ originally developed by Lanese and Phillips.
- $\vdash^{1,2,3}_{4,5,6,9}$ / $\vdash^{1,2}_{7,8}$ / $\vdash^{1,2,3}_{10}$ sound and complete for $\sim_{\mathrm{FB:ps}}$ / $\sim_{\mathrm{RB}}$ / $\sim_{\mathrm{FRB}}$.
- Third comparative remark: explicit vs. implicit idempotency.

Modal Logic Characterizations

- Hennessy-Milner logic extended with a backward modality (and init) from which suitable fragments are taken.
- Syntax:

$$\phi \ ::= \ \mathrm{true} \mid \mathrm{init} \mid \neg\phi \mid \phi \wedge \phi \mid \langle a \rangle \phi \mid \langle a^{\dagger} \rangle \phi$$

- Semantics:

$$\begin{array}{lcll}
P & \models & \mathrm{true} & \text{for all } P \in \mathbb{P} \\
P & \models & \mathrm{init} & \text{iff } \textit{initial}(P) \\
P & \models & \neg\phi & \text{iff } P \not\models \phi \\
P & \models & \phi_1 \wedge \phi_2 & \text{iff } P \models \phi_1 \text{ and } P \models \phi_2 \\
P & \models & \langle a \rangle \phi & \text{iff there exists } P \xrightarrow{a} P' \text{ such that } P' \models \phi \\
P & \models & \langle a^{\dagger} \rangle \phi & \text{iff there exists } P' \xrightarrow{a} P \text{ such that } P' \models \phi
\end{array}$$

- Fragments characterizing the four strong bisimilarities:

	true	init	$\neg$	$\wedge$	$\langle a \rangle$	$\langle a^\dagger \rangle$
$\mathcal{L}_{\mathrm{FB}}$	✓		✓	✓	✓	
$\mathcal{L}_{\mathrm{FB:ps}}$	✓	✓	✓	✓	✓	
$\mathcal{L}_{\mathrm{RB}}$	✓					✓
$\mathcal{L}_{\mathrm{FRB}}$	✓		✓	✓	✓	✓

- $\mathcal{L}_{\mathrm{FB}}$ / $\mathcal{L}_{\mathrm{FB:ps}}$ / $\mathcal{L}_{\mathrm{RB}}$ / $\mathcal{L}_{\mathrm{FRB}}$ characterizes $\sim_{\mathrm{FB}}$ / $\sim_{\mathrm{FB:ps}}$ / $\sim_{\mathrm{RB}}$ / $\sim_{\mathrm{FRB}}$:
  $P_1 \sim_B P_2$ iff $\forall \phi \in \mathcal{L}_B .\, P_1 \models \phi \iff P_2 \models \phi$.
- $\sim_{\mathrm{RB}}$ boils down to reverse trace equivalence!
- Every sequential process has at most one incoming transition in this setting with decorated actions.

Weak Bisimilarities

- Abstracting from τ-actions: $P \stackrel{\tau^*}{\Longrightarrow} P'$, $P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'$.
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a $(a \in A \setminus \{\tau\})$:
  - Weak forward bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$:
    - for each $P_1 \stackrel{\tau}{\longrightarrow} P_1'$ there exists $P_2 \stackrel{\tau^*}{\Longrightarrow} P_2'$ s.t. $(P_1', P_2') \in \mathcal{B}$;
    - for each $P_1 \stackrel{a}{\longrightarrow} P_1'$ there exists $P_2 \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2'$ s.t. $(P_1', P_2') \in \mathcal{B}$.

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    - for each $P_1 \stackrel{a}{\longrightarrow} P_1'$ there exists $P_2 \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2'$ s.t. $(P_1', P_2') \in \mathcal{B}$.
  - Weak reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - for each $P_1' \stackrel{\tau}{\longrightarrow} P_1$ there exists $P_2' \stackrel{\tau^*}{\Longrightarrow} P_2$ s.t. $(P_1', P_2') \in \mathcal{B}$;
    - for each $P_1' \stackrel{a}{\longrightarrow} P_1$ there exists $P_2' \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2$ s.t. $(P_1', P_2') \in \mathcal{B}$.

Weak Bisimilarities

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    - for each $P_1 \stackrel{\tau}{\longrightarrow} P'_1$ there exists $P_2 \stackrel{\tau^*}{\Longrightarrow} P'_2$ s.t. $(P'_1, P'_2) \in \mathcal{B}$;
    - for each $P_1 \stackrel{a}{\longrightarrow} P'_1$ there exists $P_2 \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'_2$ s.t. $(P'_1, P'_2) \in \mathcal{B}$.
  - Weak reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - for each $P'_1 \stackrel{\tau}{\longrightarrow} P_1$ there exists $P'_2 \stackrel{\tau^*}{\Longrightarrow} P_2$ s.t. $(P'_1, P'_2) \in \mathcal{B}$;
    - for each $P'_1 \stackrel{a}{\longrightarrow} P_1$ there exists $P'_2 \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2$ s.t. $(P'_1, P'_2) \in \mathcal{B}$.
  - Weak forward-reverse bisimulation iff for all $(P_1, P_2) \in \mathcal{B}$ and $a \in A$:
    - for each $P_1 \stackrel{\tau}{\longrightarrow} P'_1$ there exists $P_2 \stackrel{\tau^*}{\Longrightarrow} P'_2$ s.t. $(P'_1, P'_2) \in \mathcal{B}$;
    - for each $P_1 \stackrel{a}{\longrightarrow} P'_1$ there exists $P_2 \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P'_2$ s.t. $(P'_1, P'_2) \in \mathcal{B}$;
    - for each $P'_1 \stackrel{\tau}{\longrightarrow} P_1$ there exists $P'_2 \stackrel{\tau^*}{\Longrightarrow} P_2$ s.t. $(P'_1, P'_2) \in \mathcal{B}$;
    - for each $P'_1 \stackrel{a}{\longrightarrow} P_1$ there exists $P'_2 \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P_2$ s.t. $(P'_1, P'_2) \in \mathcal{B}$.
- Largest such relations: $\approx_{\mathrm{FB}}$, $\approx_{\mathrm{RB}}$, $\approx_{\mathrm{FRB}}$.

- Each weak bisimilarity is strictly coarser than its strong counterpart.
- $\approx_{\rm FRB} \subsetneq \approx_{\rm FB} \cap \approx_{\rm RB}$ with $\approx_{\rm FB}$ and $\approx_{\rm RB}$ being incomparable.
- $\approx_{\rm FRB} \neq \approx_{\rm FB}$ over initial processes:
  - $\tau \,.\, a \,.\, \underline{0} + a \,.\, \underline{0} + b \,.\, \underline{0}$ and $\tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0}$ are identified by $\approx_{\rm FB}$ but told apart by $\approx_{\rm FRB}$
    - Doing a on the left is matched by doing τ and then a on the right.
    - Undoing a on the right cannot be matched on the left.
  - $c \,.\, (\tau \,.\, a \,.\, \underline{0} + a \,.\, \underline{0} + b \,.\, \underline{0})$ and $c \,.\, (\tau \,.\, a \,.\, \underline{0} + b \,.\, \underline{0})$ is an analogous counterexample with non-initial τ-actions:
    - Doing c on one side is matched by doing c on the other side.
    - Doing a on the left is matched by doing τ and then a on the right.
    - Undoing a on the right cannot be matched on the left.

- Neither $\approx_{\mathrm{FB}}$ nor $\approx_{\mathrm{FRB}}$ is compositional:
  - $a^{\dagger} . b . \underline{0} \approx_{\mathrm{FB}} b . \underline{0}$ but $a^{\dagger} . b . \underline{0} + c . \underline{0} \not\approx_{\mathrm{FB}} b . \underline{0} + c . \underline{0}$ (same as $\sim_{\mathrm{FB}}$).
  - $\tau . a . \underline{0} \approx_{\mathrm{FB}} a . \underline{0}$ but $\tau . a . \underline{0} + b . \underline{0} \not\approx_{\mathrm{FB}} a . \underline{0} + b . \underline{0}$.
  - $\tau . a . \underline{0} \approx_{\mathrm{FRB}} a . \underline{0}$ but $\tau . a . \underline{0} + b . \underline{0} \not\approx_{\mathrm{FRB}} a . \underline{0} + b . \underline{0}$.
- The weak congruence construction à la Milner does not work here.

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  - $\tau . a . \underline{0} \approx_{\mathrm{FB}} a . \underline{0}$ but $\tau . a . \underline{0} + b . \underline{0} \not\approx_{\mathrm{FB}} a . \underline{0} + b . \underline{0}$.
  - $\tau . a . \underline{0} \approx_{\mathrm{FRB}} a . \underline{0}$ but $\tau . a . \underline{0} + b . \underline{0} \not\approx_{\mathrm{FRB}} a . \underline{0} + b . \underline{0}$.
- The weak congruence construction à la Milner does not work here.
- A symmetric relation $\mathcal{B}$ over $\mathbb{P}$ is a weak past-sensitive forward bisim. iff it is a weak forward bisim. in which *initial*$(P_1) \iff$ *initial*(P_2) for all $(P_1, P_2) \in \mathcal{B}$.
- A symm. rel. $\mathcal{B}$ over $\mathbb{P}$ is a weak past-sensitive forward-reverse bisim. iff it is a weak forward-reverse bisim. s.t. *initial*$(P_1) \iff$ *initial*(P_2) for all $(P_1, P_2) \in \mathcal{B}$.
- Largest such relations: $\approx_{\mathrm{FB:ps}}$, $\approx_{\mathrm{FRB:ps}}$.
- $\sim_{\mathrm{FRB}} \subsetneq \approx_{\mathrm{FRB:ps}}$ as the former satisfies the initiality condition.

- Let $P_1, P_2 \in \mathbb{P}$ be s.t. $P_1 \approx P_2$ and take arbitrary $a \in A$ and $P \in \mathbb{P}$.
- All the considered bisimilarities are congruences w.r.t. action prefix:
  - $a \,.\, P_1 \approx a \,.\, P_2$ provided that *initial*$(P_1) \wedge$ *initial*(P_2).
  - $a^{\dagger} \,.\, P_1 \approx a^{\dagger} \,.\, P_2$.
- $\approx_{\mathrm{FB:ps}}$, $\approx_{\mathrm{RB}}$, $\approx_{\mathrm{FRB:ps}}$ are congruences w.r.t. alternative composition:
  - $P_1 + P \approx P_2 + P$ and $P + P_1 \approx P + P_2$ provided that *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\approx_{\mathrm{FB:ps}}$ is the coarsest congruence w.r.t. $+$ contained in $\approx_{\mathrm{FB}}$:
  - $P_1 \approx_{\mathrm{FB:ps}} P_2$ iff $P_1 + P \approx_{\mathrm{FB}} P_2 + P$ for all $P \in \mathbb{P}$ s.t. *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.
- $\approx_{\mathrm{FRB:ps}}$ is the coarsest congruence w.r.t. $+$ contained in $\approx_{\mathrm{FRB}}$:
  - $P_1 \approx_{\mathrm{FRB:ps}} P_2$ iff $P_1 + P \approx_{\mathrm{FRB}} P_2 + P$ for all $P \in \mathbb{P}$ s.t. *initial*$(P) \vee ($*initial*$(P_1) \wedge$ *initial*$(P_2))$.

- Additional operator-specific axioms (τ-laws):

$(\mathcal{A}_1^\tau)$	$[\approx_{\mathrm{FB:ps}}]$	$a\,.\,\tau\,.\,P$	$=$	$a\,.\,P$	where $\mathit{initial}(P)$
$(\mathcal{A}_2^\tau)$	$[\approx_{\mathrm{FB:ps}}]$	$P + \tau\,.\,P$	$=$	$\tau\,.\,P$	where $\mathit{initial}(P)$
$(\mathcal{A}_3^\tau)$	$[\approx_{\mathrm{FB:ps}}]$	$a\,.\,(P + \tau\,.\,Q) + a\,.\,Q$	$=$	$a\,.\,(P + \tau\,.\,Q)$	where P, Q initial
$(\mathcal{A}_4^\tau)$	$[\approx_{\mathrm{FB:ps}}]$	$a^\dagger.\,\tau\,.\,P$	$=$	$a^\dagger.\,P$	where $\mathit{initial}(P)$
$(\mathcal{A}_5^\tau)$	$[\approx_{\mathrm{RB}}]$	$\tau^\dagger.\,P$	$=$	P	
$(\mathcal{A}_6^\tau)$	$[\approx_{\mathrm{FRB:ps}}]$	$a\,.\,(\tau\,.\,(P + Q) + P)$	$=$	$a\,.\,(P + Q)$	where P, Q initial
$(\mathcal{A}_7^\tau)$	$[\approx_{\mathrm{FRB:ps}}]$	$a^\dagger.\,(\tau\,.\,(P + Q) + P')$	$=$	$a^\dagger.\,(P' + Q)$	if $\mathit{to\_initial}(P') = P$, where P, Q initial
$(\mathcal{A}_8^\tau)$	$[\approx_{\mathrm{FRB:ps}}]$	$a^\dagger.\,(\tau^\dagger.\,(P' + Q) + P)$	$=$	$a^\dagger.\,(P' + Q)$	if $\mathit{to\_initial}(P') = P$, where $\mathit{initial}(P)$

- $\mathcal{A}_1^\tau$, $\mathcal{A}_2^\tau$, $\mathcal{A}_3^\tau$ are Milner τ-laws, $\mathcal{A}_4^\tau$ is needed for completeness.
- $\mathcal{A}_5^\tau$ is a variant of $\tau\,.\,P = P$ (not valid for weak bisim. congruence).
- $\mathcal{A}_6^\tau$ is Van Glabbeek-Weijland τ-law, $\mathcal{A}_7^\tau$ and $\mathcal{A}_8^\tau$ needed for complet.
- $\vdash^{1,2,3,4,5,6,9}_{1,2,3,4}$ / $\vdash^{1,2,7,8}_{5}$ / $\vdash^{1,2,3,10}_{6,7,8}$ is sound and complete for $\approx_{\mathrm{FB:ps}}$ / $\approx_{\mathrm{RB}}$ / $\approx_{\mathrm{FRB:ps}}$.
- $\approx_{\mathrm{FRB}}$ is branching bisimilarity over initial sequential processes!

- Modal logic with weak forward/backward modalities ($a \in A \setminus \{\tau\}$):

$$\phi \ ::= \ \text{true} \mid \text{init} \mid \neg\phi \mid \phi \wedge \phi \mid \langle\langle\tau\rangle\rangle\phi \mid \langle\langle a\rangle\rangle\phi \mid \langle\langle\tau^{\dagger}\rangle\rangle\phi \mid \langle\langle a^{\dagger}\rangle\rangle\phi$$

- Semantics:

$$\begin{array}{lll}
P & \models & \text{true} \quad \text{for all } P \in \mathbb{P} \\
P & \models & \text{init} \quad \text{iff } \textit{initial}(P) \\
P & \models & \neg\phi \quad \text{iff } P \not\models \phi \\
P & \models & \phi_1 \wedge \phi_2 \quad \text{iff } P \models \phi_1 \text{ and } P \models \phi_2 \\
P & \models & \langle\langle\tau\rangle\rangle\phi \quad \text{iff there exists } P \stackrel{\tau^*}{\Longrightarrow} P' \text{ such that } P' \models \phi \\
P & \models & \langle\langle a\rangle\rangle\phi \quad \text{iff there exists } P \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P' \text{ such that } P' \models \phi \\
P & \models & \langle\langle\tau^{\dagger}\rangle\rangle\phi \quad \text{iff there exists } P' \stackrel{\tau^*}{\Longrightarrow} P \text{ such that } P' \models \phi \\
P & \models & \langle\langle a^{\dagger}\rangle\rangle\phi \quad \text{iff there exists } P' \stackrel{\tau^*}{\Longrightarrow} \stackrel{a}{\longrightarrow} \stackrel{\tau^*}{\Longrightarrow} P \text{ such that } P' \models \phi
\end{array}$$

- Fragments characterizing the five weak bisimilarities:

	true	init	$\neg$	$\wedge$	$\langle\langle\tau\rangle\rangle$	$\langle\langle a\rangle\rangle$	$\langle\langle\tau^\dagger\rangle\rangle$	$\langle\langle a^\dagger\rangle\rangle$
$\mathcal{L}^\tau_{\mathrm{FB}}$	✓		✓	✓	✓	✓		
$\mathcal{L}^\tau_{\mathrm{FB:ps}}$	✓	✓	✓	✓	✓	✓		
$\mathcal{L}^\tau_{\mathrm{RB}}$	✓						✓	✓
$\mathcal{L}^\tau_{\mathrm{FRB}}$	✓		✓	✓	✓	✓	✓	✓
$\mathcal{L}^\tau_{\mathrm{FRB:ps}}$	✓	✓	✓	✓	✓	✓	✓	✓

- $\mathcal{L}^\tau_{\mathrm{FB}}$ / $\mathcal{L}^\tau_{\mathrm{FB:ps}}$ / $\mathcal{L}^\tau_{\mathrm{RB}}$ / $\mathcal{L}^\tau_{\mathrm{FRB}}$ / $\mathcal{L}^\tau_{\mathrm{FRB:ps}}$ characterizes $\approx_{\mathrm{FB}}$ / $\approx_{\mathrm{FB:ps}}$ / $\approx_{\mathrm{RB}}$ / $\approx_{\mathrm{FRB}}$ / $\approx_{\mathrm{FRB:ps}}$:

  $P_1 \approx_B P_2$ iff $\forall \phi \in \mathcal{L}^\tau_B .\, P_1 \models \phi \iff P_2 \models \phi$.

Expansion Laws for Reversible Concurrent Processes

- Fully-fledged process algebraic theory of reversible systems.
- Sequential specifications vs. concurrent implementations.
- Include parallel composition in the syntax:

$$P \ ::= \ \underline{0} \mid a \,.\, P \mid a^{\dagger} .\, P \mid P + P \mid P \parallel_L P$$

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- Include parallel composition in the syntax:

$$P \ ::= \ \underline{0} \mid a \,.\, P \mid a^{\dagger} .\, P \mid P + P \mid P \parallel_L P$$

- Additional operational semantic rules:

$$\frac{P_1 \xrightarrow{a} P_1' \quad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a} P_1' \parallel_L P_2} \qquad \frac{P_2 \xrightarrow{a} P_2' \quad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a} P_1 \parallel_L P_2'}$$

$$\frac{P_1 \xrightarrow{a} P_1' \quad P_2 \xrightarrow{a} P_2' \quad a \in L}{P_1 \parallel_L P_2 \xrightarrow{a} P_1' \parallel_L P_2'}$$

- The definitions of $\sim_{\mathrm{FB}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$ are unchanged.
- In forward-only process calculi $a \,.\, \underline{0} \parallel_{\emptyset} b \,.\, \underline{0}$ and $a \,.\, b \,.\, \underline{0} + b \,.\, a \,.\, \underline{0}$ are deemed equivalent: *the latter is the expansion of the former*.

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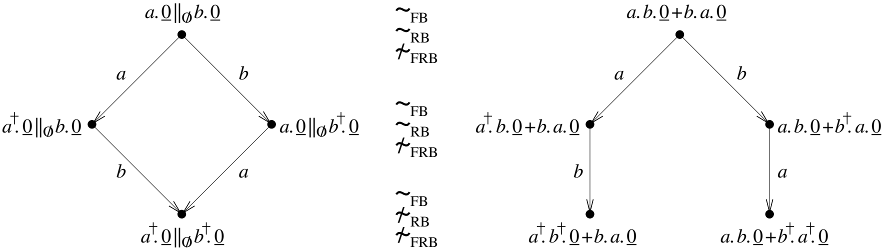
- In our reversible setting we obtain instead $(a \neq b)$:



- $\sim_{\mathrm{FB}}$ is interleaving, while $\sim_{\mathrm{RB}}$ and $\sim_{\mathrm{FRB}}$ are truly concurrent.
- What are the expansion laws for $\sim_{\mathrm{FB}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$?

- Expansion laws for forward-only calculi in the interleaving setting are used to identify $a \,.\, \underline{0} \,\|_{\emptyset}\, b \,.\, \underline{0}$ and $a \,.\, b \,.\, \underline{0} + b \,.\, a \,.\, \underline{0}$.

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  - Causal bisimilarity [DarondeauDegano90] (corresponding to history-preserving bisimilarity [RabinovichTrakhtenbrot88]): every action is enriched with the set of its causing actions each of which is expressed as a numeric backward pointer, hence we get $<a, \emptyset> \,.\, <b, \emptyset> \,.\, \underline{0} + <b, \emptyset> \,.\, <a, \emptyset> \,.\, \underline{0}$ and $<a, \emptyset> \,.\, <b, \{1\}> \,.\, \underline{0} + <b, \emptyset> \,.\, <a, \{1\}> \,.\, \underline{0}$.

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  - Location bisimilarity [BoudolCastellaniHennessyKiehn94]: every action is enriched with the name of the location in which it is executed, hence we get $<a, l_a> \,.\, <b, l_b> \,.\, \underline{0} + <b, l_b> \,.\, <a, l_a> \,.\, \underline{0}$ and $<a, l_a> \,.\, <b, l_a l_b> \,.\, \underline{0} + <b, l_b> \,.\, <a, l_b l_a> \,.\, \underline{0}$.

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  - Causal bisimilarity [DarondeauDegano90] (corresponding to history-preserving bisimilarity [RabinovichTrakhtenbrot88]): every action is enriched with the set of its causing actions each of which is expressed as a numeric backward pointer, hence we get $<a, \emptyset> . <b, \emptyset> . \underline{0} + <b, \emptyset> . <a, \emptyset> . \underline{0}$ and $<a, \emptyset> . <b, \{1\}> . \underline{0} + <b, \emptyset> . <a, \{1\}> . \underline{0}$.
  - Location bisimilarity [BoudolCastellaniHennessyKiehn94]: every action is enriched with the name of the location in which it is executed, hence we get $<a, l_a> . <b, l_b> . \underline{0} + <b, l_b> . <a, l_a> . \underline{0}$ and $<a, l_a> . <b, l_a l_b> . \underline{0} + <b, l_b> . <a, l_b l_a> . \underline{0}$.
  - Pomset bisimilarity [BoudolCastellani88a]: a prefix may contain the combination of actions that are independent of each other, hence the former process becomes $a \,.\, b \,.\, \underline{0} + b \,.\, a \,.\, \underline{0} + (a \parallel b) \,.\, \underline{0}$.

- How to uniformly derive expansion laws for $\sim_{\mathrm{FB}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$?
- Proved trees approach of [DeganoPriami92].
- Label every transition with a proof term [BoudolCastellani88b], which is an action preceded by the operators in the scope of which it occurs:

$$\theta \ ::= \ a \mid ._{a}\theta \mid \dot{+}\theta \mid +\!\!.\,\theta \mid \rfloor\!\rfloor_{L}\theta \mid \lfloor\!\lfloor_{L}\theta \mid \langle\theta,\theta\rangle_{L}$$

- The equivalence of interest then drives an observation function that maps proof terms to the required observations.

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- The equivalence of interest then drives an observation function that maps proof terms to the required observations.
- Interleaving: proof terms are reduced to the actions they contain.
- True concurrency: they are transformed into actions extended with suitable discriminating information (then encode processes accordingly).
- Information already available in the operational semantics for causal bisimilarity, location bisimilarity, pomset bisimilarity.
- Not available in the operational semantics for $\sim_{\rm RB}$ and $\sim_{\rm FRB}$!

- Proved operational semantic rules:

$$\frac{\mathit{initial}(P)}{a\,.\,P \xrightarrow{a} a^{\dagger}\,.\,P} \qquad \frac{P \xrightarrow{\theta} P'}{a^{\dagger}\,.\,P \xrightarrow{\cdot a\,\theta} a^{\dagger}\,.\,P'}$$

$$\frac{P_1 \xrightarrow{\theta} P_1' \quad \mathit{initial}(P_2)}{P_1 + P_2 \xrightarrow{{}_{\cdot}{+}\theta} P_1' + P_2} \qquad \frac{P_2 \xrightarrow{\theta} P_2' \quad \mathit{initial}(P_1)}{P_1 + P_2 \xrightarrow{{+}_{\cdot}\theta} P_1 + P_2'}$$

$$\frac{P_1 \xrightarrow{\theta} P_1' \quad \mathit{act}(\theta) \notin L}{P_1 \parallel_L P_2 \xrightarrow{\underline{\|}_L\theta} P_1' \parallel_L P_2} \qquad \frac{P_2 \xrightarrow{\theta} P_2' \quad \mathit{act}(\theta) \notin L}{P_1 \parallel_L P_2 \xrightarrow{\underline{\|}_L\theta} P_1 \parallel_L P_2'}$$

$$\frac{P_1 \xrightarrow{\theta_1} P_1' \quad P_2 \xrightarrow{\theta_2} P_2' \quad \mathit{act}(\theta_1) = \mathit{act}(\theta_2) \in L}{P_1 \parallel_L P_2 \xrightarrow{\langle \theta_1, \theta_2 \rangle_L} P_1' \parallel_L P_2'}$$

- Forward clause of bisimilarity rephrased as:
  - For each $P_1 \xrightarrow{\theta_1} P_1'$ there exists $P_2 \xrightarrow{\theta_2} P_2'$ such that $\mathit{act}(\theta_1) = \mathit{act}(\theta_2)$ and $(P_1', P_2') \in \mathcal{B}$.
- Backward clause of bisimilarity rephrased as:
  - For each $P_1' \xrightarrow{\theta_1} P_1$ there exists $P_2' \xrightarrow{\theta_2} P_2$ such that $\mathit{act}(\theta_1) = \mathit{act}(\theta_2)$ and $(P_1', P_2') \in \mathcal{B}$.

- Forward clause of bisimilarity rephrased as:
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- Observation function ℓ applied to proof terms labeling transitions, so that $\ell(\theta_1)$ and $\ell(\theta_2)$ are considered in the bisimulation game.
- May depend on other possible parameters that are present in the proved labeled transition system.
- Must preserve actions: $\ell(\theta_1) = \ell(\theta_2)$ implies $\mathit{act}(\theta_1) = \mathit{act}(\theta_2)$.
- $\sim_{\mathrm{FB:ps:}\ell_{\mathrm{F}}}$, $\sim_{\mathrm{RB:}\ell_{\mathrm{R}}}$, $\sim_{\mathrm{FRB:}\ell_{\mathrm{FR}}}$ are the three resulting equivalences.
- When do they coincide with the congruences $\sim_{\mathrm{FB:ps}}$, $\sim_{\mathrm{RB}}$, $\sim_{\mathrm{FRB}}$?
- What is the discriminating information needed by $\sim_{\mathrm{RB}}$ and $\sim_{\mathrm{FRB}}$?

- $\sim_{\rm FB:ps:\ell_F} = \sim_{\rm FB:ps}$ when $\ell_{\rm F}(\theta) = \mathit{act}(\theta)$.
- Axiomatization of $\sim_{\rm FB:ps}$ over reversible concurrent processes:

$$
\begin{array}{lrcll}
(\mathcal{A}_{\rm F,1}) & (P + Q) + R & = & P + (Q + R) & \\
(\mathcal{A}_{\rm F,2}) & P + Q & = & Q + P & \\
(\mathcal{A}_{\rm F,3}) & P + \underline{0} & = & P & \\
(\mathcal{A}_{\rm F,4}) & P + P & = & P & \text{where } \mathit{initial}(P) \\
(\mathcal{A}_{\rm F,5}) & a^{\dagger} \,.\, P & = & P & \text{if } \neg \mathit{initial}(P) \\
(\mathcal{A}_{\rm F,6}) & a^{\dagger} \,.\, P & = & b^{\dagger} \,.\, P & \text{if } \mathit{initial}(P) \\
(\mathcal{A}_{\rm F,7}) & P + Q & = & P & \text{if } \neg \mathit{initial}(P), \text{ where } \mathit{initial}(Q) \\
(\mathcal{A}_{\rm F,8}) & P_1 \parallel_L P_2 & = & [a^{\dagger}.] \Bigg(\sum\limits_{i \in I_1, a_{1,i} \notin L} a_{1,i} \,.\, (P_{1,i} \parallel_L P'_2) \; + & \\
 & & & \quad \sum\limits_{i \in I_2, a_{2,i} \notin L} a_{2,i} \,.\, (P'_1 \parallel_L P_{2,i}) \; + & \\
 & & & \quad \sum\limits_{i \in I_1, a_{1,i} \in L} \; \sum\limits_{j \in I_2, a_{2,j} = a_{1,i}} a_{1,i} \,.\, (P_{1,i} \parallel_L P_{2,j}) \Bigg) &
\end{array}
$$

- $P_k = [a_k^{\dagger}.]P'_k$ with $P'_k = \sum\limits_{i \in I_k} a_{k,i} \,.\, P_{k,i}$ for $k \in \{1, 2\}$, called F-nf.
- $[a^{\dagger}.]$ is present iff $[a_1^{\dagger}.]$ or $[a_2^{\dagger}.]$ is present (they are optional).

- $\sim_{\mathrm{RB}:\ell_{\mathrm{R}}} = \sim_{\mathrm{RB}}$ and $\sim_{\mathrm{FRB}:\ell_{\mathrm{FR}}} = \sim_{\mathrm{FRB}}$ when $\ell_{\mathrm{R}}(\theta)_{P'} = \ell_{\mathrm{FR}}(\theta)_{P'} = <\mathit{act}(\theta), \mathit{brs}(P')> \triangleq \ell_{\mathrm{brs}}(\theta)_{P'}$ for every proved transition $P \xrightarrow{\theta} P'$.
- $\mathit{brs}(P')$ is the backward ready set of P', the set of actions labeling the incoming transitions of P'.
- Thus $a\,.\,\underline{0} \,\|_{\emptyset}\, b\,.\,\underline{0}$ is encoded as:

  $<a, \{a\}>\,.<b, \{a, b\}>\,.\,\underline{0} + <b, \{b\}>\,.<a, \{a, b\}>\,.\,\underline{0}$

  while $a\,.\,b\,.\,\underline{0} + b\,.\,a\,.\,\underline{0}$ is encoded as:

  $<a, \{a\}>\,.<b, \{b\}>\,.\,\underline{0} + <b, \{b\}>\,.<a, \{a\}>\,.\,\underline{0}$

- $\sim_{\mathrm{RB}:\ell_{\mathrm{R}}} = \sim_{\mathrm{RB}}$ and $\sim_{\mathrm{FRB}:\ell_{\mathrm{FR}}} = \sim_{\mathrm{FRB}}$ when $\ell_{\mathrm{R}}(\theta)_{P'} = \ell_{\mathrm{FR}}(\theta)_{P'} = \, <\!\mathit{act}(\theta), \mathit{brs}(P')\!> \, \triangleq \ell_{\mathrm{brs}}(\theta)_{P'}$ for every proved transition $P \xrightarrow{\theta} P'$.
- $\mathit{brs}(P')$ is the backward ready set of P', the set of actions labeling the incoming transitions of P'.
- Thus $a \,.\, \underline{0} \parallel_{\emptyset} b \,.\, \underline{0}$ is encoded as:

  $<\!a, \{a\}\!> . <\!b, \{a, b\}\!> . \underline{0} + <\!b, \{b\}\!> . <\!a, \{a, b\}\!> . \underline{0}$

  while $a \,.\, b \,.\, \underline{0} + b \,.\, a \,.\, \underline{0}$ is encoded as:

  $<\!a, \{a\}\!> . <\!b, \{b\}\!> . \underline{0} + <\!b, \{b\}\!> . <\!a, \{a\}\!> . \underline{0}$

- The encoding of $a^{\dagger} . \underline{0} \parallel_{\emptyset} b^{\dagger} . \underline{0}$ (a case not addressed in [DP92]) *cannot be*:

  $<\!a^{\dagger}, \{a\}\!> . <\!b^{\dagger}, \{a, b\}\!> . \underline{0} + <\!b^{\dagger}, \{b\}\!> . <\!a^{\dagger}, \{a, b\}\!> . \underline{0}$

- It is $<\!a^{\dagger}, \{a\}\!> . <\!b^{\dagger}, \{a, b\}\!> . \underline{0} + <\!b, \{b\}\!> . <\!a, \{a, b\}\!> . \underline{0}$ or $<\!a, \{a\}\!> . <\!b, \{a, b\}\!> . \underline{0} + <\!b^{\dagger}, \{b\}\!> . <\!a^{\dagger}, \{a, b\}\!> . \underline{0}$ depending on whether trace $a\,b$ or trace $b\,a$ has been executed (initial subprocesses are needed by the forward-reverse semantics).

- Let $\widetilde{P}$ be the $\ell_{\rm brs}$-encoding of P.
- Axiomatization of $\sim_{\rm RB}$ over reversible concurrent processes:

$$
\begin{array}{llcll}
(\mathcal{A}_{\rm R,1}) & \widetilde{(P+Q)+R} & = & \widetilde{P+(Q+R)} & \\
(\mathcal{A}_{\rm R,2}) & \widetilde{P+Q} & = & \widetilde{Q+P} & \\
(\mathcal{A}_{\rm R,3}) & \widetilde{a\,.\,P} & = & \widetilde{P} & \text{where } \mathit{initial}(P) \\
(\mathcal{A}_{\rm R,4}) & \widetilde{P+Q} & = & \widetilde{P} & \text{if } \mathit{initial}(Q) \\
(\mathcal{A}_{\rm R,5}) & \widetilde{P_1 \parallel_L P_2} & = & e\ell^{\varepsilon}_{\rm brs}(\widetilde{P}_1, \widetilde{P}_2, L)_{P_1 \parallel_L P_2} &
\end{array}
$$

- $P_k = \underline{0}$ or $P_k = a^{\dagger}\,.\,P'_k$ for $k \in \{1, 2\}$, called R-nf.

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(\mathcal{A}_{\rm R,5}) & \widetilde{P_1 \parallel_L P_2} & = & e\ell^{\varepsilon}_{\rm brs}(\widetilde{P_1}, \widetilde{P_2}, L)_{P_1 \parallel_L P_2} &
\end{array}
$$

- $P_k = \underline{0}$ or $P_k = a^{\dagger}\,.\,P'_k$ for $k \in \{1, 2\}$, called R-nf.
- Axiomatization of $\sim_{\rm FRB}$ over reversible concurrent processes:

$$
\begin{array}{llcll}
(\mathcal{A}_{\rm FR,1}) & \widetilde{(P+Q)+R} & = & \widetilde{P+(Q+R)} & \\
(\mathcal{A}_{\rm FR,2}) & \widetilde{P+Q} & = & \widetilde{Q+P} & \\
(\mathcal{A}_{\rm FR,3}) & \widetilde{P+\underline{0}} & = & \widetilde{P} & \\
(\mathcal{A}_{\rm FR,4}) & \widetilde{P+Q} & = & \widetilde{P} & \text{if } \mathit{initial}(Q) \wedge \mathit{to\_initial}(P) = Q \\
(\mathcal{A}_{\rm FR,5}) & \widetilde{P_1 \parallel_L P_2} & = & e\ell^{\varepsilon}_{\rm brs}(\widetilde{P_1}, \widetilde{P_2}, L)_{P_1 \parallel_L P_2} &
\end{array}
$$

- $P_k = [a^{\dagger}\,.\,P'_k +] \sum_{i \in I_k} a_{k,i}\,.\,P_{k,i}$ for $k \in \{1, 2\}$, called FR-nf.

- How close is $\sim_{\mathrm{FRB}}$ to hereditary history-preserving bisimilarity?
- Two stable configuration structures $\mathsf{C}_i = (\mathcal{E}_i, \mathcal{C}_i, l_i)$, $i \in \{1, 2\}$, are hereditary history-preserving bisimilar, written $\mathsf{C}_1 \sim_{\mathrm{HHPB}} \mathsf{C}_2$, iff there exists a hereditary history-preserving bisimulation between C_1 and C_2, i.e., a relation $\mathcal{B} \subseteq \mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{P}(\mathcal{E}_1 \times \mathcal{E}_2)$ such that:
  - $(\emptyset, \emptyset, \emptyset) \in \mathcal{B}$.
  - Whenever $(X_1, X_2, f) \in \mathcal{B}$, then:
    - f is a *bijection* from X_1 to X_2 that preserves *labeling*, i.e., $l_1(e) = l_2(f(e))$ for all $e \in X_1$, and *causality*, i.e., $e \leq_{X_1} e' \iff f(e) \leq_{X_2} f(e')$ for all $e, e' \in X_1$.
    - For all $a \in A$ it holds that: For each $X_1 \xrightarrow{a}_{\mathsf{C}_1} X_1'$ there exist $X_2 \xrightarrow{a}_{\mathsf{C}_2} X_2'$ and f' such that $(X_1', X_2', f') \in \mathcal{B}$ and $f' \upharpoonright X_1 = f$, and vice versa. For each $X_1' \xrightarrow{a}_{\mathsf{C}_1} X_1$ there exist $X_2' \xrightarrow{a}_{\mathsf{C}_2} X_2$ and f' such that $(X_1', X_2', f') \in \mathcal{B}$ and $f \upharpoonright X_1' = f'$, and vice versa.

- $\sim_{\mathrm{HHPB}}$ [Bednarczyk91] is the finest truly concurrent equivalence preserved under action refinement that is capable of respecting causality, branching, and their interplay while abstracting from choices between identical alternatives [VanGlabbeekGoltz01].
- $\sim_{\mathrm{FRB}}$ coincides with $\sim_{\mathrm{HHPB}}$ in the absence of autoconcurrency [PhillipsUlidowski07b].

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- $\sim_{\mathrm{FRB}}$ coincides with $\sim_{\mathrm{HHPB}}$ in the absence of autoconcurrency [PhillipsUlidowski07b].
- Autoconcurrency is $a\,.\,\underline{0} \parallel_{\emptyset} a\,.\,\underline{0}$, while $a\,.\,a\,.\,\underline{0}$ is autocausation.
- $a\,.\,\underline{0} \parallel_{\emptyset} a\,.\,\underline{0} \;\sim_{\mathrm{FRB}}\; a\,.\,a\,.\,\underline{0} + a\,.\,a\,.\,\underline{0} \;\sim_{\mathrm{FRB}}\; a\,.\,a\,.\,\underline{0}$.
- Their ℓ_{brs}-encodings are basically the same:

$$<a, \{a\}> . <a, \{a, a\}> . \underline{0} + <a, \{a\}> . <a, \{a, a\}> . \underline{0}$$
$$<a, \{a\}> . <a, \{a\}> . \underline{0} + <a, \{a\}> . <a, \{a\}> . \underline{0}$$
$$<a, \{a\}> . <a, \{a\}> . \underline{0}$$

- Denotational semantics for $\mathbb{P}$ based on configuration structures in which events are proof terms.
- $a \,.\, \underline{0} \parallel_{\emptyset} a \,.\, \underline{0} \not\sim_{\mathrm{HHPB}} a \,.\, a \,.\, \underline{0}$ on their corresponding structures because events $\lfloor\!\lfloor_{\emptyset} a$ and $\rfloor\!\rfloor_{\emptyset} a$ are independent of each other while events a and $._{a} a$ are causally related, hence no bijection exists between the former and the latter that preserves causality.

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- Backward ready multisets distinguish them: $\sim_{\mathrm{FRB:brm}} = \sim_{\mathrm{HHPB}}$.
- $\sim_{\mathrm{FRB:brm}}$ *counts* the incoming a-transitions of related configurations, no bijection between identically labeled events [AubertCristescu20].

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- $\sim_{\mathrm{FRB:brm}}$ *counts* the incoming a-transitions of related configurations, no bijection between identically labeled events [AubertCristescu20].
- $\sim_{\mathrm{FRB:brm}}$ over $\mathbb{P}$ is an operational representation of $\sim_{\mathrm{HHPB}}$.
- The ℓ_{brm}-encoding of $a \,.\, \underline{0} \parallel_{\emptyset} a \,.\, \underline{0}$:

  $$<a, \{\!| a |\!\}> . <a, \{\!| a, a |\!\}> . \underline{0} + <a, \{\!| a |\!\}> . <a, \{\!| a, a |\!\}> . \underline{0}$$

  differs from its ℓ_{brs}-encoding:

  $$<a, \{a\}> . <a, \{a, a\}> . \underline{0} + <a, \{a\}> . <a, \{a, a\}> . \underline{0}$$
- Cross fertilization for their equational and logical characterizations.

Concluding Remarks and Future Work

- Our process algebraic theory encompasses most of concurrency theory:
  - Forward bisimilarity is the usual bisimilarity.
  - Reverse bisimilarity boils down to reverse trace equivalence on $\mathbb{P}_{\mathrm{seq}}$.
  - Weak forward-reverse bisimilarity is branching bisimilarity on $\mathbb{P}_{\mathrm{seq}}$.
  - Connection with hereditary history-preserving bisimilarity on $\mathbb{P}$.
  - Expansion laws addressing interleaving semantics or true concurrency.
- Applied to noninterference analysis of reversible systems (branching bisim.).
- Extended [PU07a] to stochastically timed processes in the strong case, link with ordinary/exact/strict lumpability as well as time reversibility.
- Causal reversibility of deterministic timed processes (time additivity) and probabilistic processes (alternation with nondeterminism).
- Stochastically timed processes in the weak case (W-lumpability)?
- When does time reversibility imply causal reversibility?
- What changes when admitting irreversible actions (commit)?
- Unitary transformations in quantum computing are reversible!

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