Generalized Proportional Lumpability

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VALUETOOLS – Milano, December 2024

Meeting NiRvAna – Urbino, 30 Maggio 2025

The context

- In performance evaluation of complex systems, continuous-time Markov chains (CTMCs) are fundamental models
- The main goal is to calculate stationary performance metrics like throughput, response time, and resource utilization
- This requires determining the stationary probability distribution of the CTMC

Background

Aggregation techniques

- High-level modeling formalisms often result in large state spaces, making their analysis challenging or infeasible
- State space explosion can be addressed by aggregating states with equivalent behaviors
- Lumpability enables efficient computation of performance indices for Markov chains with structural regularity
- However, lumpability is limited, as few real-world applications exhibit non-trivial lumpability

Previous work

Proportional lumpability

- Proportional lumpability[1] has been introduced to broaden the applicability of traditional lumpability
- It allows for the exact computation of stationary performance indices, unlike quasi-lumpability, which provides only bounds
- It is based on a perturbation of the Markov chain's transition rates using a proportionality function

[1] A. Marin, C. Piazza, S. Rossi: "Proportional lumpability and proportional bisimilarity". Acta Informatica 2022

Present work

Contributions

- We reformulate proportional lumpability in terms of matrix multiplications
- We introduce two matrix-based perturbation methods for Markov chains: left-perturbations and right-perturbations
- We provide a characterization for a class of left-perturbations and for a class of right-perturbations
- We generalize the notion of proportional lumpability by incorporating left- and right-perturbations

Background

Ergodic CTMC

- We refer to a finite ergodic CTMC by its infinitesimal generators $\mathbf{Q}$
- A square matrix is the infinitesimal generator of a finite ergodic CTMC, if
  - All off-diagonal elements are non-negative
  - The sum of each row is zero
  - It serves as the adjacency matrix of a strongly connected weighted directed graph

Ordinary lumpability

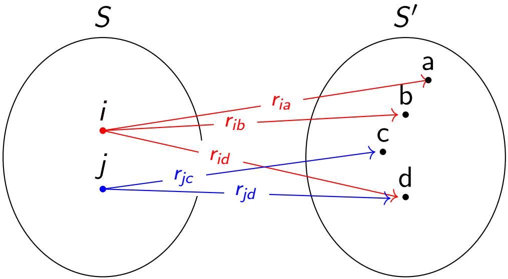
Intuition

- Lumpability is defined using equivalence relations that partition the state space of a Markov chain
- Aggregation groups equivalent states into macro-states, reducing the state space size
- If the partition satisfies ordinary lumpability criteria, the equilibrium solution of the aggregated process can provide an exact solution for the original process

Ordinary lumpability

Intuition

- An equivalence relation exhibits ordinary lumpability if it induces a partition into equivalence classes such that any two states within the same class have identical aggregated transition rates to any other class.



$$r_{ia} + r_{ib} + r_{id} = r_{jc} + r_{jd}$$

Ordinary lumpability

Notation

- $\sim$ is an equivalence relation over the state space

Original CTMC	Aggregated CTMC
$\mathbf{Q}$	$\widetilde{\mathbf{Q}}$
$q(i,j)$	$\widetilde{q}(S,S')$
$\boldsymbol{\pi}$	$\widetilde{\pi}$

- for any equivalence class S,

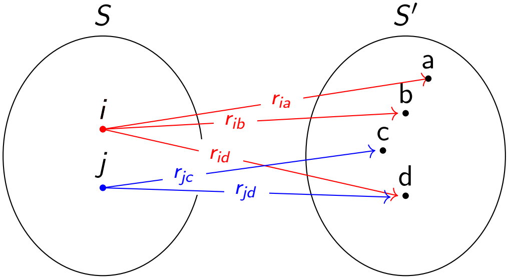
$$q(i,S) = \sum_{k \in S} q(i,k)$$

Ordinary lumpability

Definition

$\sim$ is an *ordinary lumpability* for $\mathbf{Q}$ if

$$q(i, S') = q(j, S')$$



$$r_{ia} + r_{ib} + r_{id} = r_{jc} + r_{jd}$$

Aggregated CTMC for ordinary lumpability

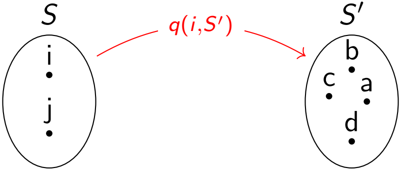
Definition

- $\widetilde{\mathbf{Q}}$ is the aggregated CTMC such that

$$\widetilde{q}(S, S') = q(i, S')$$

- $\widetilde{\boldsymbol{\pi}}$ is the equilibrium distribution of $\widetilde{\mathbf{Q}}$ such that

$$\widetilde{\pi}(S) = \sum_{i \in S} \pi(i)$$



Matrices associated to $\sim$

Notation

- N_S is the number of states
- N_C is the number of equivalence classes

Definition

The matrices V and U associated to $\sim$ are:

- V is the $N_S \times N_C$ matrix such that

$$v(s, S) = 1 \text{ iff } s \in S.$$

- U is the $N_C \times N_S$ matrix such that

$$u(S, s) = 1/|S| \text{ iff } s \in S.$$

Matrix-based characterization of ordinary lumpability

Definition of ordinary lumpability

The relation $\sim$ is an oridinary lumpability for $\mathbf{Q}$ iff

$$\mathbf{Q}V = VU\mathbf{Q}V\,.$$

Aggregated CTMC

The aggregated CTMC $\widetilde{\mathbf{Q}}$ is

$$\widetilde{\mathbf{Q}} = U\mathbf{Q}V\,.$$

Proportional lumpability

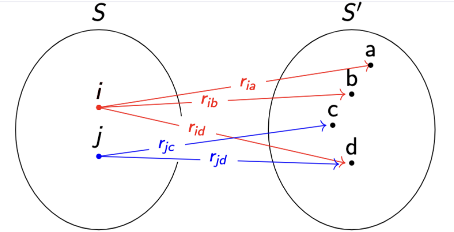
Definition

$\sim$ is an *proportional lumpability* for $\mathbf{Q}$ if there exists

$$\kappa : \mathit{States} \rightarrow \mathbb{R}^+$$

such that

$$\frac{q(i, S')}{\kappa(i)} = \frac{q(j, S')}{\kappa(j)}.$$



$$\frac{r_{ia} + r_{ib} + r_{id}}{\kappa(i)} = \frac{r_{jc} + r_{jd}}{\kappa(j)}$$

Aggregated CTMC for proportional lumpability

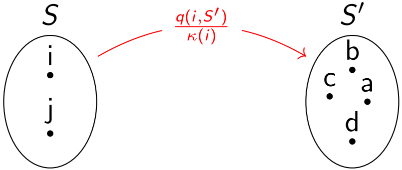
Definition

- $\widetilde{\mathbf{Q}}$ is the aggregated CTMC w.r.t. $\kappa : States \to \mathbb{R}^+$ such that

$$\widetilde{q}(S, S') = \frac{q(i, S')}{\kappa(i)} .$$

- $\widetilde{\boldsymbol{\pi}}$ is the equilibrium distribution of $\widetilde{\mathbf{Q}}$ such that

$$\widetilde{\pi}(S) = \sum_{i \in S} \pi(i)\kappa(i) .$$



Perturbed Markov chain for proportional lumpability

Perturbed Markov chain

$\mathbf{Q}'$ is a perturbation of $\mathbf{Q}$ w.r.t. $\kappa : \mathit{States} \to \mathbb{R}^+$ if for all i, j,

$$q'(i,j) = \frac{q(i,j)}{\kappa(i)}\,.$$

Equilibrium distribution of the original chain

$\mathbf{Q}'$ If $\boldsymbol{\pi}'$ is the equilibrium distribution of $\mathbf{Q}'$ then the equilibrium distribution $\boldsymbol{\pi}$ of $\mathbf{Q}$ is

$$\pi(i) = \frac{\pi'(i)}{\kappa(i)}\,.$$

Matrix-based characterization of proportional lumpability

Definition of proportional lumpability

- V and U are the matrices associated to $\sim$
- κ: $States \to \mathbb{R}^+$
- K diagonal matrix with $\kappa(i,i) = 1/\kappa(i)$

The relation $\sim$ is a proportional lumpability w.r.t. κ for $\mathbf{Q}$ iff

$$K\mathbf{Q}V = VUK\mathbf{Q}V\,.$$

Aggregated CTMC

The aggregated CTMC $\widetilde{\mathbf{Q}}$ is

$$\widetilde{\mathbf{Q}} = UK\mathbf{Q}V\,.$$

Matrix-based characterization of proportional lumpability

Results

Original CTMC	Perturbed CTMC
$\mathbf{Q}$	$K\mathbf{Q}$
$\sim$ prop. lump. w.r.t. κ	$\sim$ ordinary lump.
$\boldsymbol{\pi}$	$\boldsymbol{\pi}'$
$\boldsymbol{\pi}'K$	$\boldsymbol{\pi}K^{-1}$

- $\boldsymbol{\pi}'K\mathbf{Q} = \mathbf{0}$ implies $\boldsymbol{\pi}'K$ is an invariant measure for $\mathbf{Q}$.
- $\boldsymbol{\pi}\mathbf{Q} = \mathbf{0}$ implies $\boldsymbol{\pi}K^{-1}K\mathbf{Q} = \mathbf{0}$ implies $\boldsymbol{\pi}K^{-1}$ is an invariant measure for $K\mathbf{Q}$

Generalizing proportional lumpability

Idea

Original CTMC	Perturbed CTMC
$\mathbf{Q}$ $\sim$ prop. lump. w.r.t. L $\boldsymbol{\pi}$	$L\mathbf{Q}$ $\sim$ ordinary lump. $\boldsymbol{\pi}'$
$\mathbf{Q}$ $\sim$ prop. lump. w.r.t. R $\boldsymbol{\pi}$	$\mathbf{Q}R$ $\sim$ ordinary lump. $\boldsymbol{\pi}'$

Perturbed Markov chains

Definition

$\mathbf{Q}'$ is a *perturbation* of $\mathbf{Q}$
iff

1. All off-diagonal elements are non-negative
2. The sum of each row is zero
3. For all $s \neq s'$, $q(s, s') \neq 0$ iff $q'(s, s') \neq 0$

A class of left-perturbed Markov chains

Definition

The matrix $\mathbf{Q}' = L\mathbf{Q}$ is a *left-perturbation of* $\mathbf{Q}$ if

1. All the off-diagonal elements of $\mathbf{Q}'$ are non-negative.
2. The sum of each row of $\mathbf{Q}'$ is zero.
3. For all $s \neq s'$, $q(s,s') = 0$ iff $q'(s,s') = 0$.

A class of left-perturbed Markov chains

Theorem

Let $\mathbf{Q}$ be a CTMC and L be such that

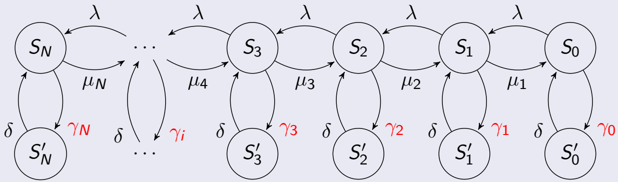
1. All the diagonal elements of L are positive.
2. All the off-diagonal elements of L are non-positive.
3. For all $s \neq s'$, if $q(s, s') = 0$, then $\ell(s, s') = 0$.
4. For all $s \neq s'$, if there exists $s'' \neq s, s'$ such that $q(s', s'') \neq 0$, then $\ell(s, s') = 0$.

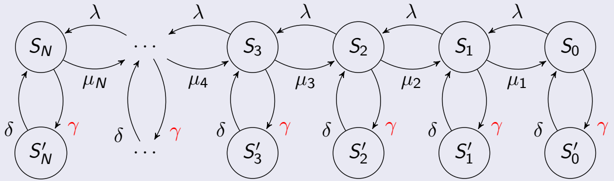
The matrix $\mathbf{Q}' = L\mathbf{Q}$ is a left perturbation of $\mathbf{Q}$ with respect to L.

Basically, if s reaches s' and s' reaches another state different from s, then $\ell(s, s') = 0$. This implies that the columns in L of the states that have more than one outgoing edge have all off-diagonal elements zero.

Left-perturbed Markov chains

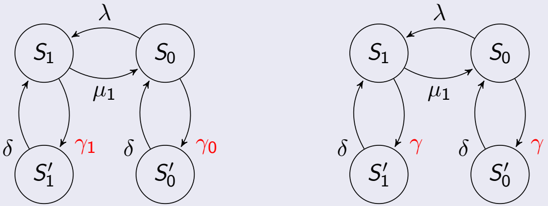
Example





Left-perturbed Markov chains

Example



$$\mathbf{Q} = \begin{pmatrix} -(\lambda+\gamma_0) & \lambda & \gamma_0 & 0 \\ \mu_1 & -\mu_1 + \gamma_1 & 0 & \gamma_1 \\ \delta & 0 & -\delta & 0 \\ 0 & \delta & 0 & -\delta \end{pmatrix} \quad L\mathbf{Q} = \begin{pmatrix} 1 & 0 & (\gamma_0 - \gamma)/\delta & 0 \\ 0 & 1 & 0 & (\gamma_1 - \gamma)/\delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\mathbf{Q}$$

Steady state distribution of left-perturbed chains

Result

Original CTMC	Perturbed CTMC
$\mathbf{Q}$	$L\mathbf{Q}$
$\boldsymbol{\pi} L$	$\boldsymbol{\pi}$

A class of right-perturbed Markov chains

Definition

The matrix $\mathbf{Q}' = \mathbf{Q}R$ is a *right-perturbation of* $\mathbf{Q}$ if

1. All the off-diagonal elements of $\mathbf{Q}'$ are non-negative.
2. The sum of each row of $\mathbf{Q}'$ is zero.
3. For all $s \neq s'$, $q(s, s') = 0$ iff $q'(s, s') = 0$.

A class of right-perturbed Markov chains

Theorem

The matrix $\mathbf{Q}' = \mathbf{Q}R$ is a *right-perturbation of* $\mathbf{Q}$ if

1. All the diagonal elements of R are positive.
2. All the off-diagonal elements of R are non-positive.
3. The sum of each row of R is zero.
4. For all $s \neq s'$, if $q(s, s') = 0$ then $r(s, s') = 0$.
5. For all $s \neq s'$, if there exists $s'' \neq s, s'$ such that $q(s'', s) \neq 0$ then $r(s, s') = 0$.

Unfortunately, only a two-state loop.
But, we have examples out of this class.

Steady state distribution of right-perturbed chains

Result

- let R be invertible

Original CTMC	Perturbed CTMC
$\mathbf{Q}$	$\mathbf{Q}R$
$\boldsymbol{\pi}$	$\boldsymbol{\pi}$

Generalized proportional lumpability

Definition

$\sim$ is an *generalized proportional lumpability* for $\mathbf{Q}$ if there exist two invertible matrices L and R such that

$$\mathbf{Q}' = L\mathbf{Q}R$$

satisfies:

1. $\mathbf{Q}'$ is a CTMC
2. $\mathbf{Q}'$ has the same topology of the original chain $\mathbf{Q}$
3. $\sim$ is an ordinary lumpability for $\mathbf{Q}'$

Generalized proportional lumpability

Result

Original CTMC	Perturbed CTMC
$\mathbf{Q}$	$L\mathbf{Q}R$
$\sim$ gen. prop. lump. w.r.t. L and R	$\sim$ ordinary lump.
$\boldsymbol{\pi} L$	$\boldsymbol{\pi}$

Conclusion

- We introduce a generalized definition of lumpability enabling perturbation of a Markov chain via matrix multiplication while retaining the original steady-state distribution.
- We plan to explore a similar generalization for the concept of exact lumpability.
- We will examine the relationships between generalized proportional lumpability and reversibility.