

LINK OF GROUPS AND HOMOGENEOUS HÖRMANDER OPERATORS

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ABSTRACT. We study a notion of link of Lie groups suggested by the structure of the partial differential operators of Kolmogorov type. As an application of our link procedure we construct explicit examples of stratified Lie groups, with dimension and step arbitrarily large. We also give a set of examples of hypoelliptic second-order operators which are left translation invariant and homogeneous of degree two on the previous groups.

INTRODUCTION

In this paper we study the notion of *link of homogeneous groups* first explicitly fixed in [9]. This notion seems not completely new and it is somehow related to the notion of *fiber product*. However, our presentation, which is inspired by the structure of the operators introduced by Kolmogorov in studying diffusion processes from a probabilistic point of view (see [12], [11]) seems more suitable to be applied to the theory of second-order partial differential operators of Hörmander type.

In order to clarify our main purpose, we first recall that the above-mentioned Kolmogorov operator is the following one:

$$(1) \quad \mathcal{K} = \Delta_x + x \nabla_y - \partial_t,$$

where $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Δ_x and ∇_y respectively denote the classical Laplace operator and the Euclidean gradient in \mathbb{R}^n . Moreover, $x \nabla_y$ stands for the first-order differential operator $\sum_{j=1}^n x_j \partial_{y_j}$.

It has been proved in [12] that \mathcal{K} is left translation invariant on the Lie group $(\mathbb{R}^{2n+1}, \circ)$ whose composition law is given by

$$(x, y, t) \circ (x', y', t') := (x + x', y + y' + t'x, t + t').$$

The dilations

$$\delta_\lambda : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{2n+1}, \quad \delta_\lambda(x, y, t) = (\lambda x, \lambda^3 y, \lambda^2 t), \quad \lambda > 0,$$

are homomorphisms of $(\mathbb{R}^{2n+1}, \circ)$, so that

$$\mathbb{K} = (\mathbb{R}^{2n+1}, \circ, \delta_\lambda)$$

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is a homogeneous Lie group. It is easy to verify that \mathcal{K} also is δ_λ -homogeneous of degree two.

We shall say that \mathbb{K} is the Kolmogorov group related to the first-order partial differential operator

$$Y := x\nabla_y - \partial_t.$$

It seems quite natural to expect that replacing Δ_x in (1) with a suitable sub-Laplacian on a Carnot group, we should obtain a new operator with an underlying structure of a homogeneous Lie group.

Actually, as an application of our general results, we shall see that if $\Delta_{\mathbb{G}}$ is any sub-Laplacian and Y is a first-order partial differential operator which is *transverse* to \mathbb{G} (in the sense of Definition 4.5), then

$$\mathcal{L} = \Delta_{\mathbb{G}} + Y$$

is left translation invariant and homogeneous of degree two on a new homogeneous group obtained by *linking* \mathbb{G} with the Kolmogorov group related to Y .

As another application of our linking procedure, we are able to construct explicit examples of homogeneous and Carnot groups of dimension and step arbitrarily large. We are also able to give new explicit examples of degenerate elliptic and ultraparabolic operators of Hörmander type.

The study of this class of operators, started with Hörmander's paper [8], has significantly improved since the works by Folland [7] and by Rothschild and Stein [13]. In the latter paper it was shown that every Hörmander operator can be locally approximated by hypoelliptic operators of the following type:

$$\sum_{j=1}^m X_j^2 + Y,$$

where the X_j 's and Y are first-order partial differential operators with polynomial coefficients, left translation invariant and homogeneous of degree one and two, respectively, on a homogeneous Lie group in a real space \mathbb{R}^N .

As is well known, in the last decades a lot of literature has been devoted to the Hörmander operators on homogeneous Lie groups. We just mention, besides the celebrated papers [8], [4], [7] and [13], the monograph [14], the recent survey article [1], the papers [2], [3] and the references therein. We would also like to explicitly mention the papers [5] and [6] containing a deep analysis of operators with non-smooth coefficients on Lie groups.

The present paper is organized as follows. In Section 1 we introduce the notion of link of groups and in Section 2 we study the algebras of the linked groups. Section 3 and Section 4 are devoted, respectively, to homogeneous and Carnot groups. In Section 3 we show that the link of homogeneous groups is always a homogeneous group. In Section 4 we give sufficient conditions for this to also hold for Carnot groups. Finally in Section 5, by using our general results, we show how to construct sequences of sub-Laplacians and of sub-Kolmogorov operators.

We close the introduction by recalling few basic facts from Lie group theory in \mathbb{R}^N . A vector field in \mathbb{R}^N is a smooth function

$$X : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad X(x) = (a_1(x), \dots, a_N(x)),$$

where $a_j \in C^\infty(\mathbb{R}^N, \mathbb{R})$ for every $j \in \{1, \dots, N\}$.

We identify X with the linear first-order partial differential operator

$$X := \sum_{j=1}^N a_j(x) \partial_{x_j}, \quad x = (x_1, \dots, x_N).$$

Given a Lie group $\mathbb{G} = (\mathbb{R}^N, \circ)$ we denote by \mathfrak{g} its Lie algebra, i.e. the Lie algebra of the vector fields in \mathbb{R}^N which are left translation invariant on \mathbb{G} . For every $j \in \{1, \dots, N\}$, the vector field

$$X_j(x) := \partial_{y_j}(x \circ y)|_{y=0}, \quad x \in \mathbb{R}^N,$$

belongs to \mathfrak{g} . The family $\{X_1, \dots, X_N\}$ is a basis of \mathfrak{g} which is called the *Jacobian basis* of \mathfrak{g} .

1. LINK OF GROUPS

Let $G_i = (M_i, \circ_i)$, $i = 1, 2$, be Lie groups. Assume the manifolds M_i can be split as follows:

$$M_i = N_0 \times N_i, \quad i = 1, 2,$$

where N_0 is the underlying manifold of a Lie group $H = (N_0, \cdot)$.

We also assume that the projection

$$\pi_i : N_0 \times N_i \longrightarrow N_0, \quad \pi_i(n_0, n_i) = n_0,$$

is a homomorphism of G_i on H , $i = 1, 2$.

Given $(n_0, n_i) \in M_i$, $i = 1, 2$, we define

$$(n_0, n_1) \triangle (n_0, n_2) := (n_0, n_1, n_2).$$

Thus, if we set $M = N_0 \times N_1 \times N_2$, we have

$$M = M_1 \triangle M_2 := \{(n_0, n_1) \triangle (n_0, n_2) \mid n_i \in N_i, i = 0, 1, 2\}.$$

In M we introduce the following composition law:

$$(n_0, n_1, n_2) \circ (n'_0, n'_1, n'_2) := ((n_0, n_1) \circ_1 (n'_0, n'_1)) \triangle ((n_0, n_2) \circ_2 (n'_0, n'_2)).$$

We explicitly remark that this definition is well posed since

$$\pi_i((n_0, n_i) \circ_i (n'_0, n'_i)) = n_0 \cdot n'_0 \quad \text{for } i = 1, 2.$$

The following proposition holds.

Proposition 1.1. $G := (M, \circ)$ is a Lie group.

Proof. We omit the details of the proof, which is completely elementary though lengthy. We just remark that the zero of M is $0 = 0_1 \triangle 0_2$, where 0_i denotes the zero of G_i , $i = 1, 2$.

Moreover, if $m \in M$, $m = m_1 \triangle m_2$, with $m_i \in M_i$, then

$$m^{-1} = m_1^{-1} \triangle m_2^{-1}.$$

□

We shall say that

$$G = (M, \circ)$$

is the *link* of G_1 and G_2 .

The following proposition will be useful in what follows.

Proposition 1.2. *Let G_i , $i = 1, 2$, be as before. Let*

$$d^{(i)} : G_i \longrightarrow G_i$$

be an automorphism of G_i , $i = 1, 2$. Assume

$$(2) \quad \pi_1 d^{(1)} = \pi_2 d^{(2)}.$$

Then

$$d := d^{(1)} \triangle d^{(2)}$$

is defined as follows:

$$d(n_0, n_1, n_2) = (d^{(1)}(n_0, n_1)) \triangle (d^{(2)}(n_0, n_2))$$

is an automorphism of G .

Proof. The proof is quite easy. We only need to notice that d is well defined thanks to condition (2). \square

2. ALGEBRAS OF LINKED GROUPS ON REAL SPACES

Let \mathbb{R}^N , $N \geq 3$, be split as follows:

$$\mathbb{R}^N = \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^r$$

and denote its points by

$$x = (x^{(p)}, x^{(q)}, x^{(r)}),$$

where $x^{(p)} \in \mathbb{R}^p$, $x^{(q)} \in \mathbb{R}^q$ and $x^{(r)} \in \mathbb{R}^r$. We shall also use the notation

$$x^{(p,q)} := (x^{(p)}, x^{(q)}) \quad \text{and} \quad x^{(p,r)} := (x^{(p)}, x^{(r)}).$$

Accordingly, for consistency of notation, we shall write $\mathbb{R}^{(p,q)}$ and $\mathbb{R}^{(p,r)}$ instead of $\mathbb{R}^p \times \mathbb{R}^q$ and $\mathbb{R}^p \times \mathbb{R}^r$, respectively.

Let

$$\mathbb{G}_1 = (\mathbb{R}^{(p,q)}, \circ_1) \quad \text{and} \quad \mathbb{G}_2 = (\mathbb{R}^{(p,r)}, \circ_2)$$

be Lie groups. Assume the composition laws in \mathbb{G}_1 and \mathbb{G}_2 take the following form:

$$(3) \quad x^{(p,q)} \circ_1 y^{(p,q)} = (x^{(p)}, x^{(q)}) \circ_1 (y^{(p)}, y^{(q)}) := (x^{(p)} + y^{(p)}, Q(x^{(p,q)}, y^{(p,q)})),$$

$$(4) \quad x^{(p,r)} \circ_2 y^{(p,r)} = (x^{(p)}, x^{(r)}) \circ_2 (y^{(p)}, y^{(r)}) := (x^{(p)} + y^{(p)}, R(x^{(p,r)}, y^{(p,r)}))$$

where Q and R are vector functions with values in \mathbb{R}^q and \mathbb{R}^r , respectively.

Then \mathbb{G}_1 and \mathbb{G}_2 are linkable and the composition law on $\mathbb{G} := \mathbb{G}_1 \triangle \mathbb{G}_2$ is given by

$$x \circ y := (x^{(p)} + y^{(p)}, Q(x^{(p,q)}, y^{(p,q)}), R(x^{(p,r)}, y^{(p,r)})).$$

We shall denote by \mathfrak{g}_i and \mathfrak{g} the Lie algebras of \mathbb{G}_i and \mathbb{G} respectively ($i = 1, 2$).

Theorem 2.1. *Let*

$$\{X_1, \dots, X_{p+q}\} \quad \text{and} \quad \{Y_1, \dots, Y_{p+r}\}$$

be the Jacobian bases of \mathfrak{g}_1 and \mathfrak{g}_2 respectively. Then

$$X_{p+1}, \dots, X_{p+q} \in \mathfrak{g}.$$

Moreover, if R is independent of $y^{(p)}$, we also have

$$(5) \quad X_1, \dots, X_p \in \mathfrak{g}.$$

Analogously,

$$(6) \quad Y_{p+1}, \dots, Y_{p+r} \in \mathfrak{g}$$

and, if Q is independent of $y^{(p)}$, then

$$Y_1, \dots, Y_p \in \mathfrak{g}.$$

Note. If $1 \leq s \leq N$, we agree to consider any vector field in \mathbb{R}^s as a vector field in \mathbb{R}^N with the last $N - s$ components equal to zero.

Proof. If R is independent of $y_j^{(p,q)}$ for some $j \in \{1, \dots, p+q\}$, then $X_j \in \mathfrak{g}$. Indeed,

$$\begin{aligned} \partial_{y_j^{(p,q)}}(x \circ y) &= \partial_{y_j^{(p,q)}}(x^{(p)} + y^{(p)}, Q(x^{(p,q)}, y^{(p,q)}), R(x^{(p,r)}, y^{(p,r)})) \\ &= \partial_{y_j^{(p,q)}}(x^{(p)} + y^{(p)}, Q(x^{(p,q)}, y^{(p,q)}), 0) \\ &= \partial_{y_j^{(p,q)}}(x^{(p)} + y^{(p)}, Q(x^{(p,q)}, y^{(p,q)})) = \partial_{y_j^{(p,q)}}(x^{(p,q)} \circ_1 y^{(p,q)}), \end{aligned}$$

so that

$$\partial_{y_j^{(p,q)}}(x \circ y)|_{y=0} = \partial_{y_j^{(p,q)}}(x^{(p,q)} \circ_1 y^{(p,q)})|_{y^{(p,r)}=0^{(p,r)}} = X_j$$

and hence $X_j \in \mathfrak{g}$.

Then (5) and (6) are proved since R is always independent of $y^{(q)}$.

The second part of the theorem can be proved exactly as the first part. \square

Example 2.2. Let $\mathbb{G}_1 = (\mathbb{R}^{(p,q)}, \circ_1)$ be a Lie group whose composition law \circ_1 is as in (3).

Let us now consider a $(p+r) \times (p+r)$ matrix B taking the following block form:

$$B = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}$$

where B_1 and B_2 are, respectively, $r \times p$ and $r \times r$ real matrices. Then the exponential matrix

$$E(t) := \exp(-tB), \quad t \in \mathbb{R},$$

takes the following block form:

$$(7) \quad E(t) = \begin{pmatrix} I_p & 0 \\ E_1(t) & E_2(t) \end{pmatrix}.$$

In $\mathbb{R}^{(p,r+1)}$, whose points will be denoted by $(x^{(p,r)}, t)$, we introduce the composition law

$$(8) \quad (x^{(p,r)}, t) \circ_2 (y^{(p,r)}, \tau) := (y^{(p,r)} + E(\tau)x^{(p,r)}, t + \tau).$$

It easy to show that

$$\mathbb{G}_2 := (\mathbb{R}^{(p,r+1)}, \circ_2)$$

is a Lie group.

Moreover, due to (7),

$$\begin{aligned} (x^{(p,r)}, t) \circ_2 (y^{(p,r)}, \tau) &= (x^{(p)} + y^{(p)}, y^{(r)} + E_1(\tau)x^{(p)} + E_2(\tau)x^{(r)}, t + \tau) \\ (9) \quad &= : (x^{(p)} + y^{(p)}, R((x^{(p,r)}, t), (y^{(p,r)}, \tau))). \end{aligned}$$

It follows that \mathbb{G}_1 and \mathbb{G}_2 are linkable and, by Theorem 2.1, since R is independent of $y^{(p)}$, the Jacobian basis of \mathfrak{g}_1 turns out to be contained in the Jacobian basis of \mathfrak{g} (the Lie algebra of $\mathbb{G} = \mathbb{G}_1 \triangle \mathbb{G}_2$).

We now compute the Jacobian basis of \mathfrak{g}_2 .

From (9) we immediately get

$$\partial_{y_j^{(p,r)}}((x^{(p,r)}, t) \circ_2 (y^{(p,r)}, \tau))|_{(y^{(p,r)}, \tau)=0} = e_j, \quad j = 1, \dots, p+r,$$

where e_j denotes the j -th element of the canonical basis of $\mathbb{R}^{(p,r+1)}$. Moreover, directly from (8), we obtain

$$\partial_\tau((x^{(p,r)}, t) \circ_2 (y^{(p,r)}, \tau))|_{(y^{(p,r)}, \tau)=0} = \begin{pmatrix} -Bx^{(p,r)} \\ 1 \end{pmatrix}.$$

Then, the Jacobian basis of \mathfrak{g}_2 is

$$\{\partial_{x_j^{(p,r)}}, j = 1, \dots, p+r\} \cup \{Y\},$$

where

$$(10) \quad Y := -\langle Bx^{(p,r)}, \nabla_{(p,r)} \rangle + \partial_t.$$

Therefore, by Theorem 2.1, the vector fields

$$Y \quad \text{and} \quad \partial_{x_j^{(p,r)}}, j = p+1, \dots, p+r,$$

are left translation invariant on \mathbb{G} .

It follows that the Jacobian basis of \mathfrak{g} is given by

$$X_1, \dots, X_{p+q}, \partial_{x_{p+1}^{(p,r)}}, \dots, \partial_{x_{p+r}^{(p,r)}}, Y$$

where we have denoted by $\{X_1, \dots, X_{p+q}\}$ the Jacobian basis of \mathfrak{g}_1 .

3. LINK OF HOMOGENEOUS GROUPS

We shall call *group of dilations in \mathbb{R}^N* any family $(d_\lambda)_{\lambda>0}$ of diagonal linear map of the form

$$d_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, d_\lambda(x_1, \dots, x_N) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_N} x_N),$$

where the α_j 's are suitable real positive numbers.

A Lie group (\mathbb{R}^N, \circ) will be called a *homogeneous Lie group* if there exists a group of dilations $(d_\lambda)_{\lambda>0}$ such that d_λ is a homomorphism of (\mathbb{R}^N, \circ) for every $\lambda > 0$.

We shall denote by

$$\mathbb{G} = (\mathbb{R}^N, \circ, d_\lambda)$$

a homogeneous Lie group endowed with a group of dilations given by $(d_\lambda)_{\lambda>0}$. Let

$$\mathbb{G}_1 = (\mathbb{R}^{(p,q)}, \circ_1, d_\lambda^{(1)}) \quad \text{and} \quad \mathbb{G}_2 = (\mathbb{R}^{(p,r)}, \circ_2, d_\lambda^{(2)})$$

be homogeneous Lie groups and suppose the composition laws \circ_1 and \circ_2 are defined as in (3) and (4). We also assume that the dilations $d_\lambda^{(1)}$ and $d_\lambda^{(2)}$ take the following form:

$$(11) \quad \begin{aligned} d_\lambda^{(1)}(x^{(p)}, x^{(q)}) &= (\lambda x^{(p)}, \rho_\lambda^{(1)}(x^{(q)})), \\ d_\lambda^{(2)}(x^{(p)}, x^{(r)}) &= (\lambda x^{(p)}, \rho_\lambda^{(2)}(x^{(r)})). \end{aligned}$$

These assumptions imply that \mathbb{G}_1 and \mathbb{G}_2 are linkable (in the sense of the previous section) and that the dilations $d_\lambda^{(1)}$ and $d_\lambda^{(2)}$ satisfy the hypotheses of Proposition 1.2. Then

$$\mathbb{G} = \mathbb{G}_1 \triangle \mathbb{G}_2$$

is a homogeneous Lie group with respect to the dilations

$$d_\lambda : \mathbb{R}^N \longrightarrow \mathbb{R}^N, \quad d_\lambda(x^{(p)}, x^{(q)}, x^{(r)}) = (\lambda x^{(p)}, \rho_\lambda^{(1)}(x^{(q)}), \rho_\lambda^{(2)}(x^{(r)})).$$

In what follows, we shall call

$$\mathbb{G} = (\mathbb{R}^N, \circ, d_\lambda)$$

the *link of the homogeneous groups* \mathbb{G}_1 and \mathbb{G}_2 .

Example 3.1. Let $\mathbb{G}_1 = (\mathbb{R}^{(p,q)}, \circ_1, d_\lambda^1)$ be a homogeneous Lie group. Assume the composition law \circ_1 and the dilation d_λ^1 as in (3) and (11), respectively.

Let us now consider a *Kolmogorov-type group*, i.e. a group

$$\mathbb{K} = (\mathbb{R}^{(p,r+1)}, \circ_2, d_\lambda^{(2)})$$

where \circ_2 is the composition law (8) related to the following $(p+r) \times (p+r)$ matrix B :

$$(12) \quad B = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ B_1 & 0 & 0 & \dots & 0 \\ 0 & B_2 & \dots & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & B_k & 0 \end{pmatrix}.$$

Assume that B_j is a maximum rank matrix of dimension $r_j \times r_{j-1}$, with $r_0 = p \geq r_1 \geq \dots \geq r_k \geq 1$ and $r_0 + r_1 + \dots + r_k = N$.

The dilations

$$d_\lambda^{(2)} : \mathbb{R}^{(p,r+1)} \longrightarrow \mathbb{R}^{(p,r+1)},$$

$$d_\lambda^{(2)}(x^{(p)}, x^{(r_1)}, \dots, x^{(r_k)}, t) = (\lambda x^{(p)}, \lambda^3 x^{(r_1)}, \dots, \lambda^{2k+1} x^{(r_k)}, \lambda^2 t), \quad \lambda > 0,$$

make \mathbb{K} a homogeneous Lie group (see [12], where the *Kolmogorov-type* groups were first introduced).

Then, \mathbb{G}_1 and \mathbb{K} are linkable homogeneous groups.

4. LINK OF CARNOT GROUPS

Let $\mathbb{G} = (\mathbb{R}^N, \circ, d_\lambda)$ be a homogeneous group and assume that

$$d_\lambda(x_1, \dots, x_N) = (\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_N} x_N),$$

where $\alpha_j \in \mathbb{N}$ for every $j \in \{1, \dots, N\}$. Let us denote by $\Lambda_{\mathbb{G}}$ the subset of the indices $j \in \{1, \dots, N\}$ such that $\alpha_j = 1$, and assume that the cardinality of $\Lambda_{\mathbb{G}}$ is greater than or equal to 2.

We say that \mathbb{G} is a Carnot group if the Lie algebra \mathfrak{g} of \mathbb{G} is generated by the left translation invariant vector fields

$$X_j, \quad j \in \Lambda_{\mathbb{G}},$$

such that $X_j(0) = \partial_{x_j}$. These vector fields are called the *generators of* \mathbb{G} . It can be proved that, if $j \in \Lambda_{\mathbb{G}}$, the j -th component of the composition law is Euclidean, i.e.

$$(x \circ y)_j = x_j + y_j \quad \forall j \in \Lambda_{\mathbb{G}}.$$

We want to show that, under suitable conditions, the link of two Carnot groups is a Carnot group.

So let

$$\mathbb{G}_1 = (\mathbb{R}^{(p,q)}, \circ_1, d_\lambda^{(1)}) \quad \text{and} \quad \mathbb{G}_2 = (\mathbb{R}^{(p,r)}, \circ_2, d_\lambda^{(2)})$$

be Carnot groups. Up to a rearrangement of the variables we may and do assume

$$\Lambda_{\mathbb{G}_1} = \{1, \dots, p\}, \quad \Lambda_{\mathbb{G}_2} = \{1, \dots, m\}, \quad \text{with } p \leq m \leq p + r.$$

As a consequence, $d_\lambda^{(1)}$ and $d_\lambda^{(2)}$ take the form

$$d_\lambda^{(1)}(x^{(p,q)}) = (\lambda x^{(p)}, \rho_\lambda^{(1)}(x^{(q)})),$$

$$d_\lambda^{(2)}(x^{(p,r)}) = (\lambda x^{(p)}, \rho_\lambda^{(2)}(x^{(r)})) = (\lambda x^{(p)}, \lambda x^{(m-p)}, \sigma_\lambda^{(2)}(x^{(p+r-m)})).$$

Then, \mathbb{G}_1 and \mathbb{G}_2 are linkable homogeneous Lie groups with dilations given by:

$$\begin{aligned} d_\lambda(x^{(p)}, x^{(q)}, x^{(r)}) &= d_\lambda(x^{(p)}, x^{(q)}, x^{(m-p)}, x^{(p+r-m)}) \\ (13) \quad &= (\lambda x^{(p)}, \rho_\lambda^{(1)}(x^{(q)}), \lambda x^{(m-p)}, \sigma_\lambda^{(2)}(x^{(p+r-m)})). \end{aligned}$$

However, in general, $\mathbb{G} := \mathbb{G}_1 \triangle \mathbb{G}_2$ is not a Carnot group.

In order to give a condition ensuring that \mathbb{G} is a Carnot group, we first remark that the composition law \circ_1 can be written as follows:

$$x^{(p,q)} \circ_1 y^{(p,q)} := (x^{(p)} + y^{(p)}, R(x^{(p,q)}, y^{(p,q)})).$$

Theorem 4.1. *Assume the following conditions are satisfied:*

(H1) *R is independent of $y^{(p)}$.*

(H2) *$\partial_{x_j^{(p)}}$ is left translation invariant on \mathbb{G}_2 for $1 \leq j \leq p$.*

Then

$$\mathbb{G} = \mathbb{G}_1 \triangle \mathbb{G}_2$$

is a Carnot group with m generators.

More precisely, if $p < m$, X_1, \dots, X_p are the generators of \mathbb{G}_1 and $\partial_{x_1^{(p)}}, \dots, \partial_{x_p^{(p)}}, Y_{p+1}, \dots, Y_m$ are the generators of \mathbb{G}_2 , then the vector fields

$$X_1, \dots, X_p, Y_{p+1}, \dots, Y_m$$

are the generators of \mathbb{G} .

The proof of this theorem will straightforwardly follow from the next lemma and corollary.

Lemma 4.2. *Let Y be a vector field in $\mathbb{R}^{(p,q,r)}$ taking the form*

$$(14) \quad Y = \sum_{k=1}^r b^{(k)} \partial_{x_k^{(r)}}$$

with $b^{(k)}$ independent of $x^{(q)}$, $k = 1, \dots, r$.

Then

$$[X_j, Y] = [\partial_{x_j^{(p)}}, Y] \quad \forall j \in \{1, \dots, p\}.$$

Proof. From the general theory of homogeneous Lie groups on Euclidean spaces it follows that X_j , $j \in \{1, \dots, p\}$, can be written as

$$X_j = \partial_{x_j^{(p)}} + \tilde{X}_j,$$

where

$$\tilde{X}_j = \sum_{k=1}^q a_j^{(k)} \partial_{x_k^{(q)}}, \quad a_j^{(k)} = a_j^{(k)}(x^{(p,q)}).$$

Then $[\tilde{X}_j, Y] = 0$ and the assertion follows. \square

Corollary 4.3. *Let*

$$(15) \quad \partial_{x_1^{(p)}}, \dots, \partial_{x_p^{(p)}}, Y_{p+1}, \dots, Y_m, \dots, Y_{p+r}$$

be the Jacobian basis of \mathfrak{g}_2 . Then

$$Y_{p+i} \in \text{Lie}\{X_1, \dots, X_p, Y_{p+1}, \dots, Y_m\} \quad \forall i \in \{1, \dots, r\}.$$

Proof. From the general theory of homogeneous Lie groups on Euclidean spaces, it follows that each vector field

$$Y_{p+1}, \dots, Y_m, \dots, Y_{p+r}$$

can be written as in (14) with coefficients independent of $x^{(q)}$.

Therefore the result follows by applying Lemma 4.2. \square

Proof of Theorem 4.1. Let us denote by

$$X_1, \dots, X_p, X_{p+1}, \dots, X_{p+q}$$

the Jacobian basis of \mathfrak{g}_1 . Moreover, let (15) be the Jacobian basis of \mathfrak{g}_2 . Hypothesis (H1) and Theorem 2.1 imply that

$$(16) \quad X_1, \dots, X_p, X_{p+1}, \dots, X_{p+q}, Y_{p+1}, \dots, Y_m, \dots, Y_{p+r} \in \mathfrak{g}.$$

On the other hand,

$$\begin{aligned} X_j(0) &= \partial_{x_j^{(p)}} & \text{if } j \in \{1, \dots, p\}, \\ X_{p+j}(0) &= \partial_{x_j^{(q)}} & \text{if } j \in \{1, \dots, q\}, \\ Y_{p+i}(0) &= \partial_{x_i^{(r)}} & \text{if } i \in \{1, \dots, r\}. \end{aligned}$$

Then, the vector fields in (16) are linearly independent since they are linearly independent at the origin.

Since $\dim \mathfrak{g} = \dim (\mathbb{R}^{(p,q)} \triangle \mathbb{R}^{(p,r)}) = p + q + r$, we get that (16) is the Jacobian basis of \mathfrak{g} . The structure of the dilations in (13) implies that

$$\Lambda_{\mathbb{G}} = \{1, \dots, p\} \cup \{p + q + 1, \dots, p + q + m\}.$$

Hence the cardinality of $\Lambda_{\mathbb{G}}$ is equal to m .

In order to complete the proof, we have to show that

$$(17) \quad \dim (\text{Lie}\{X_1, \dots, X_p, Y_{p+1}, \dots, Y_m\}) = p + q + r.$$

It is quite obvious that

$$X_j \in \text{Lie}\{X_1, \dots, X_p, Y_1, \dots, Y_m\} \quad \forall j \in \{1, \dots, p + q\}.$$

Moreover, by Corollary 4.3,

$$Y_{p+i} \in \text{Lie}\{X_1, \dots, X_p, Y_1, \dots, Y_m\} \quad \forall i \in \{1, \dots, r\}.$$

Then (17) follows from the first part of the proof. \square

Example 4.4. Let $\mathbb{G}_1 = (\mathbb{R}^{(p,q)}, \circ_1, d_{\lambda}^{(1)})$ be a Carnot group with p generators. Assume $\Lambda_{\mathbb{G}_1} = \{1, \dots, p\}$ and denote by

$$X_1, \dots, X_p$$

the generators of \mathbb{G}_1 .

Let us now consider a *Carnot-Kolmogorov type group* on $\mathbb{R}^{(p,r+1)}$, i.e. a homogeneous Lie group

$$\mathbb{CK} := (\mathbb{R}^{(p,r+1)}, \circ_2, d_{\lambda}^{(2)})$$

whose composition law is defined as in (8) (see Example 2.2) and the matrix B is given by (12). Moreover the dilation $d_\lambda^{(2)}$ is defined as follows:

$$\begin{aligned} d_\lambda^{(2)}(x^{(p,r+1)}) &= d_\lambda^{(2)}(x^{(p)}, x^{(r+1)}) \\ &= d_\lambda^{(2)}(x^{(p)}, x^{(r_1)}, \dots, x^{(r_k)}, t) \\ &= (\lambda x^{(p)}, \lambda^2 x^{(r_1)}, \dots, \lambda^{k+1} x^{(r_k)}, \lambda t). \end{aligned}$$

Then, \mathbb{CK} is a Carnot group with $p+1$ generators and

$$\Lambda_{\mathbb{CK}} = \{1, \dots, p\} \cup \{p+r+1\}$$

(see also [2]). The generators of \mathbb{CK} are the vector fields

$$\partial_{x_1^{(p)}}, \dots, \partial_{x_p^{(p)}}, Y$$

where

$$(18) \quad Y = -\langle Bx^{(p,r)}, D_{(p,r)} \rangle + \partial_t;$$

see (10). Here $\langle \cdot, \cdot \rangle$ and $D_{(p,r)}$ denote the inner product and the Euclidean gradient in $\mathbb{R}^{(p,r)}$, respectively.

Then, since the function R in (9) is independent of $y^{(p)}$, \mathbb{G}_1 and \mathbb{CK} are linkable homogeneous Lie groups and the linked group

$$\mathbb{G} := \mathbb{G}_1 \triangle \mathbb{CK}$$

is a Carnot group whose generators are

$$X_1, \dots, X_p, Y$$

(see Theorem 4.1).

Definition 4.5. We shall call the vector field Y in (18) a *derivative operator transverse to \mathbb{G}_1* .

5. AN APPLICATION

5.1. Sequences of sub-Laplacians. Let \mathbb{G}_1 be a Carnot group with generators

$$X_1, \dots, X_p.$$

Then

$$\Delta_{\mathbb{G}} := \sum_{j=1}^p X_j^2$$

is a second-order partial differential operator which is called the *canonical sub-Laplacian* on \mathbb{G}_1 . Since

$$\mathfrak{g}_1 := \text{Lie}\{X_1, \dots, X_p\},$$

\mathfrak{g}_1 being the Lie algebra of \mathbb{G}_1 , $\Delta_{\mathbb{G}}$ is hypoelliptic by the well-known theorem of Hörmander [8].

Let us now consider a derivative operator Y_1 transverse to \mathbb{G}_1 and denote by \mathbb{CK}_1 the Carnot-Kolmogorov type group related to Y_1 (see Example 4.4).

Then

$$X_1, \dots, X_p, Y_1$$

are the generators of the Carnot group

$$\mathbb{G}_2 := \mathbb{G}_1 \triangle \mathbb{CK}_1.$$

Hence

$$\begin{aligned}\Delta_{\mathbb{G}_2} &= \sum_{j=1}^p X_j^2 + Y_1^2 \\ &= \Delta_{\mathbb{G}_1} + Y_1^2.\end{aligned}$$

We can continue this procedure by choosing a derivative operator Y_2 transverse to \mathbb{G}_2 . Thus, denoting by \mathbb{CK}_2 the Carnot-Kolmogorov type group related to Y_2 , we have that

$$\Delta_{\mathbb{G}_3} = \Delta_{\mathbb{G}_2} + Y_2^2 = \Delta_{\mathbb{G}_1} + Y_1^2 + Y_2^2$$

is the canonical sub-Laplacian on the Carnot group

$$\mathbb{G}_3 := \mathbb{G}_2 \Delta \mathbb{CK}_2 \equiv (\mathbb{G}_1 \Delta \mathbb{CK}_1) \Delta \mathbb{CK}_2.$$

By iterating this argument, we can construct a sequence of Carnot groups $(\mathbb{G}_n)_{n \geq 1}$ such that

$$\mathbb{G}_{n+1} := \mathbb{G}_n \Delta \mathbb{CK}_n,$$

\mathbb{CK}_n being a Carnot-Kolmogorov type group related to a derivative operator Y_n transverse to \mathbb{G}_n .

We also have

$$\begin{aligned}\Delta_{\mathbb{G}_{n+1}} &= \Delta_{\mathbb{G}_n} + Y_n^2 \\ &= \Delta_{\mathbb{G}_1} + \sum_{k=1}^n Y_k^2.\end{aligned}$$

5.2. Sequences of sub-Kolmogorov operators. Let \mathbb{G}_1 be a Carnot group with generators X_1, \dots, X_p . Let Y_1 be a *derivative operator* transverse to \mathbb{G}_1 . Then we call

$$\mathcal{K}_1 := \Delta_{\mathbb{G}_1} + Y_1 = \sum_{j=1}^p X_j^2 + Y_1$$

a *sub-Kolmogorov operator* (see [9] where this notion was first introduced).

Then, \mathcal{K}_1 is left translation invariant on the homogeneous linked group

$$\mathbb{K}_1 = \mathbb{G}_1 \Delta \mathbb{K}_1$$

where \mathbb{K}_1 is the Kolmogorov-type group related to Y_1 (see Example 3.1). We also know that \mathcal{K}_1 is homogeneous of degree two with respect to the dilations of \mathbb{K}_2 . Moreover, the composition law on \mathbb{K}_2 is the same composition law on \mathbb{G}_2 of the previous example. Then \mathbb{K}_2 and \mathbb{G}_2 have the same Lie algebra, which is $\text{Lie}\{X_1, \dots, X_p, Y_1\}$. It follows that \mathcal{K}_2 is hypoelliptic, since it satisfies the Hörmander condition. Now, let us now consider the sequence of sub-Laplacians $(\Delta_{\mathbb{G}_n})_{n \geq 1}$ constructed in the previous section, and let $(Y_n)_{n \geq 1}$ be the related sequence of transverse derivative operators, i.e. Y_n is transverse to \mathbb{G}_n . Then

$$\mathcal{K}_n = \Delta_{\mathbb{G}_n} + Y_n$$

is a sub-Kolmogorov operator for every $n \in \mathbb{N}$.

We would like to explicitly remark that

$$\mathcal{K}_{n+1} = \Delta_{\mathbb{G}_1} + (Y_1^2 + \dots + Y_n^2) + Y_{n+1}, \quad n \geq 1.$$

All these operators are contained in the classes studied in [9] and [10].

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