AN INVERSE MEAN VALUE PROPERTY
FOR EVOLUTION EQUATIONS

Alessia E. Kogoj, Ermanno Lanconelli, and Giulio Tralli
Dipartimento di Matematica, Università di Bologna
Piazza di Porta San Donato, 5, IT-40126 Bologna - Italy

(Submitted by: Reza Aftabizadeh)

Abstract. The aim of this work is to extend a result by Suzuki and Watson concerning an inverse property for caloric functions. Our result applies, in particular, to the heat operator on stratified Lie groups and to Kolmogorov-Fokker-Planck-type operators. We show that the open sets characterizing the solutions to the involved equations, in terms of suitable average operators, have to be the level sets of the fundamental solutions of the relevant operators. The technique adopted exploits the structure of the propagation sets, i.e., the sets where the solutions to the involved equations attain their maximum.

1. Introduction

In this paper, we are concerned with an inverse problem: the characterization of bounded open subsets of $\mathbb{R}^{N+1} = \mathbb{R}^N_x \times \mathbb{R}_t$ in terms of the solutions to the PDE

$$\mathcal{H} u := (\mathcal{L} - \partial_t) u = 0,$$

where $\mathcal{L}$ is a linear second order operator with nonnegative characteristic form. We generalize some well known theorems we want to recall in the next two subsections. The main results proved in the present paper will be described in Subsection 1.3.

1.1. **Laplace equation.** The harmonic functions, namely the solutions to the Laplace equation

$$\Delta u := \sum_{j=1}^{N} \frac{\partial^2 u}{\partial x_j^2} = 0$$

in some open set $O \subset \mathbb{R}^N$, satisfy the Gauss Mean Value property

$$u(x_0) = M_r(u)(x_0) := \frac{1}{\text{meas}(B(x_0, r))} \int_{B(x_0, r)} u(x) \ dx, \qquad (1.1)$$

for every Euclidean ball $B(x_0, r) \subset\subset O$.

This very well known classical result has an equally well known vice versa: if u is a merely continuous function in O satisfying (1.1) for every Euclidean ball contained in O, then u is smooth and harmonic in O.

Thus, the average operators on Euclidean balls characterize the harmonic functions. A natural question is: are the Euclidean balls the only bounded open sets characterizing the harmonic functions? More precisely:

(Q) Let $D \subset \mathbb{R}^N$ be a bounded open set and let $x_0 \in D$. Suppose

$$u(x_0) = \frac{1}{\text{meas}(D)} \int_D u(x) dx$$

for every integrable harmonic function u in D. Then, is D a Euclidean ball centered at x_0?

As it is quite well known, the answer to this question is positive. This follows, e.g., from a spherical symmetry result by Aharonov, Shiffer and Zalcman:

Let $D \subset \mathbb{R}^N$ be a bounded open set and let $x_0 \in D$. Assume there exists a real constant $c > 0$ such that

$$\frac{1}{c} \int_D \frac{1}{|x - y|^{N-2}} \ dx = \frac{1}{|x_0 - y|^{N-2}} \quad \text{for every } y \in \mathbb{R}^N \setminus D. \qquad (1.2)$$

Then, D is a Euclidean ball centered at x_0 and $c = \text{meas}(D)$; see [2].

From this result, a positive answer to question (Q) straightforwardly follows by just recalling that the functions

$$x \mapsto \frac{1}{|x - y|^{N-2}}, \ \ y \in \mathbb{R}^N \setminus D,$$

are harmonic and integrable in D.

It is worthwhile to notice that the condition (1.2) can be written as follows:

$$\frac{1}{c} \int_D \Gamma(x - y) \ dx = \Gamma(x_0 - y) \quad \text{for every } y \in \mathbb{R}^N \setminus D,$$

where Γ is the fundamental solution at the origin of the Laplace operator Δ.

1.2. **Heat equation.** Results similar to the previous ones also hold in the context of *caloric functions*, namely, the solutions to the heat equation

$$Hu := (\Delta - \partial_t)u = 0,$$

in some open set $O \subset \mathbb{R}^{N+1} := \mathbb{R}^N_x \times \mathbb{R}_t$. Indeed, a continuous function $u : O \to \mathbb{R}$ is (smooth) and caloric in O if and only if

$$u(z_0) = \int_{\Omega_r(z_0)} u(z)K_r(z_0 - z)\,dz, \tag{1.3}$$

for every *heat ball* $\Omega_r(z_0)$ well contained in O. The heat balls "centered" at z_0 and with radius $r > 0$ are defined as follows:

$$\Omega_r(z_0) = \left\{ z \in \mathbb{R}^{N+1} : \Gamma(z_0 - z) > \tfrac{1}{r} \right\}.$$

Here, Γ stands for the fundamental solution at the origin to the heat equation:

$$\Gamma(x,t) = \tfrac{1}{(4\pi t)^{\frac{N}{2}}} \exp\left(-\tfrac{|x|^2}{4t}\right) \text{ if } t > 0, \qquad \Gamma(x,t) = 0 \text{ if } t \le 0.$$

The kernel K_r appearing in (1.3) is given by

$$K_r(z) = K_r(x,t) := \tfrac{1}{r} \tfrac{|x|^2}{4t^2}.$$

Before proceeding, we want to stress that the "center" z_0 of the heat ball $\Omega_r(z_0)$ belongs to the boundary of $\Omega_r(z_0)$.

While the history of the Mean Value property for caloric functions started with a paper by Pini dated 1951 ([18]), the formula (1.3) was discovered by Watson in 1973 ([21]). More recently, Suzuki and Watson proved a characterization of the heat balls, which is the direct analog of Aharonov, Shiffer and Zalcman's Theorem:

Let $z_0 \in \mathbb{R}^{N+1}$, and $r > 0$. Let $D \subset \mathbb{R}^{N+1}$ be a bounded open set. Assume the following conditions are satisfied:

(i) there exists a neighborhood V of z_0 such that $D \cap V = \Omega_r(z_0) \cap V$;
(ii) for every $\zeta \in \mathbb{R}^{N+1} \setminus D$, we have

$$\int_D \Gamma(z - \zeta)\,K_r(z_0 - z)\,dz = \Gamma(z_0 - \zeta).$$

Then, $D = \Omega_r(z_0)$; see [19].

We want to stress that the condition (i) corresponds to the assumption "$x_0 \in D$" of the harmonic case. Indeed, since D is open, "$x_0 \in D$" implies that D and the appropriate Euclidean ball are indistinguishable in the

vicinity of x_0.[1] From Suzuki and Watson's Theorem, one easily obtains the following characterization of the heat balls:

Let $z_0 \in \mathbb{R}^{N+1}$, and $r > 0$. Let D be a bounded open subset of $\mathbb{R}^{N+1}$. Assume the following conditions are satisfied:

(i) there exists a neighborhood V of z_0 such that $D \cap V = \Omega_r(z_0) \cap V$;
(ii) for every nonnegative caloric function u in a neighborhood of $D \cup \{z_0\}$, we have

$$u(z_0) \;=\; \int_D u(z) \, K_r(z_0 - z) \, dz.$$

Then, $D = \Omega_r(z_0)$.

1.3. **Evolution equations. Main results of the paper.** Aharonov, Shiffer and Zalcman's Theorem has been recently extended by one of us to the sub-Laplacians setting; see [14]. Even more recently, Abbondanza and Bonfiglioli proved a characterization of bounded open sets in $\mathbb{R}^N$ in terms of the solutions to "degenerate" elliptic equations endowed with well behaved fundamental solutions; see [1]. The aim of the present paper is to extend Suzuki and Watson's Theorem to a class of evolution equations of the kind

$$\mathcal{H}u := \Big(\sum_{i,j=1}^N \partial_{x_i}(a_{ij}\partial_{x_j}) + \sum_{i=1}^N b_i \partial_{x_i} - \partial_t \Big)u = 0, \tag{1.4}$$

where $A = (a_{ij})_{ij=1,\dots,N}$ is a symmetric nonnegative definite matrix whose entries $a_{ij} = a_{ij}(z)$ are smooth functions in $\mathbb{R}^{N+1}$, and the vector field

$$Y := \sum_{i=1}^N b_i \partial_{x_i} - \partial_t \quad \text{is divergence-free, i.e.,} \quad \sum_{i=1}^N \partial_{x_i} b_i \equiv 0. \tag{1.5}$$

The operator $\mathcal{H}$ is possible degenerate. However, together with some general assumptions that will be fixed in a moment, we suppose the operator $\mathcal{H}$ is not totally degenerate. Precisely:

(ND) there exists $\alpha > 0$ such that $a_{11}(z) > \alpha$ for every $z \in \mathbb{R}^{N+1}$.

Moreover, we also assume:

(HYP) the operator $\mathcal{H}$ and its formal adjoint $\mathcal{H}^*$ are hypoelliptic in $\mathbb{R}^{N+1}$.

We want to recall that:

---

[1]In [19], condition (i) is replaced with the weaker one: $(\mathbf{1}_D - \mathbf{1}_{\Omega_r(z_0)})K_r \in L^p$ for some $p > \frac{N}{2} + 1$.

(i) $\mathcal{H}$ is said to be hypoelliptic if the distributional solutions to $\mathcal{H}u = f$ are smooth whenever f is smooth;

(ii) the formal adjoint of the operator in (1.4), due to the divergence-free condition (1.5), is

$$\mathcal{H}^* := \sum_{i,j=1}^{N} \partial_{x_i}(a_{ij}\partial_{x_j}) - \sum_{i=1}^{N} b_i\partial_{x_i} + \partial_t.$$

In recent years, the classical Mean value Theorem for harmonic and caloric functions has been extended to wide classes of operators like $\mathcal{H}$, see [8], [12], [9], [16], [11]. Roughly speaking, the solutions to $\mathcal{H}u = 0$ have been characterized in terms of weighted average operators on the level sets of the fundamental solution of $\mathcal{H}^*$.

The common feature of the operators considered in the papers quoted above is the following one: they are endowed with a fundamental solution $(z,\zeta) \to \Gamma(z,\zeta)$ having the properties $(\Gamma 1) - (\Gamma 7)$ listed in Section 2. Throughout the paper, we assume that the same properties hold for our operator $\mathcal{H}$ in (1.4), i.e., together with (ND), we assume that

(Γ) $\mathcal{H}$ has a fundamental solution Γ satisfying $(\Gamma 1) - (\Gamma 7)$.

These properties also provide the validity of the condition (HYP). While we directly refer to Section 2 for the complete list of the properties of Γ. However, here, it is worthy to only mention the following ones: $(z,\zeta) \to \Gamma(z,\zeta)$ is nonnegative, smooth out of $\{z \neq \zeta\}$, strictly positive if and only if $z = (x,t), \zeta = (\xi,\tau)$ and $t > \tau$. Moreover, for any fixed $z \in \mathbb{R}^{n+1}$, $\limsup_{\zeta \to z} \Gamma(z,\zeta) = +\infty$ and $\Gamma(z,\zeta) \to 0$ as $\zeta \to \infty$.

As a consequence, the $\mathcal{H}$-ball of center $z_0 = (x_0, t_0)$ and radius $r > 0$

$$\Omega_r(z_0) := \left\{\zeta \in \mathbb{R}^{N+1} : \Gamma(z_0,\zeta) > \tfrac{1}{r}\right\} \tag{1.6}$$

is a non-empty and bounded open set. Moreover, we have

$$z_0 \in \partial\Omega_r(z_0), \quad \cap_{r>0}\overline{\Omega_r(z_0)} = \{z_0\}, \quad \overline{\Omega_r(z_0)} \smallsetminus \{z_0\} \subset \mathbb{R}^N \times (-\infty, t_0).$$

When $\mathcal{H}$ is the classical heat operator $\Delta - \partial_t$, the set $\Omega_r(z_0)$ in (1.6) coincides with the Pini-Watson ball already defined in Subsection 1.2.

Due to Sard's Lemma, $\nabla_\zeta\Gamma(z_0,\zeta) \neq 0$ when $\Gamma(z_0,\zeta) = \frac{1}{r}$, for almost every $r > 0$. Then, for all of these values of r, we have that

$$\partial\Omega_r(z_0) \smallsetminus \{z_0\} = \left\{\zeta \in \mathbb{R}^{N+1} : \Gamma(z_0,\zeta) = \tfrac{1}{r}\right\}$$

is an N-dimensional manifold of $\mathbb{R}^{n+1}$ of class C^∞.

For every $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$ with $\tau < t_0$, we also let

$$K_r(z_0, \zeta) := \frac{1}{r} \frac{\langle AD\Gamma, D\Gamma \rangle}{\Gamma^2}, \tag{1.7}$$

where $A = A(\zeta)$, $D = \nabla_\xi$, $\Gamma = \Gamma(z_0, \zeta)$.

Since A is nonnegative definite at any point, the kernel K_r is nonnegative. In what follows, we assume it is strictly positive in a dense subset of $\mathbb{R}^{N+1}$. Precisely, we suppose that

$(H) \qquad \{\zeta \in \mathbb{R}^N \times (-\infty, t_0) : \langle A(\zeta)D\Gamma(z_0, \zeta), D\Gamma(z_0, \zeta) \rangle = 0\}$

has an empty interior for every $z_0 \in \mathbb{R}^{N+1}$. In Section 2, we show that (H) is equivalent to

$(H)' \qquad \{\zeta \in \mathbb{R}^{N+1} : \Gamma(z_0, \zeta) = c\}$

has an empty interior $\forall z_0 \in \mathbb{R}^{N+1}$, $\forall c > 0$. Finally, we define

$$M_r(u)(z_0) := \int_{\Omega_r(z_0)} u(\zeta) \, K_r(z_0, \zeta) \, d\zeta. \tag{1.8}$$

With this integral operator at hand, we can characterize the solutions to $\mathcal{H}u = 0$ as follows.

Theorem 1.1. (*Gauss-Koebe-type Theorem for $\mathcal{H}$*) *Let $O \subset \mathbb{R}^{N+1}$ be open and let $u : O \to \mathbb{R}$ be a continuous function. Then, the following statements are equivalent:*

(i) *$u(z_0) = M_r(u)(z_0)$ for every $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$ such that $\overline{\Omega_r(z_0)} \subset O$.*

(ii) *u is smooth and $\mathcal{H}u = 0$ in O.*

We want to stress that the proof of this theorem does not require the assumption (H). In what follows, we call $\mathcal{H}$-*caloric functions* the (smooth) solutions to $\mathcal{H}u = 0$, i.e., equivalently, the continuous functions u satisfying the Mean Value property $u(z_0) = M_r(u)(z_0)$, on every $\mathcal{H}$-ball contained in their domain.

By applying the Mean Value property to the function $\zeta \to \Gamma(\zeta, z)$, which is $\mathcal{H}$-caloric in $\mathbb{R}^{N+1} \smallsetminus \{z\}$, we obtain

$$\int_{\Omega_r(z_0)} \Gamma(\zeta, z) \, K_r(z_0, \zeta) \, d\zeta = \Gamma(z_0, z), \qquad \forall z \in \mathbb{R}^{N+1} \smallsetminus \Omega_r(z_0). \tag{1.9}$$

The aim of this paper is to show a reverse, with respect to the domain, of the property (1.9). Our result applies to the $\mathcal{H}$-balls satisfying the following conditions:

(C1) $\Omega_r(z_0) = \operatorname{int} \overline{\Omega_r(z_0)}$ and $\partial \Omega_r(z_0)$ has Lebesgue measure equal to zero.

(C2) $\mathbb{R}^{N+1} \smallsetminus \overline{\Omega_r(z_0)}$ is *connected to infinity* by $\mathcal{H}^*$-*trajectories*.

If (C1) and (C2) are satisfied, we say that $\Omega_r(z_0)$ is a $\mathcal{CH}$-ball. The meaning of condition (C2) is explained in the following two definitions.

Definition 1.2. *We call $\mathcal{H}^*$-trajectory every continuous path $\gamma : [0, T] \to \mathbb{R}^{N+1}$ with the following property: for every open set $O \supset \{\gamma(s) : s \in [0, T]\}$, and for every $\mathcal{H}^*$-caloric function u in O such that*

$$u(\gamma(0)) = \max_O u,$$

one has $u(\gamma(s)) = u(\gamma(0))$ for every $s \in [0, T]$.

Definition 1.3. *We say that a set A is connected to infinity by $\mathcal{H}^*$-trajectories if, for every $z \in A$ and for every compact set $K \subset \mathbb{R}^{N+1}$, there exists an $\mathcal{H}^*$-trajectory $\gamma : [0, T] \to A$, such that $\gamma(0) \notin K$, $\gamma(T) = z$.*

Before stating our main results, we want to stress that they hold true by assuming, in summary, that

$$\text{the operator } \mathcal{H} \text{ satisfies (ND)}, (\Gamma) \text{ and } (H). \tag{1.10}$$

In Subsections 1.4 and 1.5, we show some explicit examples of operators $\mathcal{H}$ satisfying (1.10), and for which every $\mathcal{H}$-ball is a $\mathcal{CH}$-ball.

We are now in the position to state the main result of our paper.

Theorem 1.4. (*Suzuki-Watson-type Theorem for $\mathcal{H}$*) *Let $\Omega_r(z_0)$ be a $\mathcal{CH}$-ball, and let $D \subset \mathbb{R}^{N+1}$ be a bounded open set satisfying:*

(i) $\overline{D} \smallsetminus \{z_0\} \subset \mathbb{R}^N \times (-\infty, t_0)$,

(ii) $D \cap V = \Omega_r(z_0) \cap V$ *for a suitable neighborhood V of z_0,*

(iii) $\int_D \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta = \Gamma(z_0, z)$, $\forall z \in \mathbb{R}^{N+1} \smallsetminus D$.

Then, $D = \Omega_r(z_0)$.

Remark 1.5. Condition (iii) is trivially satisfied if $z = (x, t)$, $t \geq t_0$. Indeed, in this case, $\Gamma(z_0, z) = 0$ and, by condition (i), $\Gamma(\zeta, z) = 0$ for every $\zeta \in D$.

From this theorem, we easily get the following corollary.

Corollary 1.6. (*$\mathcal{H}$-caloric characterization of the $\mathcal{H}$-balls*) *Let $\Omega_r(z_0)$ be a $\mathcal{CH}$-ball, and let $D \subset \mathbb{R}^{N+1}$ be a bounded open set satisfying:*

(i) $\overline{D} \smallsetminus \{z_0\} \subset \mathbb{R}^N \times (-\infty, t_0)$,

(ii) $D \cap V = \Omega_r(z_0) \cap V$ *for a suitable neighborhood V of z_0,*

(iii) $u(z_0) = \int_D u(\zeta) K_r(z_0, \zeta) \, d\zeta$ *for every nonnegative $\mathcal{H}$-caloric function u in an open set containing $D \cup \{z_0\}$.*

Then, $D = \Omega_r(z_0)$.

As we have just said, we now give some examples of operators to which our results apply.

1.4. **Example 1. Heat equations on stratified Lie groups.** In $\mathbb{R}^{N+1}$ let us consider the operator

$$\mathcal{H} := \sum_{j=1}^{m} X_j^2 - \partial_t, \tag{1.11}$$

where

$$\mathcal{L} := \sum_{j=1}^{m} X_j^2$$

is a sub-Laplacian in $\mathbb{R}^N$. This means that the following conditions hold true:

(i) the X_j's are smooth vector fields in $\mathbb{R}^N$ generating a Lie algebra $\mathfrak{a}$ satisfying rank $\mathfrak{a}(x) = \dim \mathfrak{a} = N$ at any point $x \in \mathbb{R}^N$;

(ii) there exists a group of dilations $(\delta_\lambda)_{\lambda>0}$ in $\mathbb{R}^N$ such that every vector field X_j is δ_λ-homogeneous of degree one.

A group of dilations in $\mathbb{R}^N$ is a family of diagonal linear functions $(\delta_\lambda)_{\lambda>0}$ of the kind $\delta_\lambda(x_1, \ldots, x_N) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N)$, where the σ_j's are natural numbers.

The conditions (i) and (ii) imply the existence of a group law $\circ$ making $\mathbb{G} = (\mathbb{R}^N, \circ, \delta_\lambda)$ a stratified Lie group on which every vector field X_j is left translation invariant and δ_λ-homogeneous of degree one (see [4]). The rank condition in (i) also implies

rank Lie$\{X_1, \ldots, X_m, \partial_t\}(z) = N + 1$, at any point $z \in \mathbb{R}^{N+1}$.

Then, by Hörmander's Theorem in [10], $\mathcal{H}$ is hypoelliptic. On the other hand, due to homogeneity of the vector fields X_j's, we have $X_j^* = -X_j$ (see [6], Proposition 1.3.8). Then,

$$\mathcal{H}^* = \sum_{j=1}^{m} X_j^2 + \partial_t,$$

thus, the previous rank condition implies also the hypoellipticity of $\mathcal{H}^*$. Moreover, if we define

$$(x,t) \,\hat{\circ}\, (\xi, \tau) := (x \circ \xi, t + \tau) \text{ and } \hat{\delta}_\lambda(x,t) := (\delta_\lambda(x), \lambda^2 t),$$

then $\hat{\mathbb{G}} = (\mathbb{R}^{N+1}, \hat{\circ}, \hat{\delta}_\lambda)$ is a stratified Lie group with homogeneous dimension $Q + 2$, where $Q := \sigma_1 + \dots \sigma_N$ is the homogeneous dimension of $\mathbb{G}$. The operator $\mathcal{H}$ is left translation invariant on $\hat{\mathbb{G}}$ and $\hat{\delta}_\lambda$-homogeneous of degree two. Furthermore, $\mathcal{H}$ can be written as in (1.4), satisfies the divergence free condition (1.5) and the non-degeneracy condition (ND) (see [6], Proposition 1.3.8 and Section 1.5). Finally, $\mathcal{H}$ has a global fundamental solution Γ verifying all the properties $(\Gamma 1) - (\Gamma 7)$ listed in Section 2. This immediately follows from the existence and the estimates of Γ showed in [5].

We postpone to Section 4 the proof of the following proposition.

Proposition 1.7. *The fundamental solution of $\mathcal{H}$ in* (1.11) *satisfies condition* $(H)'$. *Moreover, every $\mathcal{H}$-ball is a $\mathcal{CH}$-ball.*

Then, Theorem 1.4 and Corollary 1.6 apply to every $\mathcal{H}$-ball related to the operator $\mathcal{H}$ in (1.11).

1.5. **Example 2. Kolmogorov-Fokker-Planck equations.** Let us now consider the class of operators in $\mathbb{R}^{N+1}$

$$\mathcal{H} := \operatorname{div}(A\nabla) + \langle x, B\nabla \rangle - \partial_t, \qquad (1.12)$$

where $A = (a_{i,j})_{i,j=1,\dots,N}$ and $B = (b_{i,j})_{i,j=1,\dots,N}$ are constant $N \times N$ matrices, A is symmetric and nonnegative definite. This class was introduced in [17], and subsequently studied by many authors as a basic model for general Kolmogorov-Fokker-Planck operators. If we define the matrix

$$C(t) = \int_0^t E(s) A E^T(s) \, ds, \text{ where } E(s) = \exp\left(-s B^T\right), \qquad (1.13)$$

then the operator $\mathcal{H}$ is hypoelliptic if and only if $C(t) > 0$ for every $t > 0$ (see [17]). Under these conditions, it is proved in [17] that, for some basis of $\mathbb{R}^N$, the matrices A, B take the following form:

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (1.14)$$

for some $p_0 \times p_0$ symmetric and positive definite constant matrix A_0 ($p_0 \leq N$), and

$$B = \begin{bmatrix} * & B_1 & 0 & \dots & 0 \\ * & * & B_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & * & \dots & B_r \\ * & * & * & \dots & * \end{bmatrix}, \qquad (1.15)$$

where B_j is a $p_{j-1} \times p_j$ block with rank p_j $(j = 1, 2, ..., r)$, $p_0 \geq p_1 \geq ... \geq p_r \geq 1$ and $p_0 + p_1 + ... + p_r = N$.

The operator in (1.12) satisfies the divergence-free condition (1.5) if and only if, as we assume, $\operatorname{Tr} B = 0$. Moreover, by the structure of the matrix A, $\mathcal{H}$ satisfies (ND), and the condition $C(t) > 0$ for $t > 0$ also implies the hypoellipticity of $\mathcal{H}^*$.

In [17], it is proved that the operator $\mathcal{H}$ is left-invariant with respect to the Lie group $\mathbb{K}$ whose underlying manifold is $\mathbb{R}^{N+1}$, endowed with the composition law

$$(x, t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau).$$

Note that $(\xi, \tau)^{-1} = (-E(-\tau)\xi, -\tau)$. Under the assumptions stated above, the operator $\mathcal{H}$ in (1.12) has a fundamental solution $\Gamma(z, \zeta) = \gamma(\zeta^{-1} \circ z)$ for $z, \zeta \in \mathbb{R}^{N+1}$, with

$$\gamma(x, t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \frac{(4\pi)^{-N/2}}{\sqrt{\det C(t)}} \exp\left(-\frac{1}{4}\left\langle C^{-1}(t)x, x\right\rangle\right) & \text{for } t > 0 \end{cases},$$

where $C(t)$ is as in (1.13). Recall that $C(t)$ is positive definite for all $t > 0$, hence, $\gamma \in C^\infty\left(\mathbb{R}^{N+1} \setminus \{0\}\right)$. With this explicit expression of Γ, it is easy to see that conditions $(\Gamma 1) - (\Gamma 7)$ in Section 2 are satisfied. We also have the following proposition, whose proof is postponed to Section 5.

Proposition 1.8. *The fundamental solution of $\mathcal{H}$ in* (1.12) *satisfies condition* $(H)'$. *Moreover, every $\mathcal{H}$-ball is a $\mathcal{CH}$-ball.*

Then, Theorem 1.4 and Corollary 1.6 apply to every $\mathcal{H}$-ball related to the operator $\mathcal{H}$ in (1.12).

2. Preliminary results

2.1. **The fundamental solution.** We call fundamental solution of the operator $\mathcal{H}$ in (1.4) a function $\Gamma : \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \longrightarrow \mathbb{R}$ with the following properties:

($\Gamma 1$) Γ is smooth out of $\{z = \zeta\}$;
($\Gamma 2$) Γ is nonnegative and $\Gamma((x, t), (\xi, \tau)) > 0$ if and only if $t > \tau$;
($\Gamma 3$) for any fixed $\zeta \in \mathbb{R}^{N+1}$

$$\limsup_{z \to \zeta} \Gamma(z, \zeta) = \limsup_{z \to \zeta} \Gamma(\zeta, z) = +\infty \qquad \text{and}$$

$$\lim_{|z| \to +\infty} \Gamma(z, \zeta) = \lim_{|z| \to +\infty} \Gamma(\zeta, z) = 0;$$

(Γ4) for any fixed $\zeta \in \mathbb{R}^{N+1}$, $\Gamma(\cdot, \zeta)$ and $\Gamma(\zeta, \cdot)$ are locally integrable in $\mathbb{R}^{N+1}$;

(Γ5) for every compactly supported $f \in L^\infty(\mathbb{R}^{N+1})$, the functions

$$z \mapsto \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) f(\zeta) \, d\zeta \qquad \text{and} \qquad \zeta \mapsto \int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) f(z) \, dz$$

are continuous in $\mathbb{R}^{N+1}$ and they are C^∞ if f is also C^∞;

(Γ6) for every $\varphi \in C_0^\infty(\mathbb{R}^{N+1})$ and every nonnegative compactly supported Radon measure μ, we have

$$\int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) \mathcal{H}^* \varphi(z) \, dz = -\varphi(\zeta),$$

$$\int_{\mathbb{R}^{N+1}} \Gamma(z, \zeta) \mathcal{H} \varphi(\zeta) \, d\zeta = -\varphi(z) \quad \text{and}$$

$$\mathcal{H} \int_{\mathbb{R}^{N+1}} \Gamma(\cdot, \zeta) \, d\mu(\zeta) = -\mu \qquad \text{in the sense of distributions;}$$

(Γ7) for any fixed $z = (x, t)$ and $c > 0$, we have

$$\int_{\mathbb{R}^N} \Gamma(z, (\xi, \tau)) \, d\xi = 1 \qquad \text{if } t > \tau$$

$$\text{and} \qquad \lim_{\tau \to t^-} \int_{\{\Gamma(z,(\xi,\tau)) \le c\}} \Gamma(z, (\xi, \tau)) \, d\xi = 0.$$

We note that by (Γ6), we have

$$\mathcal{H}_z \Gamma(z, \zeta) = -\delta_\zeta \quad \text{and} \quad \mathcal{H}_\zeta^* \Gamma(z, \zeta) = -\delta_z$$

in the sense of distributions, where δ_η denotes the Dirac measure supported at $\{\eta\}$. Furthermore, we want to explicitly remark that the conditions (Γ1),(Γ5) and (Γ6) make Γ a *very regular two-sided fundamental kernel*: this implies the hypoellipticity of $\mathcal{H}$ and of $\mathcal{H}^*$ (see [20], page 540).

2.2. Maximum Principles.
We start with a classical Picone-type maximum principle.

Proposition 2.1. *Let $O \subseteq \mathbb{R}^{N+1}$ be a bounded open set and let $u : O \longrightarrow \mathbb{R}$ be a C^2-function such that*

$$\begin{cases} \mathcal{H}u \ge 0 & in\ O \\ \limsup_{x \to y} u(x) \le 0 & \forall\, y \in \partial O \end{cases} \quad or \quad \begin{cases} \mathcal{H}^* u \ge 0 & in\ O \\ \limsup_{x \to y} u(x) \le 0 & \forall\, y \in \partial O \end{cases}.$$

Then, $u \le 0$ in O.

Proof. It immediately follows from the condition (ND) and Corollary 1.3 in [13]. □

Proposition 2.2. *Let $O \subseteq \mathbb{R}^{N+1}$ be a bounded open set and let $u : \overline{O} \longrightarrow \mathbb{R}$ be a continuous function such that $\mathcal{H}u \geq 0$ in O in the weak sense of distributions. Then, we get*

$$\max_{\overline{O}} u = \max_{\partial O} u. \tag{2.1}$$

Proof. Since O is bounded, there exists $T \in \mathbb{R}$ such that $O \subseteq \mathbb{R}^N \times (T, +\infty)$. For every $\varepsilon > 0$, we put

$$u_\varepsilon(x, t) := u(x, t) - \varepsilon t \ \text{ for } (x, t) \in O.$$

As a consequence, by the Poisson-Jensen-type formula proved in [16], Theorem 1.6[2], we have $u_\varepsilon(z) < M_r(u_\varepsilon)(z)$ for every $\mathcal{H}$-ball $\Omega_r(z)$ with closure contained in O. Here, M_r is the mean value operator defined in (1.8). Now, let $\tilde{z} \in \overline{O}$ be such that $u_\varepsilon(\tilde{z}) = \max_{\overline{O}} u_\varepsilon$. If $\tilde{z} \in O$, then for any $r > 0$ sufficiently small, we would have

$$0 < M_r(u_\varepsilon)(\tilde{z}) - u_\varepsilon(\tilde{z}) = M_r(u_\varepsilon - u_\varepsilon(\tilde{z}))(\tilde{z}) \leq 0$$

since $u_\varepsilon - u_\varepsilon(\tilde{z}) \leq 0$ in O. This contradiction shows that $\tilde{z} \in \partial O$, that is,

$$\max_{\overline{O}} u_\varepsilon = \max_{\partial O} u_\varepsilon.$$

By letting $\varepsilon \to 0^+$, we obtain (2.1). □

Remark 2.3. The previous proposition still holds if we replace $\mathcal{H}$ with $\mathcal{H}^*$.

2.3. **Some Potential Theory results.** A bounded open set V is called $\mathcal{H}$-regular if the boundary value problem

$$\begin{cases} \mathcal{H}u = 0 & \text{in } V \\ u = \varphi & \text{on } \partial V \end{cases}$$

has a solution $u \in C^\infty(V) \cap C(\overline{V})$ for every continuous boundary data $\varphi : \partial V \longrightarrow \mathbb{R}$. The maximum principle of Proposition 2.1 implies that the solution is unique. It will be denoted by H_φ^V. The definitions of $\mathcal{H}^*$-regular set and $(H^*)_\varphi^V$ are completely analogous.

---

[2]The operators in [15] and [16] are assumed to coincide with the heat operator outside a compact set. This is only a technical hypothesis used to prove the existence of a global fundamental solution satisfying our assumptions.

The hypoellipticity of $\mathcal{H}$ and $\mathcal{H}^*$ and the non-degeneracy condition (ND) imply the existence of families of $\mathcal{H}$-regular and $\mathcal{H}^*$-regular open sets forming a basis of the Euclidean topology (see [7], Corollary 5.2[3]; see also [15], p. 86).

2.4. $(H) \Leftrightarrow (H)'$. Assume (H) is satisfied and, by contradiction, assume $(H)'$ is false. Then, for a suitable $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ and $c > 0$, the Γ-level set $\left\{\zeta \in \mathbb{R}^{N+1} : \Gamma(z_0, \zeta) = c\right\}$ contains a non-empty open set $O \subseteq \mathbb{R}^N \times (-\infty, t_0)$. As a consequence, $\nabla_\zeta \Gamma(z_0, \zeta) = 0$ for every $\zeta \in O$. Thus, keeping in mind the very definition of K_r, $K_r(z_0, \zeta) = 0$ for every $\zeta \in O$. This contradicts (H) because $O \subseteq \mathbb{R}^N \times (-\infty, t_0)$. Then, we have proved the implication $(H) \Rightarrow (H)'$. To prove the reverse one, let us assume $(H)'$ is true and assume, by contradiction, that (H) is false. Then, for a suitable $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$, there is a non-empty open set $O \subseteq \mathbb{R}^N \times (-\infty, t_0)$ such that

$$\langle A(\zeta)\nabla_\xi \Gamma(z_0, \zeta), \nabla_\xi \Gamma(z_0, \zeta)\rangle = 0 \quad \forall \zeta \in O.$$

Since $\zeta \longmapsto \Gamma(z_0, \zeta)$ solves the equation $\mathcal{H}^* u = 0$, this implies $Y_\zeta \Gamma(z_0, \zeta) = 0$ for every $\zeta \in O$. Using the hypoellipticity of $\mathcal{H}^*$ and arguing as in [1], we can prove that $\nabla_\zeta \Gamma(z_0, \zeta) \equiv 0$ in an open set $\hat{O}$ dense in O. Then, since $\hat{O} \subseteq \mathbb{R}^N \times (-\infty, t_0)$, for a suitable $c > 0$, the level set $\{\zeta \in \mathbb{R}^N \times (-\infty, t_0) : \Gamma(z_0, \zeta) = c\}$ has non-empty interior. This contradicts (H)' and completes the proof.

2.5. **Propagations along $\mathcal{H}^*$-trajectories.** A locally-Lipschitz continuous vector field $X = (a_1, \ldots, a_N)$, identified with the first order differential operator $\sum_{j=1}^N a_j \partial_{x_j}$ is called $\mathcal{H}$-subunit if

$$\langle X(z), \xi\rangle^2 \leq \langle A(z)\xi, \xi\rangle \quad \forall z \in \mathbb{R}^{N+1}, \forall \xi \in \mathbb{R}^N.$$

It is quite well known that every integral curve of a $\mathcal{H}$-subunit vector field, as well as every integral curve of $-Y$, where Y is the drift term in (1.5), are $\mathcal{H}^*$-trajectories (see [3], see also [13]). Our proof of Theorem 1.4 will require the following lemma.

Lemma 2.4. *Let $O \subseteq \mathbb{R}^{N+1}$ be an open set and let $u : O \longrightarrow \mathbb{R}$ be a continuous function such that $\mathcal{H}^* u \geq 0$ in O in the weak sense of distributions. Assume there exists $z_0 \in O$ such that $u(z_0) = \max_O u$. Then, if*

---

[3]In [7] the involved operators are assumed to be "sum of squares plus a drift" satisfying the Hörmander rank condition. Actually, this assumption is just needed to state the hypoellipticity of the operators.

$\gamma : [0, T] \longrightarrow \mathbb{R}^{N+1}$ *is a $\mathcal{H}^*$-trajectory contained in O such that $\gamma(0) = z_0$,*
we have $u(\gamma(s)) = u(z_0)$ for every $s \in [0, T]$.

Proof. Define $I := \{s \in [0, T] : u(\gamma(s)) = u(z_0)\}$. Since $\gamma(0) = z_0, 0 \in I$.
We have to prove that $I = [0, T]$. By contradiction, assume there exists
$s_0 \in [0, T]$ such that $s_0 \notin I$. Let us put $I_{s_0} := I \cap [0, s_0]$. Since u is
continuous, I_{s_0} is closed, so that $s_1 := \sup I_{s_0} < s_0$. Therefore, it follows
that $s_1 \in I_{s_0}$ and $s \notin I_{s_0}$ for $s_1 < s \leq s_0$. Denote $z_1 := \gamma(s_1)$. Then,
$u(z_1) = \max_O u$. Let V be an open set, $\mathcal{H}^*$-regular for the Dirichlet problem,
such that $z_1 \in V \subseteq \overline{V} \subseteq O$, $\gamma(s_0) \notin \overline{V}$ (see Subsection 2.3). Let $\sigma^* \in (s_1, s_0)$
be such that $\gamma(\sigma^*) \in \partial V$. Obviously, $\sigma^* \notin I$, i.e.,

$$u(\gamma(\sigma^*)) < u(z_0) = \max_O u. \tag{2.2}$$

Consider now the function $h : \overline{V} \longrightarrow \mathbb{R}$, $h = (H^*)^V_{u|\partial V}$ in V, $h = u$ on ∂V.
Then, h is continuous on $\overline{V}$ and $\mathcal{H}^*(u - h) \geq 0$ in V, $u - h = 0$ on ∂V. The
maximum principle of Proposition 2.2 implies

$$u \leq h \text{ in } V, \quad \max_{\overline{V}} h = \max_{\partial V} h = \max_{\partial V} u \leq \max_O u.$$

Therefore, keeping in mind that $z_1 \in V$ and that $u(z_1) = \max_O u$, we have
$u(z_1) \leq h(z_1) \leq u(z_1)$, that is, $u(z_1) = h(z_1)$. This implies $\max_V h = h(z_1) \equiv$
$h(\gamma(s_1))$, so that, since γ is an $\mathcal{H}^*$-trajectory, $h(\gamma(s)) = h(z_1) \ \forall s \in [s_1, \sigma^*)$.
Since $\gamma(\sigma^*) \in \partial V$, the continuity of h up to ∂V implies $h(\gamma(\sigma^*)) = h(z_1)$. On
the other hand, $h = u$ on ∂V and $h(z_1) = u(z_1) = \max_O u$. Then, $u(\gamma(\sigma^*)) =$
$\max_O u$, in contradiction with (2.2). Thus, the proof is complete. $\square$

2.6. **A maximum principle for functions satisfying the mean value
property.** The aim of this subsection is to prove the following proposition.

Proposition 2.5. *Let $O \subseteq \mathbb{R}^{N+1}$ be a bounded open set and let $u : O \longrightarrow \mathbb{R}$
be a continuous function satisfying*

$$u(z) = M_r(u)(z) \tag{2.3}$$

for every $z \in O$ and $r > 0$ such that $\overline{\Omega_r(z)} \subseteq O$. Assume, moreover, that

$$\lim_{z \longrightarrow \zeta} u(z) \leq 0 \quad \forall \zeta \in \partial O. \tag{2.4}$$

Then, $u \leq 0$ in O.

Proof. We argue by contradiction and assume $u > 0$ somewhere in O. Then,
there exists $\max_O u := m$ with $m > 0$. Define

$$T := \{t \in \mathbb{R} \ : \ \exists \ x \in \mathbb{R}^N \text{ such that } z = (x, t) \in O \text{ and } u(z) = m\}.$$

Obviously, T is bounded and exists a sequence $\{z_j = (x_j, t_j)\}_{j \in \mathbb{N}}$ in O such that $u(z_j) = m$, $T \ni t_j \longrightarrow t^* := \inf T$, $x_j \longrightarrow x^*$. Denote $z^* := (x^*, t^*)$. Obviously, $z^* \in \overline{O}$. If $z^* \in \partial O$, we would have by (2.4) that

$$0 \geq \lim_{j \longrightarrow \infty} u(z_j) = m,$$

which is a contradiction. Then, $z^* \in O$ and for every $r > 0$ sufficiently small with $\overline{\Omega_r(z^*)} \subseteq O$, we get by (2.3) $u(z^*) = M_r(u)(z^*)$. Since $M_r(1)(z^*) = 1$, this identity implies $0 = M_r(u - u(z^*))(z^*)$. Moreover, $u - u(z^*) \leq 0$ in $\Omega_r(z^*)$. Then, keeping in mind condition (H), $u(z) = u(z^*)$ for every z in an open set dense in $\Omega_r(z^*)$. In particular, there exists a point $z' = (x', t') \in \Omega_r(z^*)$ such that $u(z') = u(z^*) = m$. The very definition of T implies $t' \in T$, but this is a contradiction, since $t' < t^* = \inf T$. □

2.7. **Proof of Theorem 1.1.** The implication (ii)$\Longrightarrow$(i) can be proved verbatim proceeding as in [16], pages 308-313.

Here, we only prove the implication (i)$\Longrightarrow$(ii). Let $u : O \longrightarrow \mathbb{R}$ be a continuous function satisfying the mean value property (i) and let V be an open set with closure contained in O, which is $\mathcal{H}$-regular for the Dirichlet problem. Consider, in V, the function $w = u|_V - H_{u|_{\partial V}}^V$. Obviously, $w(z) = M_r(w)(z)$ for every $z \in U$ and $r > 0$ such that $\overline{\Omega_r(z)} \subseteq V$. Moreover,

$$\lim_{z \longrightarrow \zeta} w(z) = 0 \qquad \forall \zeta \in \partial V.$$

Then, by the maximum principle in Proposition 2.5, $w = 0$ in V, that is, $u = u|_V - H_{u|_{\partial V}}^V$. Hence, u is smooth in V and it solves $\mathcal{H}u = 0$ in V. Since the family $\{V\}$ of the $\mathcal{H}$-regular open sets for the Dirichlet problem is a basis of the Euclidean topology, it follows that u is C^∞ in O and it solves the equation $\mathcal{H}u = 0$ in O. The proof is complete.

3. Proof of the main results

This section is mainly devoted to the proof of Theorem 1.4.

Proof of Theorem 1.4. The core idea of the proof is to compare the functions

$$v(z) := \int_D \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta \quad \text{and} \quad v_0(z) := \int_{\Omega_r(z_0)} \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta.$$

The assumption (iii) of the Theorem and the relation (1.9) imply that

$$v(z) = \Gamma(z_0, z) \quad \forall z \in \mathbb{R}^{N+1} \smallsetminus D$$

and

$$v_0(z) = \Gamma(z_0, z) \ \ \forall z \in \mathbb{R}^{N+1} \smallsetminus \Omega_r(z_0).$$

Then, if we define $u(z) := \Gamma(z_0, z) - v(z)$ and $u_0(z) := \Gamma(z_0, z) - v_0(z)$, we have

$$u = 0 \text{ in } \mathbb{R}^{N+1} \smallsetminus D \ \text{ and } \ u_0 = 0 \text{ in } \mathbb{R}^{N+1} \smallsetminus \Omega_r(z_0). \tag{3.1}$$

In what follows we also put

$$K(A) := \int_A K_r(z_0, \zeta) \ d\zeta,$$

for every measurable $A \subseteq \mathbb{R}^{N+1}$: we can consider that the kernel $K_r(z_0, \cdot)$ vanishes in the upper half-space $\mathbb{R}^N \times [t_0, +\infty)$. By condition (H), we know that

$$K(A) > 0 \ \text{ if } A \text{ is an open and non-empty subset of } \mathbb{R}^N \times (-\infty, t_0). \tag{3.2}$$

Now, we proceed by splitting our argument into several steps.

Step I. *We have $K(D) = 1 = K(\Omega_r(z_0))$.* The second identity simply follows from the Mean Value property for $\mathcal{H}$-caloric functions applied to the function $w \equiv 1$. To prove the first one, we verbatim proceed as in [19] (page 2711). Since D is bounded, there exists $\tau < t_0$ such that $D \subseteq \mathbb{R}^N \times (\tau, +\infty)$. Then, we get

$$1 = \int_{\mathbb{R}^N} \Gamma(z_0; \xi, \tau) \, d\xi = \int_{\mathbb{R}^N} \Big(\int_D \Gamma(z; \xi, \tau) K_r(z_0, z) \, dz \Big) \, d\xi$$
$$= \int_D \Big(\int_{\mathbb{R}^N} \Gamma(z; \xi, \tau) d\xi \Big) K_r(z_0, z) \, dz = \int_D K_r(z_0, \zeta) \, dz.$$

In the first and in the last equality, we have used the property $(\Gamma 7)$. The second one follows from the assumption (iii).

Step II. *The function u_0 is strictly positive in $\Omega_r(z_0)$.* Let us fix $\zeta \in \Omega_r(z_0)$ and let $w(z) := \Gamma(z, \zeta)$. In the weak sense of distributions, we have $\mathcal{H}w = -\delta_\zeta$. Then, for the Poisson-Jensen-type formula proved in [16], the following identity holds true:

$$w(z_0) = \int_{\Omega_r(z_0)} w(z) K_r(z_0, z) \ dz + \frac{1}{r} \int_0^r \Big(\int_{\Omega_\rho(z_0)} \big(\Gamma(z_0, z) - \tfrac{1}{\rho} \big) \ d\delta_\zeta(z) \Big) d\rho.$$

Since $\zeta \in \Omega_r(z_0)$, we have $\Gamma(z_0, \zeta) > \frac{1}{r}$, so that $\Gamma(z_0, \zeta) - \frac{1}{\rho} > 0$ for every $\rho \in \big(\frac{1}{\Gamma(z_0, \zeta)}, r \big)$. Then, we get

$$w(z_0) > \int_{\Omega_r(z_0)} w(z) K_r(z_0, z) \ dz.$$

This means $u_0(\zeta) > 0$, since

$$u_0(\zeta) = \Gamma(z_0, \zeta) - \int_{\Omega_r(z_0)} \Gamma(z, \zeta) K_r(z_0, z) \, dz = w(z_0) - \int_{\Omega_r(z_0)} w(z) K_r(z_0, z) \, dz.$$

Step III. *If $D \subseteq \overline{\Omega_r(z_0)}$, then $D = \Omega_r(z_0)$.* Since $\partial \Omega_r(z_0)$ has Lebesgue measure equal to zero, for every $z \in \mathbb{R}^{N+1}$, we have

$$v_0(z) = \int_{\Omega_r(z_0)} \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta = \int_{\overline{\Omega_r(z_0)}} \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta.$$

On the other hand, we have $\overline{\Omega_r(z_0)} = D \cup \left(\overline{\Omega_r(z_0)} \smallsetminus D \right)$ and

$$K\left(\overline{\Omega_r(z_0)} \smallsetminus D \right) = K\left(\overline{\Omega_r(z_0)} \right) - K(D) = K(\Omega_r(z_0)) - K(D) = 0$$

by Step I. Then, we get

$$\int_{\overline{\Omega_r(z_0)}} \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta = \int_D \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta = v(z).$$

Thus, we have proved that $v_0 = v$ in $\mathbb{R}^{N+1}$ and, thus, $u_0 = u$ in $\mathbb{R}^{N+1}$. Now, since $u_0 > 0$ in $\Omega_r(z_0)$ by Step II, the function u is strictly positive in $\Omega_r(z_0)$. However, we know that $u = 0$ in $\mathbb{R}^{N+1} \smallsetminus D$ by (3.1). Hence, $\mathbb{R}^{N+1} \smallsetminus D \subseteq \mathbb{R}^{N+1} \smallsetminus \Omega_r(z_0)$, that is, $\Omega_r(z_0) \subseteq D$. On the other hand, by the assumption of this Step we know also that $D \subseteq \text{int}(\overline{\Omega_r(z_0)}) = \Omega_r(z_0)$ since D is open and $\Omega_r(z_0)$ is a $\mathcal{CH}$-ball. It follows that $\Omega_r(z_0) = D$.

Step IV. In this final step, we prove that $D \subseteq \overline{\Omega_r(z_0)}$. Thanks to the previous Step III, this will complete the proof of the Theorem. We argue, by contradiction, by supposing

$$D \smallsetminus \overline{\Omega_r(z_0)} \neq \emptyset. \tag{3.3}$$

This assumption implies the following:
Claim. $U := u - u_0$ *is strictly positive somewhere.* We take this claim for granted, for a moment, and argue as follows. Hypothesis (ii) of the Theorem gives

$$U(z) = u(z) - u_0(z) = v_0(z) - v(z)$$
$$= \Big(\int_{\Omega_r(z_0) \smallsetminus V} - \int_{D \smallsetminus V} \Big) \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta$$

for every $z \in \mathbb{R}^{N+1}$. By (3.1), the function U is identically zero in a neighborhood of ∞, and, by our assumption ($\Gamma 5$) on the fundamental solution Γ, it is continuous in $\mathbb{R}^{N+1}$. Indeed, the set $F := (\Omega_r(z_0) \smallsetminus V) \cup (D \smallsetminus V)$ is

bounded and it has the closure contained in $\mathbb{R}^N \times (-\infty, t_0)$ (see condition (i), and the properties of the $\mathcal{H}$-balls). Thus, the function $K_r(z_0, \cdot)$ is bounded on F and we can exploit ($\Gamma 5$).

Then, there exists $m := \max_{\mathbb{R}^{N+1}} U$ and $m > 0$ (by the Claim). By following Suzuki and Watson's idea ([19], page 2712), we define

$$E := \{z \in \mathbb{R}^{N+1} : U(z) = m\}.$$

We now remark that, by *Step II* and the second identity in (3.1), the function u_0 is nonnegative everywhere. Therefore, for every $z \in E$, we have

$$u(z) = U(z) + u_0(z) \geq m,$$

so that, since $m > 0$, we have $u > 0$ in E. As a consequence, the first identity in (3.1) implies $E \subseteq D$. On the other hand, due to our assumptions on Γ,

$$\mathcal{H}^* U = (\mathbb{I} - \mathbb{I}_{\Omega_r(z_0)}) K_r(z_0, \cdot) \geq 0 \text{ in } D$$

in the weak sense of distributions. Then, by the Maximum Principle of Proposition 2.2, there exists a point $z^* \in \partial D$ such that $U(z^*) = \max_{\overline{D}} U$. Then, keeping in mind that $E \subseteq D$,

$$m \leq \max_{\overline{D}} U = U(z^*) \leq \max_{\mathbb{R}^{N+1}} U = m.$$

Hence, $U(z^*) = m$, that is, $z^* \in E$. This contradicts the inclusion $E \subseteq D$, since $z^* \in \partial D$. Thus, (3.3) is false and $D \subseteq \overline{\Omega_r(z_0)}$, as we wanted to prove. We are, thus, left with the proof of the *Claim*, under the hypothesis (3.3).

Let us argue again by contradiction and suppose $U \leq 0$ everywhere. Let $z \in D \setminus \overline{\Omega_r(z_0)}$ be fixed. By the assumption (C2), there exists an $\mathcal{H}^*$-trajectory $\gamma : [0, T] \longrightarrow \mathbb{R}^{N+1} \setminus \overline{\Omega_r(z_0)}$ such that $\gamma(0) \notin \overline{D}$ and $\gamma(T) = z$. Then, there exists $s_0 \in]0, T[$ such that $\gamma(s) \in D$ for every $s \in (s_0, T]$, $\tilde{z} := \gamma(s_0) \in \partial D$. Keeping in mind that $u_0 = 0$ in $\mathbb{R}^{N+1} \setminus \Omega_r(z_0)$, we have $U = u$ in $\mathbb{R}^{N+1} \setminus \overline{\Omega_r(z_0)}$, so that, since $\tilde{z} \in \partial D$ and $u = 0$ in $\mathbb{R}^{N+1} \setminus D$, $U(\tilde{z}) = u(\tilde{z}) = 0$. Hence, $\tilde{z}$ is a maximum point of U. We also know that U is $\mathcal{H}^*$-subharmonic outside $\overline{\Omega_r(z_0)}$. It follows by Lemma 2.4 that $U(\gamma(s)) = U(\gamma(s_0)) = U(\tilde{z}) = 0$ for any $s \in [s_0, T]$. In particular, we get $U(z) = U(\gamma(T)) = U(\gamma(s_0)) = 0$. Thus, we have proved that $U(z) = 0$ for every $z \in D \setminus \overline{\Omega_r(z_0)}$. We deduce, as a consequence, $K_r(z_0, z) = \mathcal{H}^* U(z) = 0$ in $D \setminus \overline{\Omega_r(z_0)}$. This is a contradiction, since $D \setminus \overline{\Omega_r(z_0)}$ is a non-empty open set and $\{K_r(z_0, \cdot) = 0\}$ has empty interior by (H). Therefore, the proof is complete. $\qquad\square$

As we have already stated, Theorem 1.4 has a straightforward (but remarkable) corollary. Here, we give the proof.

Proof of Corollary 1.6. By the previous theorem and Remark 1.5, it is enough to prove that

$$\int_D \Gamma(\zeta, z) K_r(z_0, \zeta) \, d\zeta = \Gamma(z_0, z) \tag{3.4}$$

for every $z = (x, t) \in \mathbb{R}^{N+1} \smallsetminus D, t < t_0$. Now, if we fix $z = (x, t) \in \mathbb{R}^{N+1} \smallsetminus D$ with $t < t_0$, the function $\zeta \longmapsto u_z(\zeta) := \Gamma(\zeta, z)$ is nonnegative and $\mathcal{H}$-harmonic in an open set containing $D \cup \{z_0\}$. Then, by the hypothesis (iii) of the Corollary,

$$u_z(z_0) = \int_D u_z(\zeta) K_r(z_0, \zeta) \, d\zeta.$$

If we replace in this identity u_z with $\Gamma(\cdot, z)$, we obtain (3.4) completing the proof. $\square$

4. Proof of Proposition 1.7

In this section, we are going to use the notations introduced in Subsection 1.4. By Sard's Lemma

$$\hat{\partial}\Omega_r(z_0) := \left\{\zeta \in \mathbb{R}^{N+1} : \Gamma(z_0, \zeta) = \tfrac{1}{r}\right\}$$

is a smooth manifold for almost every $r > 0$. On the other hand,

$$\hat{\partial}\Omega_r(z_0) := z_0 \circ \delta_r(\hat{\partial}\Omega_1(0)) \quad \text{for every } r > 0. \tag{4.1}$$

Then, $\hat{\partial}\Omega_1(z_0)$ is a smooth manifold. Using again (4.1), we deduce that $\hat{\partial}\Omega_r(z_0)$ is a smooth manifold and $\hat{\partial}\Omega_r(z_0) = \partial\Omega_r(z_0) \smallsetminus \{z_0\}$ for every $r > 0$ and $z_0 \in \mathbb{R}^{N+1}$. In particular, (H') and $(C1)$ are satisfied. In order to prove $(C2)$, we first remark that every integral curve of a vector field $X \in \text{Lie}\{X_1, \ldots, X_m\}$ is a $\mathcal{H}^*$-trajectory, as well as every path of the kind $s \mapsto (x_0, \psi(s))$ with ψ increasing (see [3], [13]). As a consequence, since $\text{Lie}\{X_1, \ldots, X_m\}$ has rank N at any point of $\mathbb{R}^N$, every path of the kind $s \mapsto (x(s), s_0 + s)$ is a $\mathcal{H}^*$-trajectory. Let us now prove that $\Omega_1(0)$ satisfies $(C2)$. By the left-translation invariance and the $\hat{\delta}_\lambda$-homogeneity of $\mathcal{H}$, it will follow that $(C2)$ holds true for every $\Omega_r(z_0)$. Let $z = (x, t) \in \mathbb{R}^{N+1} \smallsetminus \overline{\Omega_1(0)}$ and let $K \subseteq \mathbb{R}^{N+1}$ be a fixed compact set. If $t \geq 0$, we can build up a path

connecting z to the infinity keeping fixed the t-variable. Thus, assume $t < 0$. Since Γ is $\hat{\delta}_\lambda$-homogeneous of degree $-Q$ and $z \notin \overline{\Omega_1(0)}$, we have

$$\Gamma(0, \hat{\delta}_\lambda(z)) = \lambda^{-Q} \Gamma(0, z) < \lambda^{-Q}.$$

Hence, $\hat{\delta}_\lambda(z) \notin \overline{\Omega_1(0)}$ for any $\lambda \geq 1$ and we can choose $T > 0$ such that $\hat{\delta}_\lambda(z) \notin K$ for every $\lambda \geq T$. Let us now consider the path $\gamma : [0, T] \longrightarrow \mathbb{R}^{N+1}$, $\gamma(s) = \hat{\delta}_{\varphi(s)}(z) = \left(\delta_{\varphi(s)}(x), \varphi^2(s)t \right)$, where $\varphi(s) = T + 1 - s$. Since t is negative, the map $s \mapsto \varphi^2(s)t$ is strictly increasing. Hence, γ is a $\mathcal{H}^*$-trajectory. Moreover, being $\varphi \geq 1$, we get $\gamma(s) \notin \overline{\Omega_1(0)}$ for any $s \in [0, T]$. By definition, we also have $\gamma(T) = z$ and, reminding that $\varphi(0) > T$, $\gamma(0) \notin K$. Therefore, we have proved that $\Omega_1(0)$ satisfies (C2), completing the proof.

5. Proof of Proposition 1.8

As in the previous section, we exploit here the correspondent notations of Subsection 1.5. The operator $\mathcal{H}$ is left-translation invariant; therefore, it is enough to verify $(H)'$ and the conditions on the $\mathcal{H}$-balls just for $z_0 = (0, 0)$. To this end, for $\zeta = (\xi, \tau) \in \mathbb{R}^{N+1}$, we let

$$\hat{\gamma}(\zeta) := \Gamma(0, \zeta) = \gamma \left(\zeta^{-1} \right) = \gamma \left(-E(-\tau)\xi, -\tau \right).$$

Then, from the very definition of γ, we get

$$\nabla_\xi \hat{\gamma}(\zeta) = -\tfrac{1}{2} E(-\tau)^T C^{-1}(-\tau) E(-\tau) \xi.$$

Hence, keeping in mind that $C^{-1}(-\tau)$ is strictly positive definite and that $E(-\tau)$ is non-singular, we obtain

$$\{\zeta \in \mathbb{R}^N \times (-\infty, 0) : \nabla_\zeta \hat{\gamma}(\zeta) = 0\} \subseteq \{(\xi, \tau) \in \mathbb{R}^N \times (-\infty, 0) : \xi = 0\}.$$

This immediately shows that $(H)'$ and (C1) are verified. To prove (C2), we first remark that the paths

$$s \mapsto \alpha \pm s e_1, \quad e_1 = (1, 0, \ldots, 0), \quad \alpha \in \mathbb{R}^{N+1},$$

are $\mathcal{H}^*$-trajectories (see [3], [13]). From the explicit expression of $\hat{\gamma}$, we also immediately see that $\Omega_r(0) \cap \{\tau = \tau_0\}$, $\tau_0 \in \mathbb{R}$, is convex, or empty. Then, if $z = (x_1, \ldots, x_N, t) \notin \overline{\Omega_r(0)}$, one of the following half-lines

$$s \mapsto (x_1 + s, x_2, \ldots, x_N, t), \quad s \geq 0,$$
$$s \mapsto (x_1 - s, x_2, \ldots, x_N, t), \quad s \geq 0$$

is contained in $\mathbb{R}^{N+1} \setminus \overline{\Omega_r(0)}$. This proves that every $z \notin \overline{\Omega_r(0)}$ is connected to infinity by a $\mathcal{H}^*$-trajectory. Thus, the proof is complete.

Acknowledgment. The first author worked on this paper during her visit at BCAM (the Basque Center for Applied Mathematics). She is grateful to Professor Enrique Zuazua for the precious hospitality. We wish to thank the referee for his criticism that led us to improve the presentation of the paper.

References

[1] B. Abbondanza and A. Bonfiglioli, *The Dirichlet problem and the inverse mean value theorem for a class of divergence form operators*, J. Lond. Math. Soc., 87 (2013), 321-346.

[2] D. Aharonov, M.M. Schiffer, and L. Zalcman, *Potato kugel*, Israel J. Math., 40 (1981), 331-339.

[3] K. Amano, *Maximum principles for degenerate elliptic-parabolic operators*, Indiana Univ. Math. J., 28 (1979), 545-557.

[4] A. Bonfiglioli and E. Lanconelli, *On left-invariant Hörmander operators in $\mathbb{R}^N$. Applications to Kolmogorov-Fokker-Planck equations*, J. Math. Sci., 171 (2010), 22-33.

[5] A. Bonfiglioli, E. Lanconelli, andF. Uguzzoni, *Uniform Gaussian estimates of the fundamental solutions for heat operators on Carnot groups*, Adv. Differential Equations, 7 (2002), 1153-1192.

[6] A. Bonfiglioli, E. Lanconelli, and F. Uguzzoni, "Stratified Lie Groups and Potential Theory for their Sub-Laplacians," Springer Monographs in Mathematics, Springer, Berlin, 2007.

[7] J.M. Bony, *Principe du maximum, inégalité de Harnack et unicité du probleme de Cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble), 19 (1969), 277-304.

[8] E.B. Fabes and N. Garofalo, *Mean value properties of solutions to parabolic equations with variable coefficients*, J. Math. Anal. Appl., 121 (1987), 305-316.

[9] N. Garofalo and E. Lanconelli, *Wiener's criterion for parabolic equations with variable coefficients and its consequences*, Trans. Amer. Math. Soc., 308 (1988), 811-836.

[10] L. Hörmander, *Hypoelliptic second order differential equations*, Acta Math., 119 (1967), 147-171.

[11] A.E. Kogoj and E. Lanconelli, *An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations*, Mediterr. J. Math., 1 (2004), 51-80.

[12] L.P. Kupcov, *The mean property and the maximum principle for second order parabolic equations*, Soviet Math. Dokl., 19 (1978), 1140-1144.

[13] E. Lanconelli, *Maximum Principles and symmetry results in sub-Riemannian settings*, in "Symmetry for elliptic PDEs", Contemp. Math., Amer. Math. Soc., 528 (2010), 17-33.

[14] E. Lanconelli, *"Potato kugel" for sub-Laplacians*, Israel J. Math., 194 (2013), 277-283.

[15] E. Lanconelli and A. Pascucci, *On the fundamental solution for hypoelliptic second order partial differential equations with non-negative characteristic form*, Ricerche Mat., 48 (1999), 81-106.

[16] E. Lanconelli and A. Pascucci, *Superparabolic functions related to second order hypoelliptic operators*, Potential Anal., 11 (1999), 303-322.

[17] E. Lanconelli and S. Polidoro, *On a class of hypoelliptic evolution operators*, Rend. Sem. Mat. Univ. Politec. Torino, 52 (1994), 29-63.

[18] B. Pini, *Sulle equazioni a derivate parziali, lineari del secondo ordine in due variabili, di tipo parabolico*, Ann. Mat. Pura Appl., 32 (1951), 179-204.

[19] N. Suzuki and N.A. Watson, *A characterization of heat balls by a mean value property for temperatures*, Proc. Amer. Math. Soc., 129 (2001), 2709-2714.

[20] F. Treves, "Topological vector spaces, distributions and kernels", Academic Press, New-York, 1967.

[21] N.A. Watson, *A theory of subtemperatures in several variables*, Proc. London Math. Soc., 26 (1973), 385-417.