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Asymptotic Average Solutions to Linear Second Order Semi-Elliptic PDEs: A Pizzetti-Type Theorem

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Abstract

By exploiting an old idea first used by Pizzetti for the classical Laplacian, we introduce a notion of *asymptotic average solutions* making pointwise solvable every Poisson equation $\mathcal{L}u(x) = -f(x)$ with continuous data f, where $\mathcal{L}$ is a hypoelliptic linear partial differential operator with positive semidefinite characteristic form.

Keywords Asymptotic mean value formulas · Semi-elliptic operators · Hypoelliptic operators · Poisson-type equations

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1 Introduction

The Poisson-type equations related to hypoelliptic linear second order PDE's with non-negative characteristic form cannot be studied in L^p spaces due to the lack of a suitable Calderon-Zygmund theory for the relevant singular integrals. Our paper presents a result allowing to satisfactory study such equations in spaces of continuous functions. We follow a procedure introduced by Pizzetti in his 1909's paper [14] based on the asymptotic average solutions for the classical Poisson-Laplace equation.

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1.1 The Classical Pizzetti Theorem

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$, and let $f : \Omega \longrightarrow \mathbb{R}$ be a continuous bounded function. Let us denote by u_f the Newtonian potential of f, i.e.,

$$u_f : \mathbb{R}^n \longrightarrow \mathbb{R}, \qquad u_f(x) := \int_\Omega \Gamma(y-x) f(y)\, dy.$$

Here Γ denotes the fundamental solution of the Laplace equation, i.e.,

$$\Gamma(x) = c_n |x|^{2-n}, x \in \mathbb{R}^n \smallsetminus \{0\},$$

ω_n being the volume of the unit ball in $\mathbb{R}^n$ and $c_n := \dfrac{1}{n(n-2)\omega_n}$.

It is well known that $u_f \in C^1(\mathbb{R}^n, \mathbb{R})$, while, in general, $u_f|_\Omega \notin C^2(\Omega, \mathbb{R})$. However, in the weak sense of distributions,

$$\Delta u_f = -f \text{ in } \Omega. \tag{1.1}$$

As a consequence, if the continuous function f is such that

$$u_f \notin C^2(\Omega, \mathbb{R}), \tag{1.2}$$

then the Poisson equation

$$\Delta v = -f \tag{1.3}$$

has no classical solutions, i.e., there does not exist a function $v \in C^2(\Omega, \mathbb{R})$ satisfying

$$\Delta v(x) = -f(x) \text{ for every } x \in \Omega.$$

Indeed, assume by contradiction that such a function exists. Then, by Eq. 1.1,

$$\Delta(u_f - v) = 0 \text{ in } \Omega$$

in the weak sense of distributions, so that, by Caccioppoli–Weyl's Lemma, there exists a function h, harmonic in Ω, such that

$$u_f(x) - v(x) = h(x)$$

a.e. in Ω. Therefore, $u_f - v$ being continuous in Ω,

$$u_f = v + h \in C^2(\Omega, \mathbb{R}),$$

in contradiction with Eq. 1.2. This proves the existence of continuous functions f such that the Poisson Eq. 1.3 is not *pointwise* solvable. In his paper [14], Pizzetti introduced a notion of *pointwise weak Laplacian*, making pointwise solvable every Poisson equation with continuous data. Pizzetti started from the following remark. Given a function u of class C^2 in Ω one has

$$\lim_{r \longrightarrow 0} \frac{M_r(u)(x) - u(x)}{r^2} = \frac{1}{2(n+2)} \Delta u(x) \tag{1.4}$$

for every $x \in \Omega$. Here M_r denotes the *Gauss average*

$$M_r(u)(x) := \frac{1}{|B(x,r)|} \int_{\partial B(x,r)} u(y)\, dy,$$

$|B(x,r)|$ being the volume of $B(x,r)$, the Euclidean ball centered at x with radius r. Then, if $u \in C(\Omega, \mathbb{R})$ is such that the limit at the left hand side of Eq. 1.4 exists at a point $x \in \Omega$, Pizzetti defines

$$\Delta_a u(x) := 2(n+2) \lim_{r \longrightarrow 0} \frac{M_r(u)(x) - u(x)}{r^2}.$$

We call $\Delta_a u(x)$ the *asymptotic average Laplacian* of u at x. Keeping in mind Eq. 1.4, if $u \in C^2(\Omega, \mathbb{R})$, then

$$\Delta_a u(x) = \Delta u(x) \text{ for every } x \in \Omega.$$

We denote by

$$\mathcal{A}(\Omega, \Delta)$$

the class of functions $u \in C(\Omega, \mathbb{R})$, such that $\Delta_a u(x)$ exists at any point $x \in \Omega$. Obviously, $\mathcal{A}(\Omega, \Delta)$ is a (linear) sub-space of $C(\Omega, \mathbb{R})$. Moreover, by the previous remark,

$$C^2(\Omega, \mathbb{R}) \subseteq \mathcal{A}(\Omega, \Delta).$$

Pizzetti proved that the Newtonian potentials of continuous bounded functions are contained in $\mathcal{A}(\Omega, \Delta)$. Precisely he proved the following theorem.

Theorem A (Pizzetti Theorem) *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be a bounded open subset of $\mathbb{R}^n$ and let $f : \Omega \longrightarrow \mathbb{R}$ be a bounded continuous function. Then*

$$u_f \in \mathcal{A}(\Omega, \Delta)$$

and

$$\Delta_a u_f = -f \text{ in } \Omega.$$

The aim of this paper is to extend the notion of asymptotic average solution and Pizzetti's Theorem to the class of linear second order semi-elliptic partial differential operators that we will introduce in the next subsection.

1.2 Our Operators

We will deal with partial differential operators of the type

$$\mathcal{L} = \sum_{i,j=1}^{n} \partial_{x_i}(\partial_{x_j} a_{ij}(x)), \;\; x \in \mathbb{R}^n, \tag{1.5}$$

where $A(x) := (a_{ij} = a_{ji})_{i,j=1,\ldots,n}$ is a symmetric nonnegative definite matrix,

$$x \longmapsto a_{ji}(x), \qquad i, j = 1, \ldots, n$$

are smooth functions in $\mathbb{R}^n$ and

$$\sum_{i=1}^{n} a_{ii}(x) > 0 \text{ for every } x \in \mathbb{R}^n.$$

Together with these qualitative properties we assume that $\mathcal{L}$ is hypoelliptic in $\mathbb{R}^n$ and endowed with a smooth fundamental solution

$$\Gamma : \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n \mid x \neq y\} \longrightarrow \mathbb{R},$$

such that

(i) $\Gamma(x, y) = \Gamma(y, x) > 0$, for every $x \neq y$;
(ii) $\lim_{x \to y} \Gamma(x, y) = \infty$, for every $y \in \mathbb{R}^n$;
(iii) $\lim_{x \to \infty} \left(\sup_{y \in K} \Gamma(x, y)\right) = 0$, for every compact set $K \subseteq \mathbb{R}^n$;
(iv) $\Gamma(x, \cdot)$ belongs to $L^1_{\text{loc}}(\mathbb{R}^n)$, for every $x \in \mathbb{R}^n$.

We recall that when we say that Γ is a fundamental solution of $\mathcal{L}$ we mean that, for every $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ and $x \in \mathbb{R}^n$:

$$\int_{\mathbb{R}^n} \Gamma(x, y)\, \mathcal{L}\varphi(y)\, dy = -\varphi(x).$$

1.3 Examples of our Operators

Important examples of operators satisfying our assumptions are the "sum of squares"of homogeneous Hörmander vector fields. Precisely: let

$$X = \{X_1, \dots, X_m\}$$

be a family of linearly independent smooth vector fields such that

(H1) $X_1, \dots, X_m$ satisfy the Hörmander rank condition at $x = 0$, that is,

$$\dim\{Y(0) \mid Y \in \text{Lie}\{X_1, \dots, X_m\}\} = n;$$

(H2) $X_1, \dots, X_m$ are homogeneous of degree 1 with respect to a group of dilations $(\delta_\lambda)_{\lambda>0}$ of the following type

$$\delta_\lambda : \mathbb{R}^n \longrightarrow \mathbb{R}^n,$$
$$\delta_\lambda(x) = \delta_\lambda(x_1, \dots, x_n) = (\lambda^\sigma x_1, \dots, \lambda^{\sigma_n} x_n),$$

where the σ_j's are natural numbers such that $1 \le \sigma_1 \le \dots \le \sigma_n$.

Then,

$$\mathcal{L} = \sum_{j=1}^{m} X_j^2 \tag{1.6}$$

satisfies all the assumptions listed in Section 1.2 (see [1, 2]).

We stress that the sub-Laplacians on stratified Lie groups in $\mathbb{R}^n$ are particular cases of the operator $\mathcal{L}$ in Eq. 1.6.

1.4 Asymptotic Average Solutions and Main Results

The extension of Pizzetti's Theorem to the operator $\mathcal{L}$ in Eq. 1.5 rests on some representation formulas on the superlevel set of Γ. If $x \in \mathbb{R}$ and $r > 0$, define

$$\Omega_r(x) := \left\{ y \in \mathbb{R}^n \ : \ \Gamma(x, y) > \frac{1}{r} \right\}.$$

We will call $\Omega_r(x)$ the $\mathcal{L}$-ball centered at x and with radius r. It is easy to recognize that $\Omega_r(x)$ is a nonempty bounded open set of $\mathbb{R}^n$. Moreover

$$\bigcap_{r>0} \Omega_r(x) = \{x\} \tag{1.7}$$

and[1]

$$\frac{|\Omega_r(x)|}{r} \longrightarrow 0 \text{ as } r \longrightarrow 0.$$

[1] If E is a measurable set of $\mathbb{R}^n$, $|E|$ denotes its Lebesgue measure.

Remark 1.1 If $\mathcal{L} = \Delta$, then

$$\Omega_r(x) = B(x, \rho), \quad \text{with } \rho = (c_n r)^{\frac{1}{n-2}}.$$

Let $\Omega \subseteq \mathbb{R}^n$ be open and let $u \in C^2(\Omega, \mathbb{R})$. Then, for every $\mathcal{L}$-ball, $\Omega_r(x)$ such that $\overline{\Omega_r(x)} \subseteq \Omega$ and for every $\alpha > -1$ we have

$$u(x) = M_r(u)(x) - N_r(\mathcal{L}u)(x), \tag{1.8}$$

where M_r and N_r are the following average operators:

$$M_r(u)(x) := \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} u(y) K(x, y)\, dy, \tag{1.9}$$

where

$$\begin{aligned} K(x, y) &:= \frac{\langle A(y)\nabla_y \Gamma(x, y), \nabla_y \Gamma(x, y)\rangle}{(\Gamma(x, y))^{\alpha+2}}; \\ N_r(w)(x) &:= \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_r(x)} \left(\Gamma(x, y) - \frac{1}{\rho} \right) w(y)\, dy \right) d\rho. \end{aligned} \tag{1.10}$$

The proof of the representation formula Eq. 1.8 can be found in [4].

Remark 1.2 If $\mathcal{L} = \Delta$ and $\alpha = \dfrac{2}{n-2}$, then the kernel K is constant and M_r becomes the Gauss average on the Euclidean ball $B(x, \rho)$, with $\rho = (c_n r)^{\frac{1}{n-2}}$.

Letting

$$Q_r(x) := N_r(1) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_r(x)} (\Gamma(x, y) - \frac{1}{\rho}\right) dy)\, d\rho, \tag{1.11}$$

an easy computation shows that

$$Q_r(x) = \int_0^r \frac{\Omega_\rho(x)}{\rho^2} \left(1 - \left(\frac{\rho}{r}\right)^{\alpha+1} \right) d\rho.$$

Remark 1.3 If $\mathcal{L} = \Delta$ and $\alpha = \dfrac{2}{n-2}$, then, letting $\rho = (c_n r)^{\frac{1}{n-2}}$, we get

$$\frac{M_r(u)(x) - u(x)}{Q_r(x)} = 2(n+2) \frac{\frac{1}{|B(x,\rho)|} \int_{B(x,\rho)} u(y)\, dy - u(0)}{\rho^2},$$

so that, by Eq. 1.4,

$$\lim_{r \longrightarrow 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} = \Delta u(x). \tag{1.12}$$

The limit in Eq. 1.12 extends to all the operators $\mathcal{L}$ in Eq. 1.5. Indeed, if u is a C^2 function in an open set $\Omega \subseteq \mathbb{R}^n$, from the representation formula Eq. 1.8 and the identity Eq. 1.7, using Corollary 2.5 in Section 2, one immediately gets

$$\lim_{r \longrightarrow 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} = \mathcal{L}u(x).$$

Then, in analogy with the case $\mathcal{L} = \Delta$, we introduce the following definition.

Definition 1.4 Let $\mathcal{L}$ be a partial differential operator satisfying the assumptions of Section 1.2 and let u be a continuous function in an open set $\Omega \subseteq \mathbb{R}^n$. We say that

$$u \in \mathcal{A}(\Omega, \mathcal{L}),$$

if

$$\lim_{r \longrightarrow 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)}$$

exists in $\mathbb{R}$ at every point $x \in \Omega$. In this case we define

$$(\mathcal{L}_a(u))(x) := \lim_{r \longrightarrow 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)}.$$

Furthermore, if $f \in C(\Omega, \mathbb{R})$ and there exists $u \in \mathcal{A}(\Omega, \mathcal{L})$ such that

$$(\mathcal{L}_a u)(x) = f(x) \text{ for every } x \in \Omega,$$

we say that u is an *asymptotic average solution to*

$$\mathcal{L}_a u = f \text{ in } \Omega.$$

In the case $f = 0$ this definition was first introduced in the paper [6].

The main result of our paper is the following theorem which extends Pizzetti's Theorem to the operators Eq. 1.5.

Theorem 1.5 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a compactly supported continuous function. Define*

$$u_f(x) := \int_{\mathbb{R}^n} \Gamma(x, y) f(y)\, dy, \qquad x \in \mathbb{R}^n.$$

Then, $u_f \in \mathcal{A}(\mathbb{R}^n, \mathcal{L})$ and

$$\mathcal{L}_a u_f = -f \text{ in } \mathbb{R}^n.$$

We will prove this theorem in the next section. Here, by using a result in [6], we show a consequence of Theorem 1.5.

Theorem 1.6 *Let $f, u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be compactly supported continuous functions. Then,*

$$\mathcal{L}_a u = -f \text{ in } \mathbb{R}^n$$

if and only if

$$\mathcal{L} u = -f \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

Proof By the previous Theorem 1.5,

$$\mathcal{L}_a u = -f \text{ in } \mathbb{R}^n$$

if and only if

$$\mathcal{L}_a(u - u_f) = 0 \text{ in } \mathbb{R}^n.$$

Then, by Corollary 3.4 in [6], $u - u_f \in C^\infty(\mathbb{R}^n, \mathbb{R})$ and

$$\mathcal{L}(u - u_f) = 0$$

in the classical sense (and vice versa). Since $\mathcal{L}$ is hypoelliptic, this is equivalent to say that

$$\mathcal{L}(u - u_f) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^n),$$

or that

$$\mathcal{L}(u) = \mathcal{L}(u_f) \text{ in } \mathcal{D}'(\mathbb{R}^n). \tag{1.13}$$

On the other hand, Γ being a fundamental solution of $\mathcal{L}$, $\mathcal{L}(u_f) = -f$ in $\mathcal{D}'(\mathbb{R}^n)$. Then, Eq. 1.13 can be written as follows:

$$\mathcal{L}u = -f \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

This completes the proof. □

1.5 Bibliographical Note

In recent years asymptotic mean value formulas characterizing classical or viscosity solutions to linear and nonlinear second order Partial Differential Equations have been proved by many authors; we refer to [3, 5–8, 10–13]. In those papers one can find quite exhaustive bibliography on this subject.

We would also like to quote the papers [4] and [9] where the notion of asymptotic subharmonic function is introduced in sub-Riemannian settings to extend classical results by Blaschke, Privaloff, Reade and Saks.

2 Proof of Theorem 1.5

For the readers' convenience, we split this section in some subsections.

2.1 A First Lemma

Let G be a compact subset of $\mathbb{R}^n$ and let $r > 0$. Define

$$G_r := \bigcup_{x \in G} \Omega_r(x). \tag{2.1}$$

Then, we have the following lemma.

Lemma 2.1 *For every compact set $G \subseteq \mathbb{R}^n$ and for every $r > 0$, the set $\overline{G}_r$ is compact.*

Proof It is enough to prove that G_r is bounded. We argue by contradiction and assume that G_r is not bounded. Then, there exists a sequence (z_n) in G_r such that

$$|z_n| \longrightarrow \infty.$$

By the very definition of G_r, for every $n \in \mathbb{N}$, there exists $x_n \in G$ such that $z_n \in \Omega_r(x_n)$. This means that

$$\Gamma(x_n, z_n) > \frac{1}{r}.$$

As a consequence,

$$\frac{1}{r} < \Gamma(x_n, z_n) \leq \sup_{x \in G} \Gamma(x, z_n),$$

so that, by the assumption (iii) related to Γ

$$0 < \frac{1}{r} \leq \lim_{n \longrightarrow \infty} \left(\sup_{x \in G} \Gamma(x, z_n) \right) = 0.$$

This contradiction shows that G_r is bounded. □

2.2 A Second Lemma

In this subsection we prove the following lemma.

Lemma 2.2 *Let G be a compact subset of $\mathbb{R}^n$ and let $r > 0$. Then, there exists a positive constant $C_r(G)$ such that*

$$\sup_{x\in G} Q_r(x) \le C_r(G). \tag{2.2}$$

Proof Keeping in mind the definition of $Q_r(x)$ (see Eq. 1.11) for every $x \in G$ we get

$$\begin{aligned} Q_r(x) &\le \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{\Omega_r(x)} \Gamma(x,y)\, dy \right) d\rho \\ &\le \text{(by(2.1))} \quad \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \left(\int_{G_r} \Gamma(x,y)\, dy \right) d\rho. \end{aligned} \tag{2.3}$$

On the other hand, if $\varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R})$ is such that $\varphi = 1$ on G_r, $\varphi \ge 0$ (such a function exists thanks to Lemma 2.1), we have

$$\begin{aligned} \int_{G_r} \Gamma(x,y)\, dy &\le \int_{\mathbb{R}^n} \varphi(y)\Gamma(x,y)\, dy \\ &\le \sup_{x\in G} \int_{\mathbb{R}^n} \varphi(y)\Gamma(x,y)\, dy \\ &= C_\varphi(G). \end{aligned}$$

Using this estimate in Eq. 2.3 we obtain

$$\begin{aligned} \sup_{x\in G} Q_r(x) &\le C_\varphi(G) \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha\, d\rho \\ &= C_\varphi(G) := C_r(G). \end{aligned}$$

□

Remark 2.3 Since $Q_\rho(x) \subseteq Q_r(x)$ for every $\rho \in]0, r[$, we can assume

$$C_\rho(G) \le C_r(G)$$

for every $0 < \rho < r$.

2.3 A Topological Lemma

Now, we show a kind of continuity property of the $\Omega_r(x)$ balls with respect to the Euclidean topology. Precisely, we prove the following lemma.

Lemma 2.4 *For every $x \in \mathbb{R}^n$ and for every $R > 0$ there exists $r > 0$ such that*

$$\Omega_r(x) \subseteq B(x, R).$$

Proof We still argue by contradiction and assume the existence of $R > 0$ such that $\Omega_r(x) \nsubseteq B(x, R)$ for every $r > 0$. Then, if (r_n) is a sequence of real positive numbers such that $r_n \searrow 0$, for every $n \in \mathbb{N}$ there exists $y_n \in \Omega_{r_n}(x)$ such that

$$y_n \notin B(x, R).$$

This means

$$y_n \notin B(x, R) \quad \text{and} \quad \Gamma(x, y_n) > \frac{1}{r_n}.$$

Since $\Gamma(x, y) \longrightarrow 0$ as $y \longrightarrow \infty$ and $\frac{1}{r_n} \longrightarrow \infty$, the sequence (y_n) is bounded. As a consequence, we may assume

$$\lim_{n \longrightarrow \infty} y_n = y^*$$

for a suitable $y^* \in \mathbb{R}^n$. Then $y^* \notin B(x, R)$. In particular $y \neq x$ so that $\Gamma(x, y) < \infty$. On the other hand,

$$\Gamma(x, y^*) = \lim_{n \longrightarrow \infty} \Gamma(x, y_n) \geq \lim_{n \longrightarrow \infty} \frac{1}{r_n} = \infty.$$

This contradiction proves the lemma. □

From the previous lemma we obtain the following corollary.

Corollary 2.5 *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function. Then, for every $x \in \mathbb{R}^n$,*

$$\sup_{y \in \Omega_r(x)} |f(y) - f(x)| \longrightarrow 0 \textit{ as } r \longrightarrow 0.$$

Proof Since f is continuous at x, for every $\varepsilon > 0$ there exists $R > 0$ such that

$$\sup_{y \in B(x,r)} |f(y) - f(x)| < \varepsilon.$$

By the previous lemma, there exists $r_0 > 0$ such that $\Omega_{r_0}(x) \subseteq B(x, r)$. Then, for every $r < r_0$,

$$\sup_{y \in \Omega_r(x)} |f(y) - f(x)| \leq \sup_{y \in \Omega_{r_0}(x)} |f(y) - f(x)| \leq \sup_{y \in B(x,r)} |f(y) - f(x)| < \varepsilon.$$

We have so proved that for every $\varepsilon > 0$ there exists $r_0 > 0$ such that

$$\sup_{y \in \Omega_r(x)} |f(y) - f(x)| < \varepsilon$$

for every $r < r_0$. Hence,

$$\lim_{r \longrightarrow 0} \left(\sup_{y \in \Omega_r(x)} |f(y) - f(x)| \right) = 0.$$

□

2.4 A Poisson-Jensen-type Formula

Let f as in Theorem 1.5 and, to simplify the notation, let us denote u_f by u. The aim of this subsection is to prove the following identity:

$$u(x) = M_r(u)(x) + N_r(f)(x) \quad \forall x \in \mathbb{R}^n. \tag{2.4}$$

To this end we choose a sequence (f_p) in $C_0^\infty(\mathbb{R}^n, \mathbb{R})$ with the following properties:

(i) there exists a compact set $K \subseteq \mathbb{R}^n$ such that supp $f \subseteq K$ and supp $f_p \subseteq K$ for every $p \in \mathbb{N}$;
(ii) $\sup_K |f_p - f| \longrightarrow 0$ as $p \to \infty$.

For simplicity reasons, let us put $u_p = u_{f_p}$, i.e.,

$$u_p(x) = \int_{\mathbb{R}^n} \Gamma(x, y) f_p(y)\, dy = \int_K \Gamma(x, y) f_p(y)\, dy.$$

Then, by Lebesgue's dominated convergence Theorem,

$$u(x) = \lim_{p \longrightarrow \infty} u_p(x) = \int_K \Gamma(x, y) \lim_{p \longrightarrow \infty} f_p(y)\, dy,$$

for every $x \in \mathbb{R}^n$. Actually, we have a stronger result. For every compact set $G \subseteq \mathbb{R}^n$,

$$\begin{aligned}\sup_G |u_p - u| &\leq \sup_{x \in G} \left| \int_K \Gamma(x, y)(f_p(y) - f(y))\, dy \right| \\ &\leq \sup_K |f_p - f| \sup_{x \in G} \int_K \Gamma(x, y)\, dy \\ &= C(G, K) \sup_K |f_p - f|.\end{aligned}$$

We explicitly observe that $C(G, K)$ is a strictly positive finite constant.

Hence,

$$\sup_G |u_p - u| \longrightarrow 0 \text{ as } p \longrightarrow \infty. \tag{2.5}$$

Moreover, for every $p \in \mathbb{N}$,

$$u_p \in C^\infty(\mathbb{R}^n, \mathbb{R}) \quad \text{and} \quad \mathcal{L} u_p = -f_p.$$

Then, by identity Eq. 1.8,

$$\begin{aligned}u_p(x) &= M_r(u_p)(x) - N_r(\mathcal{L} u_p)(x) \\ &= M_r(u_p)(x) + N_r(f_p)(x)\end{aligned}$$

for every $p \in \mathbb{N}$.

We have already noticed that $u_p(x) \longrightarrow u(x)$ as $p \longrightarrow \infty$.

To prove Eq. 2.4 we now show that

$$\lim_{p \longrightarrow \infty} M_r(u_p)(x) = M_r(u)(x) \tag{2.6}$$

and

$$\lim_{p \longrightarrow \infty} N_r(f_p)(x) = N_r(f)(x). \tag{2.7}$$

For every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}|M_r(u_p)(x) - M_r(u)(x)| &= |M_r(u_p - u)(x)| \\ &\leq \sup_{\Omega_r(x)} |u_p - u| M_1(1)(x) \\ &= \sup_{\Omega_r(x)} |u_p - u|.\end{aligned}$$

Since $\overline{\Omega_r(x)}$ is compact (see Lemma 2.1), and keeping in mind Eq. 2.5, the last right hand side goes to zero as $p \longrightarrow \infty$. Then,

$$|M_r(u_p)(x) - M_r(u)(x)| \longrightarrow 0 \text{ as } p \longrightarrow \infty,$$

proving Eq. 2.6.

Let us now prove Eq. 2.7. For every $x \in \mathbb{R}^n$, we have

$$\begin{aligned} |N_r(f_p)(x) - N_r(f)(x)| &\leq |N_r(|f_p - f|)(x)| \\ &\leq \sup_K |f_p - f| Q_r(x). \end{aligned}$$

Then, for every compact set $G \subseteq \mathbb{R}^n$,

$$\begin{aligned} \sup_G |N_r(f_p) - N_r(f)| &\leq \sup_K |f_p - f| \sup_{x \in G} |Q_r(x)| \\ &\leq \text{(by(2.2))} \quad C_r(G) \sup_K |f_p - f|. \end{aligned}$$

So we have proved that $(N_r(f_p))$ is uniformly convergent to $N_r(f)$ on every compact subset of $\mathbb{R}^n$. This, in particular, implies Eq. 2.7.

2.5 Conclusion

In this subsection we complete the proof of Theorem 1.5. To this end we first remark that, thanks to Eq. 2.4, for every $x \in \mathbb{R}^n$, we have

$$\frac{M_r(u)(x) - u(x)}{Q_r(x)} = -\frac{N_r(f)(x)}{Q_r(x)},$$

so that, as $f(x)$ is constant with respect to $y \in \Omega_r(x)$,

$$\begin{aligned} \left| \frac{M_r(u)(x) - u(x)}{Q_r(x)} + f(x) \right| &= \frac{1}{Q_r(x)} |N_r(f(x) - f)(x)| \\ &\leq \sup_{y \in \Omega_r(x)} |f(u) - f(y)| Q_r(x). \end{aligned}$$

By Corollary 2.5 and Remark 2.3, the left hand side of the previous inequality goes to zero as $r \longrightarrow 0$. Hence,

$$\lim_{r \longrightarrow 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} = -f(x)$$

for every $x \in \mathbb{R}^n$. This completes the proof of Theorem 1.5.

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Declarations

Conflict of Interests The authors declare that they have no conflict of interest.

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