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Instantaneous blowup and singular potentials on Heisenberg groups

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Abstract. In this paper we generalize the instantaneous blowup result from the 1984 paper by Baras and Goldstein and the 2001 paper by Goldstein and Zhang to the heat equation perturbed by singular potentials on the Heisenberg group.

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1. Introduction

The problem of existence and nonexistence of non-negative solutions to the heat equation with singular potentials $V_c^*(x) = \dfrac{c}{|x|^2}$, $x \in \Omega_N$,

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t) + V_c^*(x)u(x,t) & (x,t) \in \Omega_N \times (0,\infty) \\ u(x,0) = u_0(x), & x \in \Omega_N, \end{cases} \tag{1.1}$$

where $\Omega_N = \begin{cases} \mathbb{R}^N & \text{if } N \geq 2 \\ (0,\infty) & \text{if } N = 1, \end{cases}$ was settled and solved by Baras and Goldstein [3]. For $\Omega_1 = (0,\infty)$ one has to add a Dirichlet boundary condition at 0. For simplicity we assume in the sequel that $N \geq 3$ and set $C_*(N) := \left(\frac{N-2}{2}\right)^2$.

Obviously, the phenomenon of existence and nonexistence is caused by the singular potential V_c^*, which is controlled by Hardy's inequality

$$C_*(N) \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla\varphi(x)|^2 \, dx, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N),$$

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together with its optimal constant $C_*(N)$. Moreover V_c^* belongs to a borderline case where the strong maximum principle and Gaussian bounds fail, *cf.* [2].

Let $W_n(x) = \inf\{V_c^*(x), n\}$ be the cutoff potential, with $c > C_*(N)$. Let u_n be the unique solution of

$$\begin{cases} \dfrac{\partial_t u_n}{\partial t} - \Delta u_n - W_n u_n = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ u_n(x,0) = u(x,0) = u_0(x) \geq 0. \end{cases}$$

Here $0 \not\equiv u_0 \in L^2(\mathbb{R}^N)$ or, more generally, u_0 grows no faster than $e^{|x|^{2-\varepsilon}}$ at infinity. Since W_n is bounded, u_n exists. If a positive solution u to (1.1) were to exist, then $0 < u_n \leq u$ which is a contradiction, since $u_n(x,t)$ tends to infinity at all spatial points and at all positive times (see [3, Theorem 2.2.(ii)]). This is called *instantaneous blowup*.

Given non-negative functions $u_0 \in L^1(\mathbb{R}^N)$, $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^N)$ and $0 \leq f \in L^1(\mathbb{R}^N \times (0,T))$, Baras and Goldstein [3] considered the problem of finding a function u such that

$$(P) \begin{cases} 0 \leq u \text{ on } \mathbb{R}^N \times (0,T) \\ V(\cdot)u \in L^1_{\text{loc}}(\mathbb{R}^N \times (0,T)) \\ \dfrac{\partial u}{\partial t} = \Delta u + Vu + f & \text{in } \mathcal{D}'(\mathbb{R}^N \times (0,T)) \\ \operatorname*{esslim}_{t\to 0^+} \displaystyle\int_{\mathbb{R}^N} u(x,t)\psi(x)\,dx = \int_{\mathbb{R}^N} u_0(x)\psi(x)\,dx & \text{for all } \psi \in \mathcal{D}(\mathbb{R}^N). \end{cases}$$

Here $\mathcal{D}(\mathbb{R}^N) := C_c^\infty(\mathbb{R}^N)$, $\mathcal{D}(\mathbb{R}^N \times (0,T)) := C_c^\infty(\mathbb{R}^N \times (0,T))$ with the usual topology and $\mathcal{D}'(\mathbb{R}^N \times (0,T))$, the dual of $\mathcal{D}(\mathbb{R}^N \times (0,T))$, is the space of all distributions on $\mathbb{R}^N \times (0,T)$.

Consider the potential

$$W_0(x) = \begin{cases} \dfrac{c}{|x|^2} & \text{if } x \in B_1 \\ 0 & \text{if } x \in \mathbb{R}^N \setminus B_1. \end{cases}$$

Here B_1 can be replaced by B_δ for every fixed $\delta > 0$, where B_r denotes the ball in $\mathbb{R}^N$ of center 0 and radius $r > 0$. Baras and Goldstein [3] proved the following result:

Theorem 1.1.

(i) *Let $0 \leq c \leq C_*(N)$ and let $V \geq 0$ be a measurable potential satisfying $V \in L^\infty(\mathbb{R}^N \setminus B_1)$, where B_1 denotes the unit ball in $\mathbb{R}^N$. Let $0 \leq f \in L^1(\mathbb{R}^N \times (0,T))$. If $V \leq W_0$ in B_1, then (P) has a positive solution if*

$$\int_{\mathbb{R}^N} |x|^{-\alpha} u_0(x)\,dx < \infty\,, \quad \int_0^T \int_{\mathbb{R}^N} f(x,s)|x|^{-\alpha}\,dx\,ds < \infty, \tag{1.2}$$

where α is the smallest root of $\alpha(N-2-\alpha) = c$. If $V \geq W_0$ in B_1, and if (P) has a solution u, then

$$\int_{\Omega'} |x|^{-\alpha} u_0(x)\, dx < \infty, \qquad \int_0^{T-\varepsilon} \int_{\Omega'} f(x,s)|x|^{-\alpha} dx ds < \infty,$$

for each $\varepsilon \in (0,T)$ and each $\Omega' \subset\subset \mathbb{R}^N$ with α as above. If either $u_0 \neq 0$ or $f \neq 0$ in $\mathbb{R}^N \times (0,\varepsilon)$ for each $\varepsilon \in (0,T)$, then given $\Omega' \subset\subset \mathbb{R}^N$, there is a $C = C(\varepsilon, \Omega') > 0$ such that

$$u(x,t) \geq \frac{C}{|x|^\alpha} \text{ if } (x,t) \in \Omega' \times [\varepsilon, T); \tag{1.3}$$

(ii) *If $c > C_*(N)$, $V \geq W_0$ and either $u_0 \not\equiv 0$ or $f \not\equiv 0$, then (P) does not have a positive solution.*

Many extensions of the above result have been done by several authors, *cf.* [6, 7, 9, 10, 12–17]. In this article we present a new result of this type replacing the Laplacian on $\mathbb{R}^N$ by the sub-Laplacian $\Delta_{\mathbb{H}}$ (also known as the Kohn Laplacian) on the Heisenberg group $\mathbb{H}^N$. For the definitions see Section 2.

For this purpose let us consider, for $w = (z,l) \in \mathbb{H}^N$, the problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(w,t) = \Delta_{\mathbb{H}} u(w,t) + V_*(w)u(w,t) + f(w,t) & t > 0,\ w \in \mathbb{H}^N \\ u(w,0) = u_0(w), & w \in \mathbb{H}^N. \end{cases} \tag{1.4}$$

Assume $u_0 \geq 0$, $f \geq 0$ and as V_* choose the corresponding critical potential in the case of the Heisenberg group $\mathbb{H}^N$

$$V_*(w) = c\frac{|z|^2}{|z|^4 + l^2}, \quad w = (z,l) \in \mathbb{H}^N.$$

We thus look at the problem

$$(P)_{\mathbb{H}^N} \quad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{\mathbb{H}} u + V_* u + f & \text{in } \mathbb{H}^N \times (0,T) \\ u(w,0) = u_0(w) & w \in \mathbb{H}^N, \end{cases}$$

with $u_0 \geq 0$ and $u_0 \not\equiv 0$ a.e. Set $V_n(w) = \min\{V_*(w), n\}$, $f_n(w,t) = \min\{f(w,t), n\}$. Let u_n be the unique non-negative solution of

$$(P_n)_{\mathbb{H}^N} \quad \begin{cases} \dfrac{\partial u_n}{\partial t} = \Delta_{\mathbb{H}} u_n + V_n u_n + f_n & \text{in } \mathbb{H}^N \times (0,T) \\ u_n(w,0) = u_0(w) & w \in \mathbb{H}^N, \end{cases}$$

and assume that u_n exists. We only need to assume that the heat equation with no potential has a global positive solution when u_0 is the initial value, see (2.6) and

Theorem 2.2 below. It is sufficient that $u_0 \in L^2_{\text{loc}}(\mathbb{H}^N)$ and u_0 grows no faster than $e^{d^2(w)-\varepsilon}$ at infinity, where $d(\cdot)$ is the function given by (2.4).

Let

$$C^*(N) = N^2.$$

We will prove, for u_n the solution of $(P_n)_{\mathbb{H}^N}$, that:

(I) If $0 < c \le C^*(N)$, then

$$\lim_{n\to\infty} u_n(w,t) = u(w,t), \quad (w,t) \in \mathbb{H}^N \times (0,T),$$

exists and is a solution of $(P)_{\mathbb{H}^N}$;

(II) If $c > C^*(N)$, then

$$\lim_{n\to\infty} u_n(w,t) = +\infty \tag{1.5}$$

for all $(w,t) \in \mathbb{H}^N \times (0,T)$.

The conclusion in (II), namely (1.5), is known as *instantaneous blowup*, or (IBU) for short.

In the existence case (I), by the maximum principle for $\Delta_{\mathbb{H}}$, it is clear that we can replace V_n, V_* by $\tilde{V}_n, \tilde{V}_*$ where $\tilde{V}_n \le V_n$, $\tilde{V}_* \le V_*$ a.e. for each n. Similarly, for the nonexistence result (II), we can replace V_n, V_* by $\tilde{V}_n, \tilde{V}_*$ where $\tilde{V}_n \ge V_n$, $\tilde{V}_* \ge V_*$ a.e. (at least in a neighborhood of the origin).

The paper is organized as follows. In the next section we recall the definitions of the Heisenberg group $\mathbb{H}^N$ and the sub-Laplacian $\Delta_{\mathbb{H}}$ on $\mathbb{H}^N$. We also give some known properties of $\Delta_{\mathbb{H}}$ that we need in this paper. In Section 3 we state and prove the main results of this paper. In the Appendix we prove some technical lemmas that we use in the proof of the main results.

This paper treats the same basic problem as did [15]. There, the existence part of Theorem 3.4 was proved, using a different method. But part (ii) of Theorem 3.4 is much stronger than the corresponding result of [15].

In 1999, X. Cabré and Y. Martel [5] gave a different approach to a more general problem. The paper [3] treated a potential $V \ge 0$ with only one singularity, at the origin, while [5] allowed for a much more general potential which one takes to be $0 \le W \in L^1_{\text{loc}}(\mathbb{R}^N \setminus \{0\})$. In [5] the authors defined the "generalized first eigenvalue" of the Schrödinger operator $-\Delta - W$ as

$$\sigma_W = \inf_{u\in C_c^1(\mathbb{R}^N),\, \|u\|_{L^2}=1} \left\{ \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - W(x)|u(x)|^2)\, dx \right\}$$

$$\left(\text{or } \sigma_W = \inf_{u\in C_c^1(\mathbb{R}^N\setminus\{0\}),\, \|u\|_{L^2}=1} \left\{ \int_{\mathbb{R}^N} (|\nabla u(x)|^2 - W(x)|u(x)|^2)\, dx \right\} \text{ if } N \le 2 \right).$$

Note that for $W(x) = \frac{c}{|x|^2}$, $x \in \mathbb{R}^N$, one has $\sigma_W = -\infty$ if $c > C_*(N)$ and $\sigma_W > -\infty$ if $c \le C_*(N)$. Roughly speaking, in [5] the existence of positive solutions, when $\sigma_W > -\infty$ and for $\sigma_W = -\infty$, was obtained; further the authors proved that there is no globally defined pointwise solution that is exponentially bounded in time. This is a much weaker conclusion than the instantaneous blowup (IBU).

In the $\mathbb{H}^N$ setting, the authors in [15] used the method of [5] and proved nonexistence of globally defined (in (x,t)) positive solutions that grow at most exponentially for $c > C^*(N)$. But the question of (IBU) remained open until now.

2. Notation and preliminaries

The Heisenberg group and its sub-Laplacian play a crucial role in several branches of harmonic analysis, complex geometry and partial differential equations (see, *e.g.*, [8, 11, 19, 20]; see also the survey papers [18, 21]).

The Heisenberg group $\mathbb{H}^N$, $N \in \mathbb{N}$, is the stratified Lie group of step two

$$(\mathbb{R}^{2N+1}, \circ, D_\lambda). \tag{2.1}$$

If we denote the generic point of $\mathbb{R}^{2N+1}$ by $w = (z,l) = (x,y,l)$, with $x, y \in \mathbb{R}^N$ and $l \in \mathbb{R}$, the composition law $\circ$ is defined by

$$(x,y,l) \circ (x',y',l') = (x + x', y + y', l + l' + 2(x' \cdot y - y' \cdot x)),$$

where $x \cdot y$ denotes the inner product in $\mathbb{R}^N$.

In (2.1), D_λ, $\lambda > 0$ denotes the anisotropic dilation

$$D_\lambda : \mathbb{R}^{2N+1} \longrightarrow \mathbb{R}^{2N+1}, D_\lambda(z,l) = (\lambda z, \lambda^2 l).$$

The family $(D_\lambda)_{\lambda>0}$ is a group of automorphisms of $\mathbb{H}^N$, that is,

$$D_\lambda((z,l) \circ (z',l')) = (D_\lambda(z,l) \circ D_\lambda(z',l')).$$

The real number

$$Q := 2N + 2$$

is called the *homogeneous dimension* of $\mathbb{H}^N$ since it appears in the formula

$$|D_\lambda(A)| = \lambda^Q |A|,$$

where $A \subseteq \mathbb{R}^{2N+1}$ is a Lebesgue measurable set and $|A|$ stands for the Lebesgue measure of A.

A basis for the Lie algebra of left invariant vector fields on $\mathbb{H}^N$ is given by

$$X_j = \partial_{x_j} + 2y_j \partial_l, \quad Y_j = \partial_{y_j} - 2x_j \partial_l, \quad j = 1, \dots, N.$$

One easily calculates that

$$[X_j, X_k] = [Y_j, Y_k] = 0 \text{ for every } j,k = 1, \dots, N, \text{ and } [X_j, Y_k] = -4\delta_{jk}\partial_l. \tag{2.2}$$

These are the canonical commutation relations of Quantum Mechanics for position and momentum, whence $\mathbb{H}^N$ is called the Heisenberg group.

The subelliptic gradient is the gradient taken with respect to the horizontal directions $\nabla_{\mathbb{H}} := (X_1, \dots, X_N, Y_1, \dots, Y_N)$ and the sub-Laplacian on $\mathbb{H}^N$ is

$$\Delta_{\mathbb{H}} := \sum_{j=1}^{N} (X_j^2 + Y_j^2) = \nabla_{\mathbb{H}} \cdot \nabla_{\mathbb{H}},$$

and it can be explicitly also written as

$$\Delta_{\mathbb{H}} = \Delta_z + 4|z|^2 \partial_l^2 + 4\partial_l T,$$

where

$$\Delta_z = \sum_{j=1}^{N} (\partial_{x_j}^2 + \partial_{y_j}^2)$$

and

$$T = \sum_{j=1}^{N} (y_j \partial_{x_j} - x_j \partial_{y_j}).$$

From (2.2) it immediately follows that

$$\text{rank Lie } (X_1, \dots, X_N, Y_1, \dots, Y_N)(z, l) = 2N + 1$$

at any point $(z, l) \in \mathbb{R}^{2N+1}$. Then, by a celebrated theorem of Hörmander, $\Delta_{\mathbb{H}}$ is hypoelliptic, that is, every distributional solution of $\Delta_{\mathbb{H}} u = f$ is smooth whenever f is smooth.

The operator $\Delta_{\mathbb{H}}$ is left translation invariant on $\mathbb{H}^N$ and D_λ-homogeneous of degree two. Moreover $\Delta_{\mathbb{H}}$ has a fundamental solution (with a pole at the origin) given by

$$\gamma(w) = c_N \left(\frac{1}{d(w)} \right)^{Q-2} = c_N \left(\frac{1}{d(w)} \right)^{2N}, \quad w \neq (0, 0), \tag{2.3}$$

where

$$d(w) = (|z|^4 + l^2)^{\frac{1}{4}} \text{ for } w = (z, l) \in \mathbb{H}^N \tag{2.4}$$

defines the metric $\rho(w, \tilde{w}) := d(\tilde{w}^{-1} \circ w)$ on $\mathbb{H}^N$, and $\tilde{w}^{-1}$ denotes the inverse of $\tilde{w}$ in the group $\mathbb{H}^N$.

In the following lemma we summarize some properties of d and its gradient $\nabla_{\mathbb{H}}$ which one can obtain by simple computations, see [4, Proposition 5.4.3].

Lemma 2.1. *For $d(w) = (|z|^4 + l^2)^{\frac{1}{4}}$, $w = (z, l) \in \mathbb{H}^N$, the following hold:*

$$\begin{aligned} |\nabla_{\mathbb{H}} d(w)|^2 &= |z|^2(|z|^4 + l^2)^{-\frac{1}{2}}, \\ \Delta_{\mathbb{H}} d(w) &= \frac{Q-1}{d(w)} |\nabla_{\mathbb{H}} d(w)|^2, \\ -\Delta_{\mathbb{H}} d^{-\alpha}(w) &= C d^{-\alpha}(w) \frac{|z|^2}{|z|^4 + l^2} \end{aligned} \tag{2.5}$$

for $w \in \mathbb{H}^N \setminus \{(0,0)\}$, where $C := \alpha(Q - 2 - \alpha) = \alpha(2N - \alpha)$. So, $\Delta_{\mathbb{H}} d^{-\alpha} \in L^1_{\mathrm{loc}}(\mathbb{H}^N)$ if and only if $2N - \alpha > 0$.

It is known that the left translation invariance of $\Delta_{\mathbb{H}}$ implies that the semigroup $e^{t\Delta_{\mathbb{H}}}$ is given by a right convolution

$$e^{t\Delta_{\mathbb{H}}} f(w) = \int_{\mathbb{H}^N} f(\tilde{w}) p_t(\tilde{w}^{-1} \circ w)\, d\tilde{w}, \quad t > 0,\ w \in \mathbb{H}^N, \tag{2.6}$$

where $(w, t) \mapsto p_t(w)$ is the fundamental solution of $\left(\frac{\partial}{\partial t} + \Delta_{\mathbb{H}}\right) u = 0$. Hence, by hypoellipticity, $p_t(w)$ is a C^∞ function on $\mathbb{H}^N \times (0, \infty)$ and $\|p_t\|_1 = 1$. Moreover, p_t satisfies the following Gaussian estimates, *cf.* [22, Theorem IV.4.2 and Theorem IV.4.3].

Theorem 2.2. *The heat kernel p_t satisfies*

$$C t^{-\frac{Q}{2}} \exp\left(-c \frac{d^2(w)}{t}\right) \le p_t(w) \le C_\varepsilon t^{-\frac{Q}{2}} \exp\left(\frac{-d^2(w)}{4(1+\varepsilon)t}\right)$$

for some positive constants C, c, C_ε, any $\varepsilon > 0$, $w \in \mathbb{H}^N$ and $t > 0$.

3. The main results

In this section we make the following hypotheses.

Hypotheses 3.1.

- $0 \le V \in L^1_{\mathrm{loc}}(\mathbb{H}^N)$;
- $0 \le f \in L^1(\mathbb{H}^N \times (0, T))$;
- $0 \le u_0 \in L^1(\mathbb{H}^N)$ (or more generally u_0 can be a positive finite Radon measure).

We consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{\mathbb{H}} u + V u + f & \text{in } \mathcal{D}'(\mathbb{H}^N \times (0, T)) \\ \operatorname*{esslim}_{t \to 0+} \displaystyle\int_{\mathbb{H}^N} u(w, t) \psi(w)\, dw = \int_{\mathbb{H}^N} u_0(w) \psi(w)\, dw & \text{for all } \psi \in \mathcal{D}(\mathbb{H}^N) \\ u \ge 0 & \text{on } \mathbb{H}^N \times (0, T) \\ V u \in L^1_{\mathrm{loc}}(\mathbb{H}^N \times (0, T)). \end{cases} \tag{3.1}$$

Here $\mathcal{D}(\mathbb{H}^N) = C_c^\infty(\mathbb{H}^N)$ (respectively $\mathcal{D}(\mathbb{H}^N \times (0,T)) = C_c^\infty(\mathbb{H}^N \times (0,T))$) with the usual topologies and $\mathcal{D}' := \mathcal{D}'(\mathbb{H}^N)$ (respectively $\mathcal{D}'_T := \mathcal{D}'(\mathbb{H}^N \times (0,T))$) is its dual space. We also consider the approximating problem

$$\begin{cases} \dfrac{\partial u_n}{\partial t} = \Delta_{\mathbb{H}} u_n + V_n u_n + f_n & \text{in } \mathcal{D}'_T \\ \lim\limits_{t\to 0^+} \displaystyle\int_{\mathbb{H}^N} u_n(w,t)\psi(w)\,dw = \int_{\mathbb{H}^N} u_0(w)\psi(w)\,dw & \text{for all } \psi \in \mathcal{D}(\mathbb{H}^N). \end{cases} \tag{3.2}$$

Here

$$f_n = \min\{f, n\},$$
$$V_n \in L^\infty,\ 0 \le V_n \le V,\ V_n \uparrow V \text{ a.e.}$$

By the variation of parameters formula, (3.2) has a unique bounded non-negative solution obtained by solving the integral equation

$$u_n(t) = e^{t\Delta_{\mathbb{H}}} u_0 + \int_0^t e^{(t-s)\Delta_{\mathbb{H}}} V_n(\cdot) u_n(s)\,ds + \int_0^t e^{(t-s)\Delta_{\mathbb{H}}} f_n(s)\,ds, \tag{3.3}$$

where $\left(e^{t\Delta_{\mathbb{H}}}\right)_{t\ge 0}$ is the semigroup generated by $\Delta_{\mathbb{H}}$ on $\mathbb{H}^N$. We note that V_n is a bounded multiplication operator on $L^p\left(\mathbb{H}^N\right)$ for all $p \in [1, +\infty)$. Since $\{V_n\}$ is an increasing sequence, clearly $\{u_n\}$ is an increasing sequence, as well.

Proposition 3.2. *Suppose there is a* $(w_0,t_0) \in \mathbb{H}^N \times (0,T)$ *with* $\lim\limits_{n\to\infty} u_n(w_0,t_0) < \infty$. *Then* (3.1) *has a non-negative solution on* $\mathbb{H}^N \times (0,T_0)$ *for all* $0 < T_0 < t_0$ *given by*

$$u(w,t) = \lim_{n\to\infty} u_n(w,t) \quad a.e.\ in\ \mathbb{H}^N \times (0,T_0). \tag{3.4}$$

Moreover, if (3.1) *has a non-negative solution in* $\mathbb{H}^N \times (0,T)$, *then* $\lim\limits_{n\to\infty} u_n(w,t) < \infty$ *a.e. in* $\mathbb{H}^N \times (0,T)$.

Proof. Clearly, if $u \ge 0$ is a solution of (3.1), then $u_n \le u$ for all n, so $\lim\limits_{n\to\infty} u_n(w,t) \le u(w,t)$ a.e. in $\mathbb{H}^N \times (0,T)$. This establishes the last part of the proposition.

For the main part, we start by considering

$$U_n = e^t u_n, \quad t > 0.$$

Then

$$\frac{\partial U_n}{\partial t} = \Delta_{\mathbb{H}} U_n + (V_n + 1)U_n + e^t f_n,$$

and, using the variation of parameters formula, we obtain

$$e^{t_0} u_n(w_0,t_0) \ge \int_0^{t_0} e^s \left(e^{(t_0-s)\Delta_{\mathbb{H}}}(V_n+1)u_n(s)\right)(w_0)\,ds, \quad (w_0,t_0) \in \mathbb{H}^N \times (0,T), \tag{3.5}$$

since $e^{\Delta_{\mathbb{H}}}u_0 \geq 0$ and $f_n \geq 0$. On the other hand, it follows from the Gaussian estimates in Theorem 2.2 that

$$\int_0^{t_0} e^s (e^{(t_0-s)\Delta_{\mathbb{H}}}(V_n+1)u_n(s))(w_0)\,ds$$
$$\geq C\int_0^{t_0}\int_{\mathbb{H}^N} e^s(V_n(\tilde{w})+1)u_n(\tilde{w},s)(t_0-s)^{-\frac{Q}{2}}\exp\left(-c\frac{d^2(\tilde{w}^{-1}\circ w_0)}{t_0-s}\right)d\tilde{w}ds.$$

So if $\Omega' \subset\subset \mathbb{H}^N$ and $\varepsilon \in (0,T)$, it follows that, for $(w_0,t_0) \in \mathbb{H}^N \times (0,T)$, there is $c_0 > 0$ such that

$$c_0\int_0^{t_0-\varepsilon}\int_{\Omega'} V_n(\tilde{w})u_n(\tilde{w},s)\,d\tilde{w}ds + c_0\int_0^{t_0-\varepsilon}\int_{\Omega'} u_n(\tilde{w},s)\,d\tilde{w}ds \leq e^{t_0}u_n(w_0,t_0). \tag{3.6}$$

By our hypothesis u_n increases, moreover the right-hand side of (3.6) is clearly bounded, so by the monotone convergence theorem, $u_n \uparrow u$ and $V_n u_n \uparrow Vu$ in $L^1(\Omega' \times (0, t_0-\varepsilon))$ and u is a solution of (3.1) in the sense of distributions. □

Remark 3.3. Notice that the solution of (3.1) satisfies the integral equation

$$u(w,t) = e^{t\Delta_{\mathbb{H}}}u_0(w) + \int_0^t e^{(t-s)\Delta_{\mathbb{H}}}V(w)u(w,s)\,ds$$
$$+\int_0^t e^{(t-s)\Delta_{\mathbb{H}}}f(w,s)\,ds, \qquad (w,t) \in \mathbb{H}^N \times (0,t_0).$$

Also, since $u_n(w,t) \to u(w,t) < \infty$ a.e. on $\mathbb{H}^N \times (0,t_0)$, we get, using (3.5), $s \longmapsto (e^{(t_0-s)\Delta_{\mathbb{H}}}V(\cdot)u(\cdot,s))(w) \in L^1(0,t_0)$ for a.e. $w \in \mathbb{H}^N$.

The inverse square potential in the Euclidean case of $x \in \mathbb{R}^N$ is $V_c^*(x) = \dfrac{c}{|x|^2}$ and the critical constant is the best constant

$$C_*(N) = \left(\frac{N-2}{2}\right)^2$$

in Hardy's inequality

$$\int_{\mathbb{R}^N} |\nabla u(x)|^2\,dx \geq C_*(N)\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,dx$$

for $u \in C_c^1(\mathbb{R}^N)$ if $N \geq 3$ and for $u \in C_c^1(\mathbb{R}^N \setminus \{0\})$ if $N = 1, 2$.

The multiplication operator V_c^* and the Laplacian both have the same scaling property, namely

$$U(\lambda)^{-1}\mathcal{L}U(\lambda) = \lambda^2\mathcal{L}$$

for $\mathcal{L} = V_c^*$ or $\mathcal{L} = \Delta$, where $U(\lambda)f(x) = \lambda^{\frac{N}{2}} f(\lambda x)$, for $\lambda > 0$, defines a unitary operator on $L^2(\mathbb{R}^N)$.

In the case of the Heisenberg group $\mathbb{H}^N$, the corresponding critical potential is

$$\tilde{V}_c^*(w) = \frac{c|z|^2}{|z|^2 + l^2}$$

for $w = (x, y, l) = (z, l) \in \mathbb{H}^N$ and $c > 0$. The corresponding Hardy's inequality, due to Garofalo and Lanconelli ([11], see also [4, 15]), is

$$\int_{\mathbb{H}^N} |\nabla_{\mathbb{H}^N} u(w)|^2 \, dw \geq C^*(N) \int_{\mathbb{H}^N} \tilde{V}_1^*(w) |u(w)|^2 \, dw,$$

with the best constant being $C^*(N) = N^2$, for all $N \geq 1$. Both $\Delta_{\mathbb{H}^N}$ and multiplication by $\tilde{V}_c^*$ scale in the same way. Let

$$\tilde{U}(\lambda) f(z, l) = \lambda^{N+1} f(\lambda z, \lambda^2 l);$$

$\tilde{U}(\lambda)$ is unitary on $L^2(\mathbb{H}^N)$ for all $\lambda > 0$ and

$$\tilde{U}(\lambda)^{-1} \mathcal{L} \tilde{U}(\lambda) = \lambda^2 \mathcal{L}$$

for $\mathcal{L} = \Delta_{\mathbb{H}^N}$ or $\mathcal{L} = \tilde{V}_c^*$, and all $\lambda > 0$.

As in the Euclidean case, the critical potential is $C^*(N)\tilde{V}_1^* = \tilde{V}_{C^*(N)}^*$ near the origin. That is, by localizing to the unit ball B_1 in $\mathbb{H}^N$ (or to B_ρ for any $\rho > 0$), let

$$V_0^*(w) = \begin{cases} \dfrac{c|z|^2}{|z|^4 + l^2} & w \in B_1 \\[2ex] 0 & w \in \mathbb{H}^N \setminus B_1, \end{cases} \tag{3.7}$$

where B_1 is the unit ball centered at the origin in $\mathbb{H}^N$ with respect to the metric $\rho(w, w') = d(w'^{-1} \circ w)$, w, $w' \in \mathbb{H}^N$.

Finally, notice that the smallest root of

$$\alpha(Q - 2 - \alpha) = c$$

is given by

$$\alpha = \frac{Q-2}{2} - \sqrt{\left(\frac{Q-2}{2}\right)^2 - c} = N - \sqrt{N^2 - c},$$

when $c \leq C^*(N)$.

The following theorem is the main result of this paper. It is an extension of [15, Theorem 1.1] and a generalization of [3].

Theorem 3.4.

(i) *Let* $0 \leq c \leq C^*(N)$ *and let* $V \geq 0$ *be a measurable potential satisfying* $V \in L^\infty(\mathbb{H}^N \setminus B_1)$. *Let* $0 \leq f \in L^1(\mathbb{H}^N \times (0,T))$. *If* $V \leq V_0^*$ *in* B_1, *then* (3.1) *has a solution if*

$$\int_{\mathbb{H}^N} d(w)^{-\alpha} u_0(w)\, dw < \infty\,, \quad \int_0^T \int_{\mathbb{H}^N} f(w,s) d(w)^{-\alpha} dw ds < \infty, \tag{3.8}$$

where α *is the smallest root of* $\alpha(2N-\alpha) = c$. *If* $V \geq V_0^*$ *in* B_1, *and if* (3.1) *has a solution* u, *then*

$$\int_{\mathbb{H}^N} d(w)^{-\alpha} u_0(w)\, dw < \infty, \quad \int_0^{T-\varepsilon} \int_{\Omega'} f(w,s) d(w)^{-\alpha} dw ds < \infty$$

for each $\varepsilon \in (0,T)$ *and each* $\Omega' \subset\subset \mathbb{H}^N$ *with* α *as above. If either* $u_0 \not\equiv 0$ *or* $f \not\equiv 0$ *in* $\mathbb{H}^N \times (0,\varepsilon)$ *for each* $\varepsilon \in (0,T)$, *then given* $\Omega' \subset\subset \mathbb{H}^N$ *with* $0 \in \Omega'$, *there is a constant* $C = C(\varepsilon, \Omega') > 0$ *such that*

$$u(w,t) \geq \frac{C}{d^\alpha(\omega)}, \quad (w,t) \in \Omega' \times [\varepsilon, T]; \tag{3.9}$$

(ii) *If* $c > C^*(N)$, $V \geq V_0^*$ *and either* $u_0 \not\equiv 0$ *or* $f \not\equiv 0$, *then* (3.1) *does not have a positive solution. Moreover, we have instantaneous blowup.*

Proof.
(i) We can assume that $V \leq V_0^*$ in $\mathbb{H}^N$. Otherwise consider $V = \tilde{V} + B =: V\chi_{B_1} + V\chi_{B_1^c}$ with $\tilde{V} \leq V_0^*$ in $\mathbb{H}^N$, $B \in L^\infty(\mathbb{H}^N)$ and use Proposition A.5 in the Appendix.

Let $\phi(w) := d(w)^{-\alpha}$, and choose a convex function $\rho \in C^2(\mathbb{R})$ with $\rho(0) = \rho'(0) = 0$. Next, multiply (3.2) by $\rho'(u_n)\phi$ and integrate over $\mathbb{H}^N \times [\delta, t)$ for $0 < \delta < t < T$. Then, letting $\int$ denote $\int_{\mathbb{H}^N}$,

$$\int_\delta^t \int \frac{\partial u_n}{\partial s} \rho'(u_n)\phi = \int_\delta^t \int \Delta_{\mathbb{H}} u_n \rho'(u_n)\phi + \int_\delta^t \int (V_n u_n + f_n)\rho'(u_n)\phi,$$

and so

$$\int \int_\delta^t \frac{\partial}{\partial s}(\rho(u_n))\phi = -\int_\delta^t \int \nabla_{\mathbb{H}} u_n \cdot \nabla_{\mathbb{H}}(\rho'(u_n)\phi) + \int_\delta^t \int (V_n u_n + f_n)\rho'(u_n)\phi.$$

Then,

$$\begin{aligned}\int \rho(u_n(t))\phi &= -\int_\delta^t \int \rho''(u_n)|\nabla_{\mathbb{H}} u_n|^2\phi + (\nabla_{\mathbb{H}} u_n \cdot \nabla_{\mathbb{H}}\phi)\rho'(u_n) \\ &\qquad + \int_\delta^t \int (V_n u_n + f_n)\rho'(u_n)\phi + \int \rho(u_n(\delta))\phi \\ &= \int_\delta^t \int -\rho''(u_n)|\nabla_{\mathbb{H}} u_n|^2\phi + \int_\delta^t \int \rho(u_n)\Delta_{\mathbb{H}}\phi \\ &\qquad + \int_\delta^t \int (V_n u_n + f_n)\rho'(u_n)\phi + \int \rho(u_n(\delta))\phi,\end{aligned}$$

since $\rho'(u_n)\nabla_{\mathbb{H}} u_n = \nabla_{\mathbb{H}}(\rho(u_n))$. Hence

$$\begin{aligned}\int \rho(u_n(t))\phi \le& \int_\delta^t \int \rho(u_n)\Delta_{\mathbb{H}}\phi \\ &+ \int_\delta^t \int (V_n u_n + f_n)\rho'(u_n)\phi + \int \rho(u_n(\delta))\phi.\end{aligned} \tag{3.10}$$

Replace ρ in (3.10) with the convex function $\rho_\varepsilon(r) = \sqrt{r^2+\varepsilon^2} - \varepsilon^2$, $r \ge 0$, and let $\varepsilon \to 0$ to obtain, by the monotone convergence theorem,

$$\int u_n(t)\phi \le \int_\delta^t \int u_n \Delta_{\mathbb{H}}\phi + \int_\delta^t \int (V_n u_n + f_n)\phi + \int u_n(\delta)\phi. \tag{3.11}$$

Next we want to let $\delta \to 0$. Notice that

$$e^{\delta\Delta_{\mathbb{H}}} u_0 \le u_n(\delta) = e^{\delta(\Delta_{\mathbb{H}}+V_n)} u_0 + \int_0^\delta e^{(\delta-s)(\Delta_{\mathbb{H}}+V_n)} f_n(s)ds. \tag{3.12}$$

Since $\|V_n\|_\infty =: c_n < \infty$, it follows from the Daletskii–Trotter product formula that

$$\begin{aligned}e^{\delta(\Delta_{\mathbb{H}}+V_n)} u_0 &= \lim_{m\to\infty} \left(e^{\delta\Delta_{\mathbb{H}}/m} e^{\frac{\delta}{m}V_n}\right)^m u_0 \\ &\le e^{\delta c_n} e^{\delta\Delta_{\mathbb{H}}} u_0,\end{aligned}$$

by the positivity of the semigroup $\{e^{\delta\Delta_{\mathbb{H}}}\}$. So (3.12) becomes

$$e^{\delta\Delta_{\mathbb{H}}} u_0 \le u_n(\delta) \le e^{\delta c_n} e^{\delta\Delta_{\mathbb{H}}} u_0 + \int_0^\delta e^{c_n(\delta-s)} e^{(\delta-s)\Delta_{\mathbb{H}}} f_n(s)ds,$$

and by the contractivity of $e^{t\Delta_{\mathbb{H}}}$ we have

$$\int \left(e^{\delta\Delta_{\mathbb{H}}} u_0\right)\phi \le \int u_n(\delta)\phi \le e^{\delta c_n} \int \left(e^{\delta\Delta_{\mathbb{H}}} u_0\right)\phi + e^{\delta c_n}\|f_n\|_\infty \delta \int \phi.$$

The strong continuity of the semigroup implies

$$\lim_{\delta\to 0}\int \left(e^{\delta\Delta_{\mathbb{H}}}u_0\right)\phi = \int \phi u_0.$$

Thus we have shown that

$$\lim_{\delta\to 0}\int u_n(\delta)\phi = \int \phi u_0.$$

Now let $\delta \to 0$ in (3.11), using (2.5), to deduce

$$\begin{aligned}\int u_n(t)\phi &\le \int_0^t\int u_n\Delta_{\mathbb{H}}\phi + \int_0^t\int V_n u_n\phi + \int_0^t\int f_n\phi + \int u_0\phi\\ &= \int_0^t\int\left(-c\frac{|z|^2}{|z|^4+l^2}+V_n\right)u_n\phi + \int_0^t\int f_n\phi + \int u_0\phi\\ &\le \int_0^t\int f_n\phi + \int u_0\phi,\end{aligned}$$

since $V_n \le V_0^*$. It follows that if $\int_0^t\int f\phi + \int \phi u_0 < \infty$, then, by Proposition 3.2, $u_n(w,t) \uparrow u(w,t)(< +\infty)$ as $n\to\infty$ for all $t\in(0,T]$ and a.e. $w\in\mathbb{H}^N$, which gives the first part of (i) of the theorem.

Let us now prove the second part of (i). The inequality (3.9) is proved in Lemma 3.5 below. On the other hand, by the first part of (i) we have that, for each $w\in\mathbb{H}^N\setminus\{0\}$, (3.1) has a solution with $u_0=\phi^{-1}(w)\delta_w$, $f\equiv 0$ and $V=V_0^*$, where δ_w denotes the Dirac measure at w. We denote this solution by u_w. We define

$$h_w(\tilde w,t)=u_w(\tilde w,t)\phi(\tilde w)^{-1},\quad (\tilde w,t)\in\mathbb{H}^N\times(0,T],$$

and set $h=u\phi^{-1}$ and $h_n=u_n\phi^{-1}$ with u (respectively u_n) the solution of (3.1) (respectively (3.2)) obtained by Proposition 3.2.

We now prove

$$\begin{aligned}h(w,t) \ge &\int_{\mathbb{H}^N} h_w(\tilde w,t)\phi(\tilde w)u_0(\tilde w)\,d\tilde w\\ &+\int_0^t\int_{\mathbb{H}^N} h_w(\tilde w,t-s)f(\tilde w,s)\phi(\tilde w)\,d\tilde w\,ds\end{aligned}\tag{3.13}$$

holds for $w\in\mathbb{H}^N\setminus\{0\}$ and $t\in(0,T]$.

To this end let u_n be the solution of (3.2), and let v_n be the solution of

$$\begin{cases}\dfrac{\partial v_n}{\partial t}=\Delta_{\mathbb{H}}v_n+V_{0,n}v_n\\ v_n(0)=\phi(w)^{-1}\delta_w,\end{cases}$$

where $V_{0,n} = \min\{V_0^*, n\}$. Note that by the above construction, $v_n(\tilde{w}, t) \uparrow u_w(\tilde{w}, t)$ as $n \to \infty$ for all $t \in (0, T]$ and a.e. $\tilde{w} \in \mathbb{H}^N$.

On the other hand, we have

$$\begin{aligned}
&\frac{\partial}{\partial s} \int_{\mathbb{H}^N} u_n(\tilde{w}, s) v_n(\tilde{w}, t-s)\, d\tilde{w} \\
&= \int_{\mathbb{H}^N} \left[\frac{\partial u_n}{\partial s}(\tilde{w}, s) v_n(\tilde{w}, t-s) - u_n(\tilde{w}, s) \frac{\partial v_n}{\partial s}(\tilde{w}, t-s) \right] d\tilde{w} \\
&= \int_{\mathbb{H}^N} \left[v_n(\tilde{w}, t-s) \Delta_{\mathbb{H}} u_n(\tilde{w}, s) - \Delta_{\mathbb{H}} v_n(\tilde{w}, t-s) u_n(\tilde{w}, s) + f_n(\tilde{w}, s) v_n(\tilde{w}, t-s) \right] d\tilde{w} \\
&\quad + \int_{\mathbb{H}^N} (V_n - V_{0,n})\, u_n(\tilde{w}, s) v_n(\tilde{w}, t-s)\, d\tilde{w} \\
&\geq \int_{\mathbb{H}^N} f_n(\tilde{w}, s) v_n(\tilde{w}, t-s)\, d\tilde{w}.
\end{aligned}$$

Hence, integration from δ to $t - \delta$ yields

$$\begin{aligned}
&\int_{\mathbb{H}^N} u_n(\tilde{w}, t-\delta) v_n(\tilde{w}, \delta)\, d\tilde{w} \\
&\geq \int_{\delta}^{t-\delta} \int_{\mathbb{H}^N} f_n(\tilde{w}, s) v_n(\tilde{w}, t-s)\, d\tilde{w}\, ds + \int_{\mathbb{H}^N} u_n(\tilde{w}, \delta) v_n(\tilde{w}, t-\delta)\, d\tilde{w}.
\end{aligned} \tag{3.14}$$

Letting $\delta \to 0$ in (3.14) and noting that, as $\delta \to 0$, $u_n(t-\delta) \to u_n(t)$ weakly, $v_n(t-\delta) \to v_n(t)$ weakly, $u_n(\delta) \to u_0$ weakly and $v_n(\delta) \to \phi(w)^{-1}\delta_w$ weakly, we get

$$\begin{aligned}
u_n(w, t)\phi^{-1}(w) \geq &\int_0^t \int_{\mathbb{H}^N} f_n(\tilde{w}, s) v_n(\tilde{w}, t-s)\, d\tilde{w}\, ds \\
&+ \int_{\mathbb{H}^N} v_n(\tilde{w}, t) u_0(\tilde{w})\, d\tilde{w}.
\end{aligned} \tag{3.15}$$

Letting $n \to \infty$ in (3.15) and noting that $u_n(w, t) \uparrow u(w, t) = h(w, t)\phi(w)$ and $v_n(\tilde{w}, t) \uparrow u_w(\tilde{w}, t) = h_w(\tilde{w}, t)\phi(\tilde{w})$, we obtain (3.13).

Applying (3.9) to u_w for a fixed $w \in \mathbb{H}^N \setminus \{0\}$, we obtain that there exists a constant $C > 0$ such that

$$h_w(\tilde{w}, t) \geq C \text{ for } (\tilde{w}, t) \in \Omega' \times [\varepsilon, T].$$

It follows from (3.13) that

$$h(w, t) \geq C \int_{\Omega'} \phi(\tilde{w}) u_0(\tilde{w})\, d\tilde{w} + C \int_0^{T-\varepsilon} \int_{\Omega'} f(\tilde{w}, s)\phi(\tilde{w})\, d\tilde{w} ds.$$

If a solution u exists, we must have $h(w, t) < \infty$ for a.e. $w \in \mathbb{H}^N$ and all $t \in (0, T]$. Thus, necessary conditions for the existence of a solution u are

$$\int_{\Omega'} \phi(w) u_0(w)\, dw < \infty, \text{ and } \int_0^{T-\varepsilon} \int_{\Omega'} f(w, s)\phi(w)\, dw\, ds < \infty.$$

This completes the proof of (i).

(ii) Let $c > C^*(N)$ and let $u \not\equiv 0$ be a solution of (3.1). Then,

$$\frac{\partial u}{\partial t} - \Delta_{\mathbb{H}} u = C^*(N)\frac{|z|^2}{|z|^4 + l^2}u + (c - C^*(N))\frac{|z|^2}{|z|^4 + l^2}u.$$

From part (i), a solution exists only if

$$(c - C^*(N))\frac{|z|^2}{|z|^4 + l^2}u\phi \in L^1(\Omega' \times (0, T - \varepsilon)),$$

for Ω' any compact set in $\mathbb{H}^N$ and $\varepsilon > 0$. (Here we have assumed $0 \in \Omega'$.) But by the preceding proof (see (3.9) with $\alpha = N$), we have

$$u \geq C_\varepsilon d^{-N}(w)$$

in $\Omega' \times [\varepsilon, T)$, and so we would need $\frac{|z|^2}{|z|^4+l^2} d^{-N} \in L^1(\Omega')$, which is false. □

Lemma 3.5. *Assume* $0 \leq c \leq C^*(N)$*,* α *the smallest root of* $\alpha(2N - \alpha) = c$*, and* $0 \leq V \in L^\infty(\mathbb{H}^N \setminus B_1)$ *with* $V \geq V_0^*$ *in* B_1*. If* u *is a solution of* (3.1) *with* $u_0 \not\equiv 0$ *in* $\mathbb{H}^N$*, then given* $\Omega' \subset\subset \mathbb{H}^N$*,* $\varepsilon \in (0, T)$*, there is* $C = C(\varepsilon, \Omega') > 0$ *such that* (3.9) *holds.*

Proof. Assume that $\Omega' \subset\subset \mathbb{H}^N$ with $0 \in \Omega'$. Since $u_0 \not\equiv 0$, it follows from Theorem 2.2 that there is a constant $C_0 > 0$ with

$$e^{t\Delta_{\mathbb{H}}} u_0(\tilde{w}) \geq C_0, \tag{3.16}$$

for $\tilde{w} \in \Omega'$ and $\frac{\varepsilon}{2} \leq t < T$. Since $u \geq e^{t\Delta_{\mathbb{H}}} u_0$, by the Maximum Principle, (3.9) follows from (3.16) for the case $\alpha = 0$. So from now on we assume that α is strictly positive.

Let as before $V_{0,n} := \inf\{V_0^*, n\}$, and consider the problems

$$\begin{cases} \dfrac{\partial z}{\partial t} = \Delta_{\mathbb{H}} z + V_0^* z & \text{in } \mathcal{D}'\left(\mathbb{H}^N \times [\frac{\varepsilon}{2}, T]\right) \\ z\left(\tilde{w}, \frac{\varepsilon}{2}\right) = C_0\, \chi_{\Omega'}(\tilde{w}) & \tilde{w} \in \mathbb{H}^N, \end{cases} \tag{3.17}$$

$$\begin{cases} \dfrac{\partial z_n}{\partial t} = \Delta_{\mathbb{H}} z_n + V_{0,n} z_n & \text{in } \mathcal{D}'\left(\mathbb{H}^N \times [\frac{\varepsilon}{2}, T]\right) \\ z_n\left(\tilde{w}, \frac{\varepsilon}{2}\right) = C_0\, \chi_{\Omega'}(\tilde{w}) & \tilde{w} \in \mathbb{H}^N, \end{cases} \tag{3.18}$$

and for $B_{r_0} \subset \Omega'$ a ball centered at the origin with radius $r_0 \in (0, 1)$,

$$\begin{cases} \dfrac{\partial v_n}{\partial t} = \Delta_{\mathbb{H}} v_n + V_{0,n} v_n & \text{in } \mathcal{D}'\left(B_{r_0} \times [\frac{\varepsilon}{2}, T]\right) \\ v_n = 0 & \text{on } \partial B_{r_0} \\ v_n\left(\tilde{w}, \frac{\varepsilon}{2}\right) = C_0 & \tilde{w} \in B_{r_0}. \end{cases} \tag{3.19}$$

Notice that (3.18) has a unique solution $z_n \geq 0$, also that $z_n(\tilde{w}, t) \uparrow z(\tilde{w}, t)$, for almost every $(\tilde{w}, t) \in \mathbb{H}^N \times [\frac{\varepsilon}{2}, T)$, where z is the unique solution of (3.17). It is also clear that $z_n \geq v_n$, the solution of (3.19), and that v_n is a radial function.[1] Finally, we note that u is bounded below by the solution of (3.17) since $V \geq V_0^*$.

Multiply the equation in (3.19) by $v_n^{p-1}\phi^{2-p}$, $p \geq 2$, where we recall that $\phi(w) = d(w)^{-\alpha}$ only depends on w, and integrate to get

$$\int_{B_{r_0}} \frac{\partial v_n}{\partial t} v_n^{p-1}\phi^{2-p} = \int_{B_{r_0}} (\Delta_{\mathbb{H}} v_n) v_n^{p-1}\phi^{2-p} + \int_{B_{r_0}} V_{0,n} v_n^p \phi^{2-p},$$

so

$$\begin{aligned} \frac{\partial}{\partial t}\int_{B_{r_0}} \frac{1}{p}\left(\frac{v_n}{\phi}\right)^p \phi^2 = &-\int_{B_{r_0}} \nabla_{\mathbb{H}} v_n \cdot \nabla_{\mathbb{H}} \left(v_n^{p-1}\phi^{2-p}\right) \\ &+ \int_{B_{r_0}} V_{0,n}\left(\frac{v_n}{\phi}\right)^p \phi^2. \end{aligned} \tag{3.20}$$

Set $g_n = \frac{v_n}{\phi}$. Then equation (3.20) becomes

$$\begin{aligned} \frac{\partial}{\partial t}\int_{B_{r_0}} \frac{1}{p} g_n^p \phi^2 = &-\frac{4(p-1)}{p^2}\int_{B_{r_0}} |\nabla_{\mathbb{H}} g_n^{p/2}|^2 \phi^2 \\ &+ \int_{B_{r_0}} g_n^p \left(\Delta_{\mathbb{H}}\phi\right)\phi + \int_{B_{r_0}} V_{0,n} g_n^p \phi^2. \end{aligned}$$

Using (2.5) and the fact that $\alpha(2N - \alpha) = c$, we obtain $V_{0,n} \leq V_0^* \leq -\frac{\Delta_{\mathbb{H}}\phi}{\phi}$, so that $V_{0,n}\phi^2 \leq (-\Delta_{\mathbb{H}}\phi)\,\phi$. Hence we have shown

$$\frac{\partial}{\partial t}\int_{B_{r_0}} g_n^p \phi^2 \leq 0,$$

and we thus have, for $\frac{\varepsilon}{2} \leq t \leq T$, that

$$\left(\int_{B_{r_0}} v_n^p \phi^{2-p}\right)^{1/p} \leq C_0 \left(\int_{B_{r_0}} \phi^{2-p}\right)^{1/p}. \tag{3.21}$$

Letting $p \to \infty$ in (3.21) we get

$$g_n \leq C_0 \text{ a.e. in } B_{r_0}, \tag{3.22}$$

which is equivalent to $v_n \leq C_0\phi$ a.e. in B_{r_0}. So we can make sense of

$$v = \lim_{n\to\infty} v_n \text{ and } g = \lim_{n\to\infty} g_n.$$

[1] Recall that a function $g(w)$ is radial on $\mathbb{H}^n$ if $w = (z, l)$ and $g(z, l) = g(|z|, l)$. In fact, our function v_n is even more special, since $v_n = v_n(d(w))$. Notice that this gives $\nabla_{\mathbb{H}} v_n = v_n'(d(w))\nabla_{\mathbb{H}} d(w)$.

Now we claim that

$$0 < C_1 \le g(w,t) \le C_0 \tag{3.23}$$

for $t \in [\varepsilon, T]$ and a.e. $w \in B_{\frac{r_0}{2}}$. Once (3.23) is proved, and since

$$u \ge z \ge z_n \ge v_n = g_n\phi,$$

(3.9) follows directly in the case $\Omega' = B_{\frac{r_0}{2}}$. Otherwise, we observe that for almost every $\tilde{w} \in \Omega' \setminus B_{\frac{r_0}{2}}$ we have

$$h(\tilde{w},t) = \phi(\tilde{w})^{-1}u(\tilde{w}) \ge \phi(\tilde{w})^{-1}\left(e^{t\Delta_{\mathbb{H}}}u_0\right)(\tilde{w}) \ge C_2 > 0$$

from Theorem 2.2 since

$$\phi(\tilde{w})^{-1} \ge C_3 > 0$$

for all $t \in [\varepsilon, T]$ and some constants $C_2, C_3 > 0$. This concludes the proof of (3.9).

Now we must prove (3.23). By (3.22), the right inequality is proved. For the remaining part of (3.23), let $\mathcal{I} \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ be convex. Multiply equation (3.19) by $\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2$ and integrate over $Q = B_{r_0} \times \left(\frac{\varepsilon}{2}, t\right)$, $t \in [\frac{\varepsilon}{2}, T]$, where $\psi \in \mathcal{D}(B_{r_0} \times (\frac{\varepsilon}{2}, T])$, to get

$$\int_Q \mathcal{I}'(g_n)\mathcal{I}(g_n)\frac{\partial v_n}{\partial t}\phi\psi^2 = \int_Q \left\{\Delta_{\mathbb{H}} v_n \mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2 + V_{0,n}v_n\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2\right\},$$

$$\begin{aligned}\frac{1}{2}\int_Q \frac{\partial}{\partial t}(\mathcal{I}(g_n))^2\phi^2\psi^2 &= -\int_Q \nabla_{\mathbb{H}}(g_n\phi)\cdot\nabla_{\mathbb{H}}(\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2)\\ &\quad + \int_Q V_{0,n}g_n\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi^2\psi^2.\end{aligned}$$

Notice that

$$\begin{aligned}&\int_{B_{r_0}} \nabla_{\mathbb{H}}(g_n\phi)\cdot\nabla_{\mathbb{H}}(\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2)\\
&= \int_{B_{r_0}} \left\{(\nabla_{\mathbb{H}}g_n\cdot\nabla_{\mathbb{H}}(\mathcal{I}'(g_n))\mathcal{I}(g_n)\phi\psi^2)\phi + g_n(\nabla_{\mathbb{H}}\phi\cdot\nabla_{\mathbb{H}}(\mathcal{I}'(g_n))\mathcal{I}(g_n)\phi\psi^2)\right\}\\
&= \int_{B_{r_0}} \left\{\mathcal{I}''(g_n)|\nabla_{\mathbb{H}}g_n|^2\mathcal{I}(g_n)\phi^2\psi^2 + |\nabla_{\mathbb{H}}\mathcal{I}(g_n)|^2\phi^2\psi^2\right\}\\
&\quad + \int_{B_{r_0}} \left\{(\nabla_{\mathbb{H}}\mathcal{I}(g_n)\cdot\nabla_{\mathbb{H}}\phi)\psi^2\phi\mathcal{I}(g_n) + (\nabla_{\mathbb{H}}\mathcal{I}(g_n)\cdot\nabla_{\mathbb{H}}\psi^2)\mathcal{I}(g_n)\phi^2\right\}\\
&\quad + \int_{B_{r_0}} (-\Delta_{\mathbb{H}}\phi)g_n\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2 + \int_{B_{r_0}} -(\nabla_{\mathbb{H}}\phi\cdot\nabla_{\mathbb{H}}\mathcal{I}(g_n))\,\mathcal{I}(g_n)\phi\psi^2\\
&= \int_{B_{r_0}} \left\{\mathcal{I}''(g_n)|\nabla_{\mathbb{H}}g_n|^2\mathcal{I}(g_n)\phi^2\psi^2 + |\nabla_{\mathbb{H}}\mathcal{I}(g_n)|^2\phi^2\psi^2\right\}\\
&\quad + \int_{B_{r_0}} (\nabla_{\mathbb{H}}\mathcal{I}(g_n)\cdot\nabla_{\mathbb{H}}\psi^2)\mathcal{I}(g_n)\phi^2 + \int_{B_{r_0}} (-\Delta_{\mathbb{H}}\phi)g_n\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2,\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{2}\int_Q \frac{\partial}{\partial t}\left(\mathcal{I}(g_n)^2\right)\phi^2\psi^2 + \int_Q \nabla_{\mathbb{H}}\mathcal{I}(g_n)\cdot\nabla_{\mathbb{H}}\psi^2\mathcal{I}(g_n)\phi^2 \\
&= -\int_Q \mathcal{I}''(g_n)|\nabla_{\mathbb{H}} g_n|^2\left(\mathcal{I}(g_n)\phi^2\psi^2\right) \\
&\quad -\int_Q |\nabla_{\mathbb{H}}\mathcal{I}(g_n)|^2\phi^2\psi^2 + \int \Delta_{\mathbb{H}}\phi g_n\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2 \\
&\quad +\int_Q V_{0,n} g_n\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi^2\psi^2.
\end{aligned} \tag{3.24}$$

Using Hölder's inequality,

$$\begin{aligned}
&\left|2\int_{B_{r_0}}\left(\nabla\mathcal{I}(g_n)\cdot\nabla_{\mathbb{H}}\psi\right)\mathcal{I}(g_n)\phi^2\psi\right| \\
&\leq \frac{1}{2}\int_{B_{r_0}}|\nabla_{\mathbb{H}}\mathcal{I}(g_n)|^2\phi^2\psi^2 + 2\int_{B_{r_0}}|\nabla_{\mathbb{H}}\psi|^2|\mathcal{I}(g_n)|^2\phi^2,
\end{aligned}$$

on the second term of the left-hand side of (3.24), using the convexity assumption on $\mathcal{I}$, and integrating by parts on the first term in (3.24), we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{B_{r_0}}\left(\mathcal{I}(g_n)^2\psi^2\phi^2\right)(t) + \frac{1}{2}\int_Q |\nabla_{\mathbb{H}}\mathcal{I}(g_n)|^2\phi^2\psi^2 \\
&\leq \int_Q (V_{0,n}\phi + \Delta_{\mathbb{H}}\phi) g_n\mathcal{I}'(g_n)\mathcal{I}(g_n)\phi\psi^2 \\
&\quad + \int_Q \mathcal{I}(g_n)^2\left(2|\nabla_{\mathbb{H}}\psi|^2 + \psi\frac{\partial\psi}{\partial t}\right)\phi^2.
\end{aligned} \tag{3.25}$$

Now, we make a key observation. Since $\alpha < N$, we have $\phi\Delta_{\mathbb{H}}\phi \in L^1(B_{r_0})$. Assume r_0 to be sufficiently small. Since $V_0^* = \frac{-\Delta_{\mathbb{H}}\phi}{\phi}$, the first term on the right-hand side of (3.25) converges to 0 as $n \to \infty$ by Lebesgue's dominated convergence theorem, since $\|g_n\| \leq C_0$ in B_{r_0} and $\mathcal{I}$ is convex, C^2 and non-negative. Letting $n \to \infty$, (3.25) gives

$$\begin{aligned}
&\int_{B_{r_0}}\mathcal{I}(g)^2\psi^2\phi^2 + \int_Q |\nabla_{\mathbb{H}}\mathcal{I}(g)|^2\psi^2\phi^2 \\
&\leq 2\int_Q \mathcal{I}(g)^2\left(2|\nabla_{\mathbb{H}}\psi|^2 + \psi\frac{\partial\psi}{\partial t}\right)\phi^2.
\end{aligned} \tag{3.26}$$

Choose ψ so that $0 \leq \psi \leq 1$ for $s \geq \frac{\varepsilon}{2}$, $r < r_0$ and $0 < \delta < r$, and

$$\psi(w,t) = \begin{cases} 1 & B_{r-\delta}\times[s+\delta,T] \\ 0 & \left(B_{r_0}\times[0,s]\right)\cup\left(B_{r_0}\setminus B_{r-\frac{\delta}{2}}\times[0,T]\right), \end{cases}$$

so that

$$\left|\frac{\partial\psi}{\partial t}\right| \le \frac{\tilde{C}}{\delta}, \qquad |\nabla_{\mathbb{H}}\psi|^2 \le \frac{\tilde{C}}{\delta^2},$$

for some constant $\tilde{C}$ independent of s, δ. Then for all $s+\delta \le t \le T$ (3.26) becomes

$$\begin{aligned} &\int_{B_{r-\delta}} \mathcal{I}(g(t))^2\phi^2 + \int_{s+\delta}^{T}\int_{B_{r-\delta}} |\nabla_{\mathbb{H}}\mathcal{I}(g)|^2\phi^2 \\ &\le 6\tilde{C}\delta^{-2}\int_s^T\int_{B_{r_0}} \mathcal{I}(g)^2\phi^2. \end{aligned} \tag{3.27}$$

Note that for fixed t, $w \mapsto \mathcal{I}(g(w,t))$ is a radial function; in fact as we noted earlier, $\mathcal{I}(g(w,t))$ is a function of $d(w)$. Applying (A.4), with β as in Lemma A.4, and (3.27), one obtains

$$\begin{aligned} &\int_{s+\delta}^{T}\int_{B_{r-\delta}} \mathcal{I}(g)^{2+2\beta}\phi^2 \\ &\le \hat{\hat{C}}\int_{s+\delta}^{T}\left[\left(\int_{B_{r-\delta}} |\nabla_{\mathbb{H}}\mathcal{I}(g(t))|^2\phi^2 + \mathcal{I}(g(t))^2\phi^2\right)\left(\int_{B_{r-\delta}} \mathcal{I}(g(t))^2\phi^2\right)^{\beta}\right] dt \\ &\le \hat{\hat{C}}\left[\int_{s+\delta}^{T}\int_{B_{r-\delta}}|\nabla_{\mathbb{H}}\mathcal{I}(g)|^2\phi^2 + \int_{s+\delta}^{T}\int_{B_{r-\delta}} I(g)^2\phi^2\right]\left(6\tilde{C}\delta^{-2}\int_s^T\int_{B_{r_0}} I(g)^2\phi^2\right)^{\beta} \\ &\le \hat{\hat{C}}(6\tilde{C}\delta^{-2}+1)\left(\int_s^T\int_{B_{r_0}}\mathcal{I}(g)^2\phi^2\right)\left(6\tilde{C}\delta^{-2}\int_s^T\int_{B_{r_0}}\mathcal{I}(g)^2\phi^2\right)^{\beta}. \end{aligned}$$

Since $0 < \delta < r < 1$, it follows that

$$\begin{aligned} &\left(\int_{s+\delta}^{T}\int_{B_{r-\delta}} \mathcal{I}(g)^{2+2\beta}\phi^2\right)^{\frac{1}{2+2\beta}} \\ &\le \hat{\hat{C}}^{\frac{1}{2+2\beta}}(6\tilde{C}+1)^{1/2}\delta^{-1}\left(\int_s^T\int_{B_{r_0}}\mathcal{I}(g)^2\phi^2\right)^{1/2} \\ &\le \overline{C}\delta^{-1}\left(\int_s^T\int_{B_{r_0}}\mathcal{I}(g)^2\phi^2\right)^{1/2}. \end{aligned} \tag{3.28}$$

Let $b > 0$ be sufficiently small, and set

$$\delta = \frac{b}{2^n}, \quad r_{n+1} = r_n - \frac{b}{2^n}, \quad \mathcal{I}_{n+1} = \mathcal{I}_n^{1+\beta}, \quad s_{n+1} = s_n + \frac{b}{2^n},$$

$$\text{and } k_n = \left(\int_{s_n}^{T}\int_{B_{r_n}} \mathcal{I}_n(g)^2\phi^2\right)^{1/2}.$$

Here $\mathcal{I}_1 = \mathcal{I}$ and $r_1, s_1 > 0$ with $s_1 \geq \frac{\varepsilon}{2}$ and $r_1 < 1$ are given. Applying (3.28) yields

$$k_{n+1}^{\frac{1}{1+\beta}} \leq \overline{C} 2^n b^{-1} k_n. \tag{3.29}$$

Applying Lemma A.1 in the Appendix, we have

$$k_n^{\frac{1}{(1+\beta)^{n-1}}} \leq \left(\frac{\overline{C}}{b}\right)^{\frac{a_n}{(1+\beta)^{n-2}}} 2^{\frac{d_n}{(1+\beta)^{n-2}}} k_1,$$

where $a_n = \sum_{j=0}^{n-2}(1+\beta)^j$ and $d_n = \sum_{j=0}^{n-2}(j+1)(1+\beta)^{n-2-j}$ for $n \geq 2$. Letting $n \to \infty$ we have

$$s_n \to s_1 + b,\ r_n \to r_1 - b,\ \frac{a_n}{(1+\beta)^{n-2}} \to \left(\frac{1+\beta}{\beta}\right),\ \frac{d_n}{(1+\beta)^{n-2}} \to \left(\frac{1+\beta}{\beta}\right)^2$$

and taking into account that $\mathcal{I}_n = \mathcal{I}^{(1+\beta)^{n-1}}$ we obtain

$$k_n^{\frac{1}{(1+\beta)^{n-1}}} = \left(\int_{s_n}^T \int_{B_{r_n}} \mathcal{I}(g)^{2(1+\beta)^{n-1}} \phi^2\right)^{\frac{1}{2(1+\beta)^{n-1}}} \to \sup_{B_{r_1-b}\times[s_1+b,T]} \mathcal{I}(g).$$

Finally, we have

$$\sup_{B_{r_1-b}\times[s_1+b,T]} \mathcal{I}(g) \leq \left(\frac{\overline{C}}{b} 2^{\frac{1+\beta}{\beta}}\right)^{\frac{1+\beta}{\beta}} \left(\int_{s_1}^T \int_{B_{r_1}} \mathcal{I}(g)^2 \phi^2\right)^{1/2}. \tag{3.30}$$

Now, consider a sequence $\mathcal{I}_n(r) \in C^2(\mathbb{R}_+, \mathbb{R}_+)$ of convex functions converging to $\frac{1}{r^\gamma}$ as $n \to \infty$, where $\gamma > 0$ is a parameter to be chosen later. Replacing $\mathcal{I}$ by a $\mathcal{I}_n$ in (3.30), we obtain

$$\sup_{B_{r_1-b}\times[s_1+b,T]} g^{-\gamma} \leq \left(\frac{\overline{C}}{b} 2^{\frac{1+\beta}{\beta}}\right)^{\frac{1+\beta}{\beta}} \left(\int_{s_1}^T \int_{B_{r_1}} g^{-2\gamma} \phi^2\right)^{1/2}.$$

Set $s_1 = \frac{3}{4}\varepsilon$, $b = \frac{\varepsilon}{4}$ and $r_1 < r_0$, where r_0 is the one chosen in the beginning of the proof. We have

$$g(\omega, t) = \frac{v}{\phi} \geq \phi^{-1}(w) \left(e^{t\Delta_{\mathbb{H}}} C_0\right)(w) = C_0 \phi^{-1}(w),$$

for almost every $w \in B_{r_0}$ where C_0 is the constant given in (3.16). So

$$\sup_{B_{r_1-\varepsilon/4}\times[\varepsilon,T]} g^{-\gamma} \leq C_2 C_0^{-\gamma} \varepsilon^{-\frac{1+\beta}{\beta}} \left(\int_{\frac{3}{4}\varepsilon}^T \int_{B_{r_1}} \phi^{2+2\gamma}\right)^{1/2},$$

and it follows that

$$g(w,t) \geq C_2^{-\frac{1}{\gamma}} C_0 \varepsilon^{\left(1+\frac{1}{\beta}\right)\frac{1}{\gamma}} \left(\int_{B_{r_1}} \phi^{2+2\gamma} \right)^{-\frac{1}{2\gamma}}, \tag{3.31}$$

for almost every $w \in B_{r_1 - \frac{\varepsilon}{4}}$ and for all $t \in [\varepsilon, T]$, where C_2 is a positive constant independent of ε and r_1. A simple computation shows that $\int_{B_{r_1}} \phi^{2+2\gamma} < \infty$ by choosing $0 < \gamma < \frac{N+1}{\alpha} - 1$, which is possible since $\alpha \in (0, N]$.

Thus (3.23) follows by taking $\gamma \in (0, \frac{N+1}{\alpha} - 1)$ and $\varepsilon = 2(2r_1 - r_0)$ with $\frac{r_0}{2} < r_1 < r_0$.

This concludes the proof. □

We end this section by the following remark.

Remark 3.6. The arguments used are based on the explicit form of the fundamental solution and on the existence of an underlying group of dilations; thus, the results would likely extend to the setting of H-type groups.

A. Appendix

In this appendix we collect all technical lemmas that we needed for proving the main result.

Lemma A.1. *For $\beta > 0$ and $n \geq 2$, define $a_n = \sum_{j=0}^{n-2} (1+\beta)^j$ and $d_n = \sum_{j=0}^{n-2} (j+1)(1+\beta)^{n-2-j}$ and $k_n \geq 0$ for $n \geq 1$ such that*

$$k_n^{\frac{1}{1+\beta}} \leq \overline{C} 2^{n-1} b^{-1} k_{n-1}.$$

Then

$$k_n^{\frac{1}{1+\beta}} \leq \left(\frac{\overline{C}}{b} \right)^{a_n} 2^{d_n} k_1^{(1+\beta)^{n-2}}. \tag{A.1}$$

Proof. We use an induction argument. Assume (A.1) is true for $1 \leq k \leq n$. We will show

$$k_{n+1}^{\frac{1}{1+\beta}} \leq \left(\frac{\overline{C}}{b} \right)^{a_{n+1}} 2^{d_{n+1}} k_1^{(1+\beta)^{n-1}}.$$

Clearly (3.29) gives (A.1) if $n = 1$. By (3.29),

$$k_{n+1}^{\frac{1}{1+\beta}} \leq \overline{C} 2^n b^{-1} k_n \leq \left(\frac{\overline{C}}{b} \right) 2^n \left(\frac{\overline{C}}{b} \right)^{a_n(1+\beta)} 2^{d_n(1+\beta)} k_1^{(1+\beta)^{n-1}},$$

by the induction hypothesis. Now it is easy to check that

$$a_n(1+\beta)+1=\sum_{j=0}^{n-2}(1+\beta)^{j+1}+1=\sum_{j=0}^{n-1}(1+\beta)^j=a_{n+1}$$

and

$$\begin{aligned} d_n(1+\beta)+n &= \sum_{j=0}^{n-2}(j+1)(1+\beta)^{n-1-j}+n=\sum_{j=0}^{n-1}(j+1)(1+\beta)^{n-1-j} \\ &= d_{n+1}. \end{aligned}$$

□

The following two lemmas can be found in [3, Appendix].

Lemma A.2. *If $0 \le h \in C^1[0,2r]$ and $h(2r)=0$, then*

$$\left(\int_0^{2r} h^p(s)s^{\gamma-1}ds\right)^{1/p} \le M_0\left(\int_0^{2r}\left|\frac{dh}{ds}\right|^2 s^{\gamma-1}ds\right)^{1/2}, \tag{A.2}$$

where $\gamma=2N+2-2\alpha>2$, $\frac{1}{p}=\frac{1}{2}-\frac{1}{\gamma}$ and M_0 is a constant depending only on γ.

Lemma A.3. *If $0<r'\le r\le 1$, $0\le h\in C^1[0,r]$ then*

$$\left(\int_0^r |h(s)|^p s^{2N-2\alpha+1}ds\right)^{2/p} \le \hat{C}\left(\int_0^r [|h'(s)|^2+|h(s)|^2]s^{2N-2\alpha+1}ds\right), \tag{A.3}$$

where $\frac{1}{p}\ge\frac{1}{2}-\frac{1}{2N+2-2\alpha}$ and $p=\infty$ if $N=\alpha-1$, $\hat{C}$ depends on r' but not r.

The following lemma is needed for the proof of Lemma 3.5.

Lemma A.4. *If $k\in C^1(\mathbb{R}_+,\mathbb{R}_+)$, $\tilde{k}(w):=k(d(w))$, $\phi(w)=d(w)^{-\alpha}$, $w=(z,l)\in\mathbb{H}^N$, and $0<\beta$ is such that $\beta+\frac{2}{p}=1$, where $\frac{1}{p}=\frac{1}{2}-\frac{1}{2N-2\alpha+2}$, then*

$$\begin{aligned} &\int_{B_r}\tilde{k}^{2+2\beta}(w)\phi^2(w)\,dw \\ &\le \hat{\hat{C}}\left(\int_{B_r}\left(|\nabla_{\mathbb{H}}\tilde{k}(w)|^2+\tilde{k}^2(w)\right)\phi^2(w)\,dw\right)\left(\int_{B_r}\tilde{k}^2(w)\phi^2(w)\,dw\right)^{\beta}. \end{aligned} \tag{A.4}$$

Proof. We first prove that

$$\int_{B_r}|\nabla_{\mathbb{H}}k(d(\omega))|^2\phi^2(d(w))\,dw=C_N\int_0^r\left|k'(s))\right|^2 s^{2N-2\alpha+1}\,ds, \tag{A.5}$$

where $C_N := S_{2N-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^N\varphi\, d\varphi$ and S_{2N-1} the surface area of the unit ball in $\mathbb{R}^{2N}$.

Consider the change of variables $\rho = |z|$ in polar coordinates, and take

$$\begin{cases} \rho^2 = r^2\cos\varphi \\ l = r^2\sin\varphi \end{cases}$$

with $\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Recalling from Lemma 2.1 that $|\nabla_{\mathbb{H}} d(w)|^2 = |z|^2 d(w)^{-2}$, $w = (z,l) \in \mathbb{H}^N$ and since $\nabla_{\mathbb{H}} k(d(w)) = k'(d(w))\nabla_{\mathbb{H}} d(w)$ we obtain

$$\begin{aligned}
&\int_{B_r} |\nabla_{\mathbb{H}} k(d(w))|^2 \phi^2(d(w))\, dw \\
&= S_{2N-1}\int |\nabla_{\mathbb{H}} k(\sqrt[4]{\rho^4+l^2})|^2\phi^2(\sqrt[4]{\rho^4+l^2})\rho^{2N-1}\, d\rho dl \\
&= S_{2N-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^r |k'(s)|^2 (s^2\cos\varphi)s^{-2}s^{-2\alpha}(s\sqrt{\cos\varphi})^{2N-1}\left(\frac{s^2}{\sqrt{\cos\varphi}}\right) ds d\varphi \\
&= C_N\int_0^r |k'(s)|^2 s^{2N-2\alpha+1}\, ds
\end{aligned}$$

for $C_N := S_{2N-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^N\varphi\, d\varphi$.

It follows from Hölder's inequality, (A.5) and Lemma A.3 that

$$\begin{aligned}
&\int_{B_r} k^{2+2\beta}(w)\phi^2(w)\, dw \\
&= C_N\int_0^r k^{2+2\beta}(s)\phi^2(s)s^{2N+1} ds \\
&\leq \left(C_N\int_0^r k^p(s)\phi^2(s)s^{2N+1} ds\right)^{2/p}\left(C_N\int_0^r k^2(s)\phi^2(s)s^{2N+1} ds\right)^\beta \\
&= \left(C_N\int_0^r k^p(s)s^{2N-2\alpha+1} ds\right)^{2/p}\left(\int_{B_r} k^2\phi^2\right)^\beta \\
&\leq \hat{C} C_N^{\frac{2}{p}-1} C_N\int_0^r \left(|k'(s)|^2 + k^2(s)\right)s^{2N-2\alpha+1} ds\left(\int_{B_r} k^2\phi^2\right)^\beta \\
&= \hat{\hat{C}}\left(\int_{B_r}|\nabla_{\mathbb{H}} k|^2\phi^2 + \int_{B_r} k^2\phi^2\right)\left(\int_{B_r} k^2\phi^2\right)^\beta .
\end{aligned}$$

This completes the proof. □

We conclude this Appendix by proving the following perturbation result:

Proposition A.5. *Assume that the problem* (3.1) *has a solution for some* $u_0 \geq 0$, $f \geq 0$ *and let* $B \in L^\infty(\mathbb{H}^N)$. *Then the problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta_{\mathbb{H}} u + V(\cdot)u + B(\cdot)u + f & \text{in } \mathcal{D}'(\mathbb{H}^N \times (0,T)) \\ \operatorname*{esslim}_{t\to 0^+} \displaystyle\int_{\mathbb{H}^N} u(w,t)\psi(w)dw = \int_{\mathbb{H}^N} u_0(w)\psi(w)\,dw & \forall\ \psi \in \mathcal{D}(\mathbb{H}^N) \\ u \geq 0 & \text{on } \mathbb{H}^N \times (0,T) \\ Vu \in L^1_{\text{loc}}(\mathbb{H}^N \times (0,T)), & \end{cases} \tag{A.6}$$

has a solution.

Proof. Let u_n be the solution of (3.2). We know that $u_n \uparrow u$, u being a solution of (3.1). Suppose that v_n solves

$$\begin{cases} \dfrac{\partial v_n}{\partial t} = \Delta_{\mathbb{H}} v_n + (V_n(\cdot) + B(\cdot))\, v_n + f_n & \text{in } \mathcal{D}'_T \\ \displaystyle\lim_{t\to 0^+} \int_{\mathbb{H}^N} v_n(w,t)\psi(w)\,dw = \int_{\mathbb{H}^N} u_0(w)\psi(w)\,dw & \forall \psi \in \mathcal{D}(\mathbb{H}^N), \end{cases} \tag{A.7}$$

where $f_n = \min\{f, n\}$ and $V_n = \min\{V, n\}$. Fix $\lambda \geq \|B\|_\infty$, and consider

$$U_n = e^{\lambda t} u_n.$$

So U_n satisfies

$$\frac{\partial U_n}{\partial t} = \Delta_{\mathbb{H}} U_n + (V_n + \lambda)U_n + e^t f_n.$$

By the Maximum Principle we have

$$v_n(w,t) \leq U_n(w,t) \leq e^{\lambda t} u(w,t) \text{ for a.e. } (w,t) \in \mathbb{H}^N \times (0,T).$$

Clearly $\{v_n\}$ is an increasing sequence and since $u, Vu \in L^1_{\text{loc}}(\mathbb{H}^N \times (0,T))$, it follows by the Monotone Convergence theorem that $v_n \uparrow v$ and $(V_n + B)v_n \uparrow (V + B)v$ in $L^1_{\text{loc}}(\mathbb{H}^N \times (0,T))$, and v gives a solution of (A.6). □

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