Measurement-based Quantum Computation

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Quantum Information Processing

A cross-disciplinary field of great importance from both fundamental and technological perspectives.

It has changed our perspective on the foundation of

Information Theory, Computation and Physics.

Birth of QIP

It is possible to perform computation both logically and thermodynamically reversible.

Quantum physics is also reversible, as the reverse-time evolution specified by the unitary operator always exists.

Quantum Mechanics in a nutshell

- **Data:** Unit vector in a Hilbert space (**qubit**)
- **Processing:** Unitary transformation
- **Result:** Projective measurement
- **Composite System:** Tensor product

Models of QC

Quantum Circuit Model
Quantum Cellular Automata
Quantum Turing Machine

Measurement-based QC
Adiabatic QC
Topological QC

Quantum Categorical Framework
Quantum Processes Calculus
Quantum Programming Languages

An end-to-end Story

- Physics - Ising Hamiltonian, one-way QC

  Raussendorf and Briegel 2000

- Formal Methods - Measurement Calculus

  Danos, Kashefi, Panangaden 2004

- Parallelism and Determinism

  Broadbent, Browne, Danos, Kashefi, Mhalla, Perdrix, 2006, 2007, 2009

- Protocol Design - Universal Blind QC

  Broadbent, Fitzsimons, Kashefi 2009

- Implementation - Foundation of Quantum Mechanics

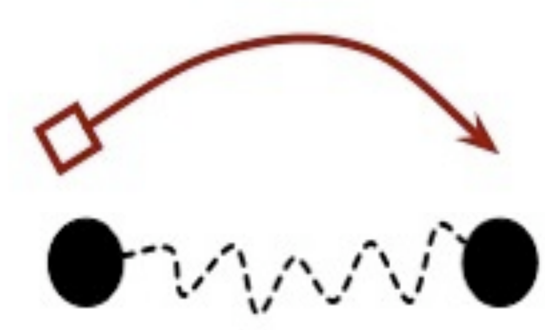
  Philip Walther experimental Lab 2010

Measurement-based QC

Measurements play a central role.

Scalable implementation

Clear separation between classical and quantum parts of computation



Entanglement

Clear separation between creation and consumption of resources

Basic Commands

- **New qubits,** to prepare the auxiliary qubits: N
- **Entanglements,** to build the quantum channel: E
- **Measurements,** to propagate(manipulate) qubits: M
- **Corrections,** to make the computation deterministic: C

2-state System $\mathbb{C}^2$

The canonical basis, $(1, 0)$, $(0, 1)$, also called the computational basis, is usually denoted $|0\rangle$, $|1\rangle$. It is orthonormal by definition of $\langle x, y\rangle_{\mathbb{C}^2}$.

$$|\pm\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle) \qquad |\pm_\alpha\rangle := \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\alpha}|1\rangle)$$

The *preparation* map N_i^α is defined to be:

$$|+_\alpha\rangle \otimes \_ : \mathfrak{H}_n \to \mathbb{C}^2 \otimes \mathfrak{H}_n$$

Maps over $\mathbb{C}^2$

Pauli Spin Matrices

$$X := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Other Single qubit gates

$$H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad P(\alpha) := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

$$P(\alpha)^{\star} = P(-\alpha)$$

The two qubit state $\mathbb{C}^2 \otimes \mathbb{C}^2$

Canonical basis

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

Bases need not be made of decomposable elements, they can consists of entangled states.

Graph basis

$$\mathcal{G}_{00} = \tfrac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$$
$$\mathcal{G}_{01} = \tfrac{1}{2}(|00\rangle - |01\rangle + |10\rangle + |11\rangle)$$
$$\mathcal{G}_{10} = \tfrac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$$
$$\mathcal{G}_{11} = \tfrac{1}{2}(|00\rangle - |01\rangle - |10\rangle - |11\rangle)$$

Maps on $\mathbb{C}^2 \otimes \mathbb{C}^2$

- In general if $f : A \longrightarrow B$ and $g : A' \longrightarrow B'$, one defines $f \otimes g : A \otimes A' \longrightarrow B \otimes B' : \psi \otimes \phi \mapsto f(\psi) \otimes g(\phi)$.

- Or given $f : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$, one defines $\wedge f$ (read controlled-f) a new map on $\mathbb{C}^2 \otimes \mathbb{C}^2$:

$$\begin{aligned} \wedge f|0\rangle|\psi\rangle &:= |0\rangle|\psi\rangle \\ \wedge f|1\rangle|\psi\rangle &:= |1\rangle f(|\psi\rangle) \end{aligned}$$

Entangling Map

$$\wedge Z(|+\rangle \otimes |+\rangle) = \mathcal{G}_{00}$$

Pauli and Clifford

Define the *Pauli group* over A as the closure of $\{X_i, Z_i \mid 1 \leq i \leq n\}$ under composition and $\otimes$. These are all local maps (corrections).

Define the *Clifford group* over A as the normalizer of the Pauli group, that is to say the set of unitaries f over A such that for all g in the Pauli group, fgf^{-1} is also in the Pauli group.

Entangling Map is in Clifford

$$\wedge Z_{ij} X_i = X_i Z_j \wedge Z_{ij}$$
$$\wedge Z_{ij} Z_i = Z_i \wedge Z_{ij}$$

Projective Measurement on $\mathfrak{H}_n$

A complete measurement is given by an orthonormal basis

$$\mathcal{B} = \{\psi_a\}$$

which defines a decomposition into orthogonal 1-dimensional subspaces

$$\mathfrak{H}_n = \oplus_a E_a$$

Define $|\psi_a\rangle\langle\psi_a| : \mathfrak{H}_n \to E_a$ to be projection to E_a

Outcome

$$M^{\mathcal{B}} : \mathfrak{H}_n \to \oplus_a E_a : |\phi\rangle \mapsto \oplus_a \langle\psi_a, \phi\rangle |\psi_a\rangle$$

Probability

Destructive Measurement

Given a complete measurement over A, as $\mathcal{A} = \{\psi_a\}$, one can extend it to an incomplete measurement on $A \otimes B$, with components given by $|\psi_a\rangle\langle\psi_a| : A \otimes B \to B$.

1-qubit destructive measurement

M^α associated to $\{|+_\alpha\rangle\}$

Unitary Action

If U maps orthonormal basis $\mathcal{B}$ to $\mathcal{A}$ then

$$M^{\mathcal{A}} = UM^{\mathcal{B}}U^{\dagger}$$

- X-action:

$$X|+_{\alpha}\rangle = |+_{-\alpha}\rangle$$
$$X|-_{\alpha}\rangle = -|-_{-\alpha}\rangle$$

- Z-action:

$$Z|+_{\alpha}\rangle = |+_{\alpha+\pi}\rangle$$
$$Z|-_{\alpha}\rangle = |-_{\alpha+\pi}\rangle$$

A formal language

- N_i prepares qubit in $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

- M_i^α projects qubit onto basis states $|\pm_\alpha\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\alpha}|1\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ \pm e^{i\alpha}\end{pmatrix}$
  (measurement outcome is $s_i = 0, 1$)

- E_{ij} creates entanglement $\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$

- Local Pauli corrections $X_i = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, $Z_i = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

- **Feed forward:** measurements and corrections commands are allowed to depend on previous measurements outcomes.

$$C_i^s \qquad [M_i^\alpha]^s = M_i^{(-1)^s\alpha} \qquad {}_s[M_i^\alpha] = M_i^{\alpha+s\pi}$$

Dependent Commands

The measurement outcome $s_i \in \mathbb{Z}_2$:
— 0 refers to the $\langle +_\alpha |$ projection,
— 1 refers to the $\langle -_\alpha |$ projection.

measurements and corrections may be parameterised by signal $\sum_i s_i$

- $[M_i^\alpha]^s = M_i^{(-1)^s \alpha} = M_i^\alpha X_i^s$
- ${}_t[M_i^\alpha] = M_i^{t\pi + \alpha} = M_i^\alpha Z_i^s$

with $X^0 = Z^0 = I$, $X^1 = X$, $Z^1 = Z$.

$${}_t[M_i^\alpha]^s \; = \; M_i^{t\pi + (-1)^s \alpha}$$

Patterns of Computation

$$(V, I, O, A_n \dots A_1)$$

$$\mathfrak{H} := (\{1,2\},\{1\},\{2\}, X_2^{s_1} M_1^0 E_{12} N_2^0)$$

Sequential or Parallel Composition

$$X_3^{s_2} M_2^0 E_{23} \quad X_2^{s_1} M_1^0 E_{12}$$

Definiteness Conditions

no command depends on outcomes not yet measured

no command acts on a qubit already measured

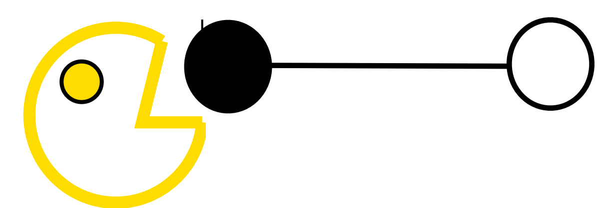
a qubit i is measured if and only if i is not an output

Example

$$\mathfrak{H} := (\{1,2\},\{1\},\{2\},X_2^{s_1}M_1^0E_{12}N_2^0)$$

Starting with the **input state** $(a|0\rangle + b|1\rangle)|+\rangle$

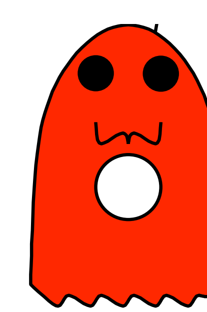
$$(a|0\rangle + b|1\rangle)|+\rangle \xrightarrow{E_{12}} \frac{1}{\sqrt{2}}(a|00\rangle + a|01\rangle + b|10\rangle - b|11\rangle)$$



$$\xrightarrow{M_1^0} \begin{cases} \frac{1}{2}((a+b)|0\rangle + (a-b)|1\rangle) & s_1 = 0 \\ \frac{1}{2}((a-b)|0\rangle + (a+b)|1\rangle) & s_1 = 1 \end{cases}$$

H

$$\xrightarrow{X_2^{s_1}} \frac{1}{2}((a+b)|0\rangle + (a-b)|1\rangle)$$



State Space

$$\mathcal{S} \; := \; \bigcup_{V,W} \mathfrak{H}_V \times \mathbb{Z}_2^W$$

In other words a computation state is a pair q, Γ, where q is a quantum state and Γ is a map from some W to the outcome space $\mathbb{Z}_2$. We call this classical component Γ an *outcome map* and denote by $\varnothing$ the unique map in $\mathbb{Z}_2^{\varnothing}$.

Operational Semantics

$$
\begin{aligned}
q, \Gamma &\xrightarrow{N_i^\alpha} q \otimes |+_\alpha\rangle_i, \Gamma \\
q, \Gamma &\xrightarrow{E_{ij}} \wedge Z_{ij} q, \Gamma \\
q, \Gamma &\xrightarrow{X_i^s} X_i^{s_\Gamma} q, \Gamma \\
q, \Gamma &\xrightarrow{Z_i^s} Z_i^{s_\Gamma} q, \Gamma \\
q, \Gamma &\xrightarrow{{}_t[M_i^\alpha]^s} \langle +_{\alpha_\Gamma}|_i q, \Gamma[0/i] \\
q, \Gamma &\xrightarrow{{}_t[M_i^\alpha]^s} \langle -_{\alpha_\Gamma}|_i q, \Gamma[1/i]
\end{aligned}
$$

where $\alpha_\Gamma = (-1)^{s_\Gamma}\alpha + t_\Gamma \pi$.

Denotational Semantics

$$\begin{array}{ccccc} \mathfrak{H}_I & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\rightarrow & & & \mathfrak{H}_O \\ \downarrow & & & & \uparrow \\ \mathfrak{H}_I \times \mathbb{Z}_2^{\varnothing} & \xrightarrow{prep} & \mathfrak{H}_V \times \mathbb{Z}_2^{\varnothing} & \longrightarrow & \mathfrak{H}_O \times \mathbb{Z}_2^{V \smallsetminus O} \end{array}$$

Let $A_{\mathbf{s}} = C_{\mathbf{s}} \Pi_{\mathbf{s}} U$ be a branch map, the pattern realises the cptp-map

$$T(\rho) := \sum_{\mathbf{s}} A_{\mathbf{s}} \rho A_{\mathbf{s}}^{\dagger}$$

Density operator : A probability distribution over quantum states

Denotational Semantics

$$\begin{array}{ccccc} \mathfrak{H}_I & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \mathfrak{H}_O \\ \downarrow & & & & \uparrow \\ \mathfrak{H}_I \times \mathbb{Z}_2^{\varnothing} & \xrightarrow{prep} & \mathfrak{H}_V \times \mathbb{Z}_2^{\varnothing} & \longrightarrow & \mathfrak{H}_O \times \mathbb{Z}_2^{V \setminus O} \end{array}$$

A pattern is strongly deterministic if all the branch maps are equal.

Theorem. A strongly determinist pattern realises a unitary embedding.

Universal Gates

$$\wedge Z \quad := \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$J(\alpha) \quad := \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & e^{i\alpha} \\ 1 & -e^{i\alpha} \end{pmatrix}$$

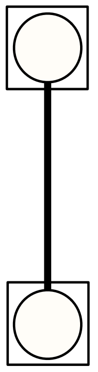
$$U = e^{i\alpha} J(0) J(\beta) J(\gamma) J(\delta)$$

$$\begin{aligned} P(\alpha) &= J(0)J(\alpha) \\ H &= J(0) \\ H^i &= J(\tfrac{\pi}{2}) \end{aligned}$$

Generating Patterns

$$\wedge Z \;:=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\wedge\mathfrak{Z} \;:=\; E_{12}$$



$$J(\alpha) \;:=\; \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & e^{i\alpha} \\ 1 & -e^{i\alpha} \end{pmatrix}$$

$$\mathfrak{J}(\alpha) \;:=\; X_2^{s_1} M_1^{-\alpha} E_{12}$$

Example (ctrl-U)

$$U = e^{i\alpha} J(0) J(\beta) J(\gamma) J(\delta)$$

$$\wedge U_{12} = J_1^0 J_1^{\alpha'} J_2^0 J_2^{\beta+\pi} J_2^{-\frac{\gamma}{2}} J_2^{-\frac{\pi}{2}} J_2^0 \wedge Z_{12} J_2^{\frac{\pi}{2}} J_2^{\frac{\gamma}{2}} J_2^{\frac{-\pi-\delta-\beta}{2}} J_2^0 \wedge Z_{12} J_2^{\frac{-\beta+\delta-\pi}{2}}$$

$$\alpha' = \alpha + \frac{\beta+\gamma+\delta}{2}$$

Example (ctrl-U)

Wild Pattern

$$X_C^{s_B} M_B^{0} E_{BC} X_B^{s_A} M_A^{-\alpha'} E_{AB} X_k^{s_j} M_j^{0} E_{jk} X_j^{s_i} M_i^{-\beta-\pi} E_{ij}$$
$$X_i^{s_h} M_h^{\frac{\gamma}{2}} E_{hi} X_h^{s_g} M_g^{\frac{\pi}{2}} E_{gh} X_g^{s_f} M_f^{0} E_{fg} E_{Af} X_f^{s_e} M_e^{-\frac{\pi}{2}} E_{ef}$$
$$X_e^{s_d} M_d^{-\frac{\gamma}{2}} E_{de} X_d^{s_c} M_c^{\frac{\pi+\delta+\beta}{2}} E_{cd} X_c^{s_b} M_b^{0} E_{bc} E_{Ab} X_b^{s_a} M_a^{\frac{\beta-\delta+\pi}{2}} E_{ab}$$

Standard Pattern

$$Z_k^{s_i+s_g+s_e+s_c+s_a} X_k^{s_j+s_h+s_f+s_d+s_b} X_C^{s_B} Z_C^{s_A+s_e+s_c}$$
$$M_B^{0} M_A^{-\alpha'} M_j^{0} [M_i^{\beta-\pi}]^{s_h+s_f+s_d+s_b} [M_h^{-\frac{\gamma}{2}}]^{s_g+s_e+s_c+s_a} [M_g^{\frac{\pi}{2}}]^{s_f+s_d+s_b}$$
$$M_f^{0} [M_e^{-\frac{\pi}{2}}]^{s_d+s_b} [M_d^{\frac{\gamma}{2}}]^{s_c+s_a} [M_c^{\frac{\pi-\delta-\beta}{2}}]^{s_b} M_b^{0} M_a^{\frac{-\beta+\delta+\pi}{2}}$$
$$E_{BC} E_{AB} E_{jk} E_{ij} E_{hi} E_{gh} E_{fg} E_{Af} E_{ef} E_{de} E_{cd} E_{bc} E_{ab} E_{Ab}$$

Measurement Calculus

Pushing entanglement to the beginning

$$
\begin{array}{rcl}
E_{ij} X_i^s & = & X_i^s Z_j^s E_{ij} \\
E_{ij} X_j^s & = & X_j^s Z_i^s E_{ij} \\
E_{ij} Z_i^s & = & Z_i^s E_{ij} \\
E_{ij} Z_j^s & = & Z_j^s E_{ij}
\end{array}
$$

Pushing correction to the end

$$
\begin{array}{rcl}
{}^t[M_i^\alpha]^s X_i^r & = & {}^t[M_i^\alpha]^{s+r} \\
{}^t[M_i^\alpha]^s Z_i^r & = & {}^{t+r}[M_i^\alpha]^s
\end{array}
$$

Theorem. The re-writing system is confluent and terminating.

Theorem. An MQC model admits a standardisation procedure iff the ***E*** operator is normaliser of all the ***C*** operators.

Algorithm

$$U = e^{i\alpha} J(0) J(\beta) J(\gamma) J(\delta)$$

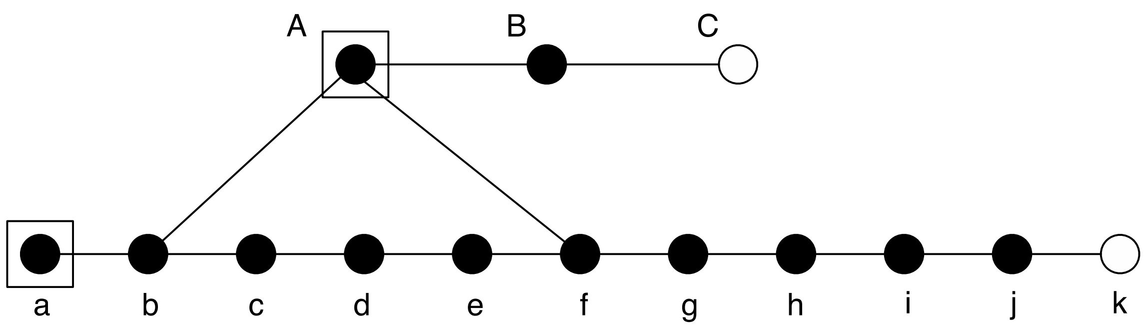
$$\mathfrak{J}(0)(4,5)\mathfrak{J}(\alpha)(3,4)\mathfrak{J}(\beta)(2,3)\mathfrak{J}(\gamma)(1,2) \quad =$$

$$X_5^{s_4} M_4^0 E_{45} X_4^{s_3} M_3^{\alpha} E_{34} X_3^{s_2} M_2^{\beta} E_{23} X_2^{s_1} M_1^{\gamma} E_{12} \quad \Rightarrow_{EX}$$
$$X_5^{s_4} M_4^0 E_{45} X_4^{s_3} M_3^{\alpha} E_{34} X_3^{s_2} M_2^{\beta} X_2^{s_1} Z_3^{s_1} M_1^{\gamma} E_{123} \quad \Rightarrow_{MX}$$
$$X_5^{s_4} M_4^0 E_{45} X_4^{s_3} M_3^{\alpha} E_{34} X_3^{s_2} Z_{s_1}^3 [M_2^{\beta}]^{s_1} M_1^{\gamma} E_{123} \quad \Rightarrow_{EXZ}$$
$$X_5^{s_4} M_4^0 E_{45} X_4^{s_3} M_3^{\alpha} X_3^{s_2} Z_{s_1}^3 Z_4^{s_2} [M_2^{\beta}]^{s_1} M_1^{\gamma} E_{1234} \quad \Rightarrow_{MXZ}$$
$$X_5^{s_4} M_4^0 E_{45} X_4^{s_3} Z_4^{s_2} {}_{s_1}[M_3^{\alpha}]^{s_2} [M_2^{\beta}]^{s_1} M_1^{\gamma} E_{1234} \quad \Rightarrow_{EXZ}$$
$$X_5^{s_4} M_4^0 X_4^{s_3} Z_4^{s_2} Z_5^{s_3} {}_{s_1}[M_3^{\alpha}]^{s_2} [M_2^{\beta}]^{s_1} M_1^{\gamma} E_{12345} \quad \Rightarrow_{MXZ}$$

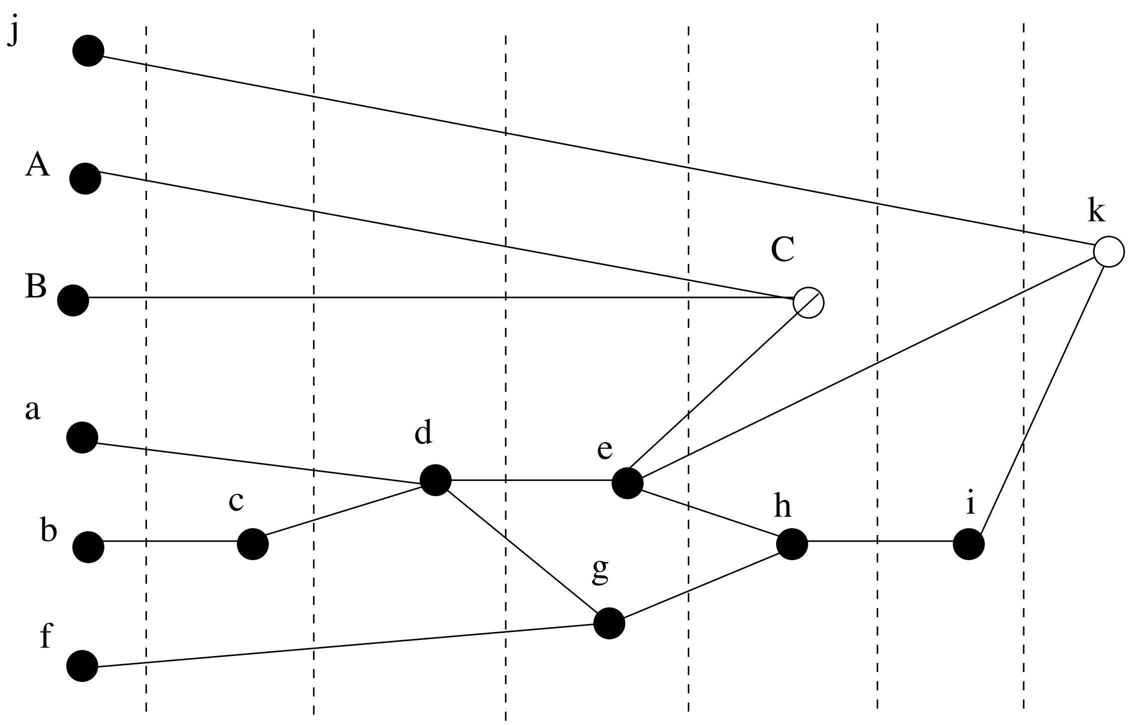
Worst Case Complexity: $O(N^5)$ where N is the number of qubits in the given pattern

The Key Feature of MBQC

A clean separation between Classical and Quantum Control



Entanglement Graph



Execution Graph

No dependency Theorems

Pauli Measurements

Theorem. A unitary map is in Clifford iff $\exists$ a pattern implementing it with measurement angles 0 and $\frac{\pi}{2}$

Theorem. If pattern *P* with no dependent commands implements unitary U, then U is in Clifford

Gottesman Knill Theorem

Efficient representation in terms of Pauli Operators

If the states of computation are restricted to the stabiliser states
and the operation over them to the Clifford group then
the corresponding quantum computation
can be efficiently simulated using Classical Computing

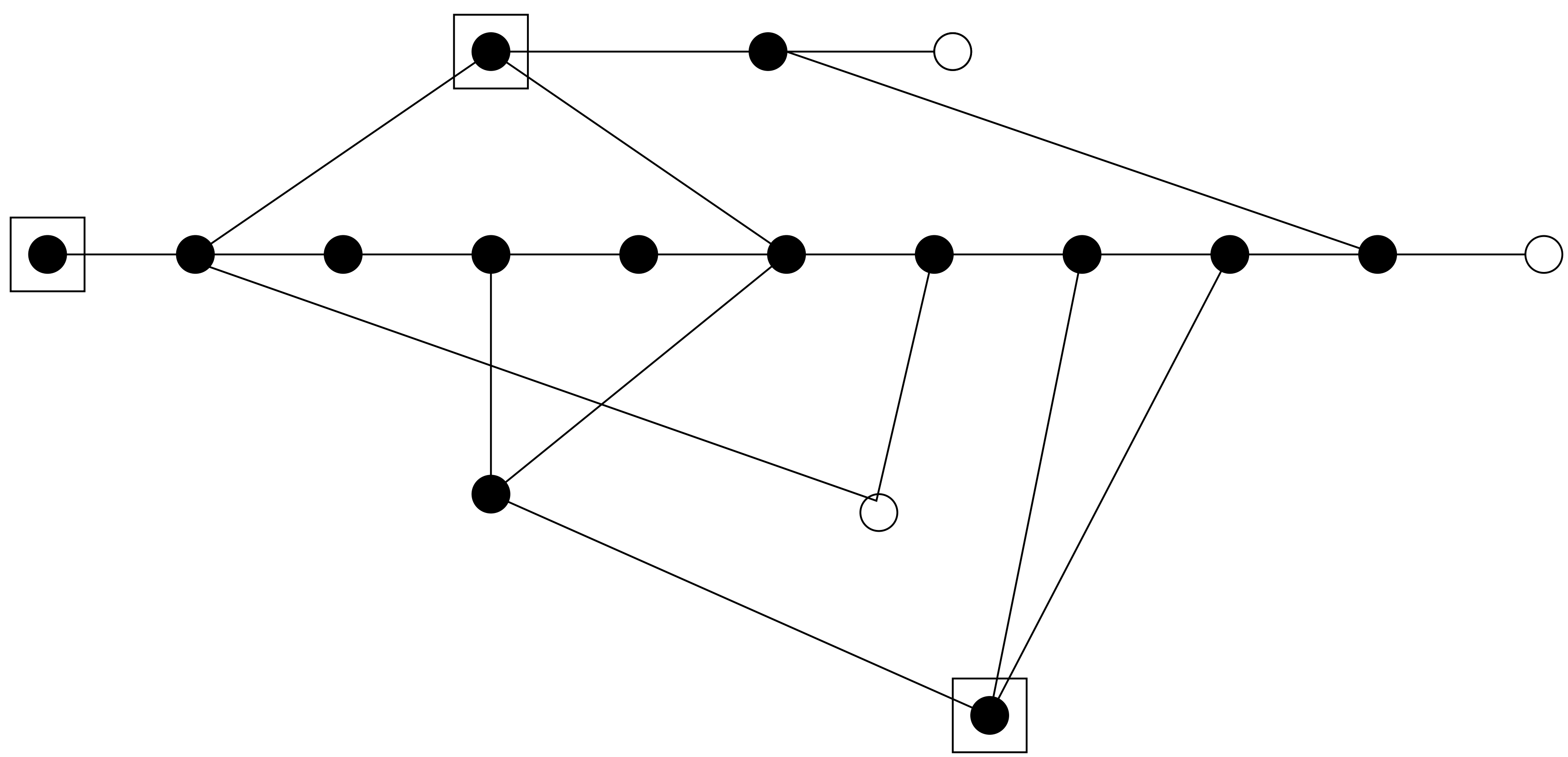
Preserves the efficient representation

Graph State as Stabiliser States

Graph Stabilisers:

$$K_i := X_i (\prod_{j \in N_G(i)} Z_j)$$

$$K_i E_G N_{I^c} = E_G N_{I^c}$$



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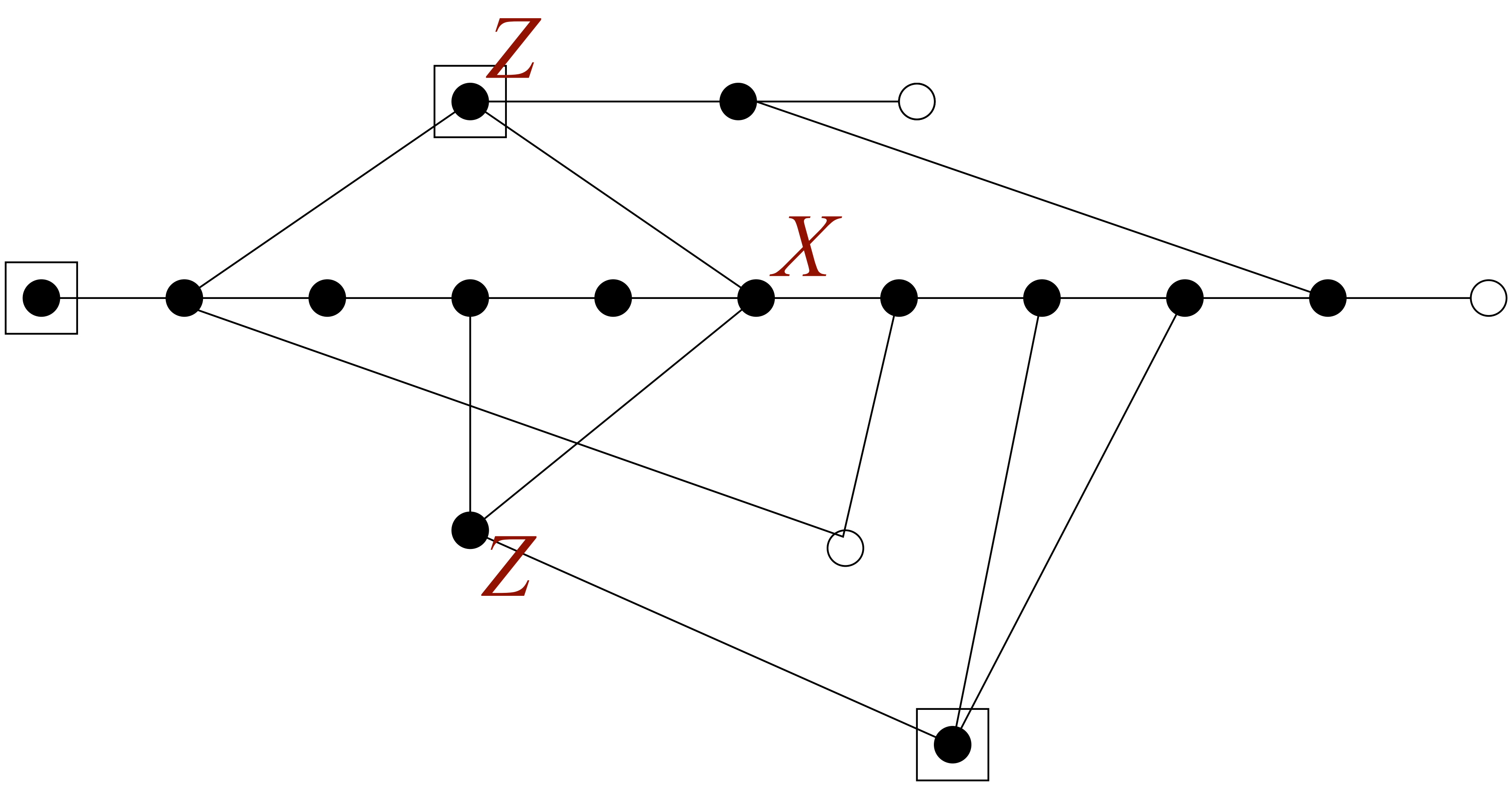

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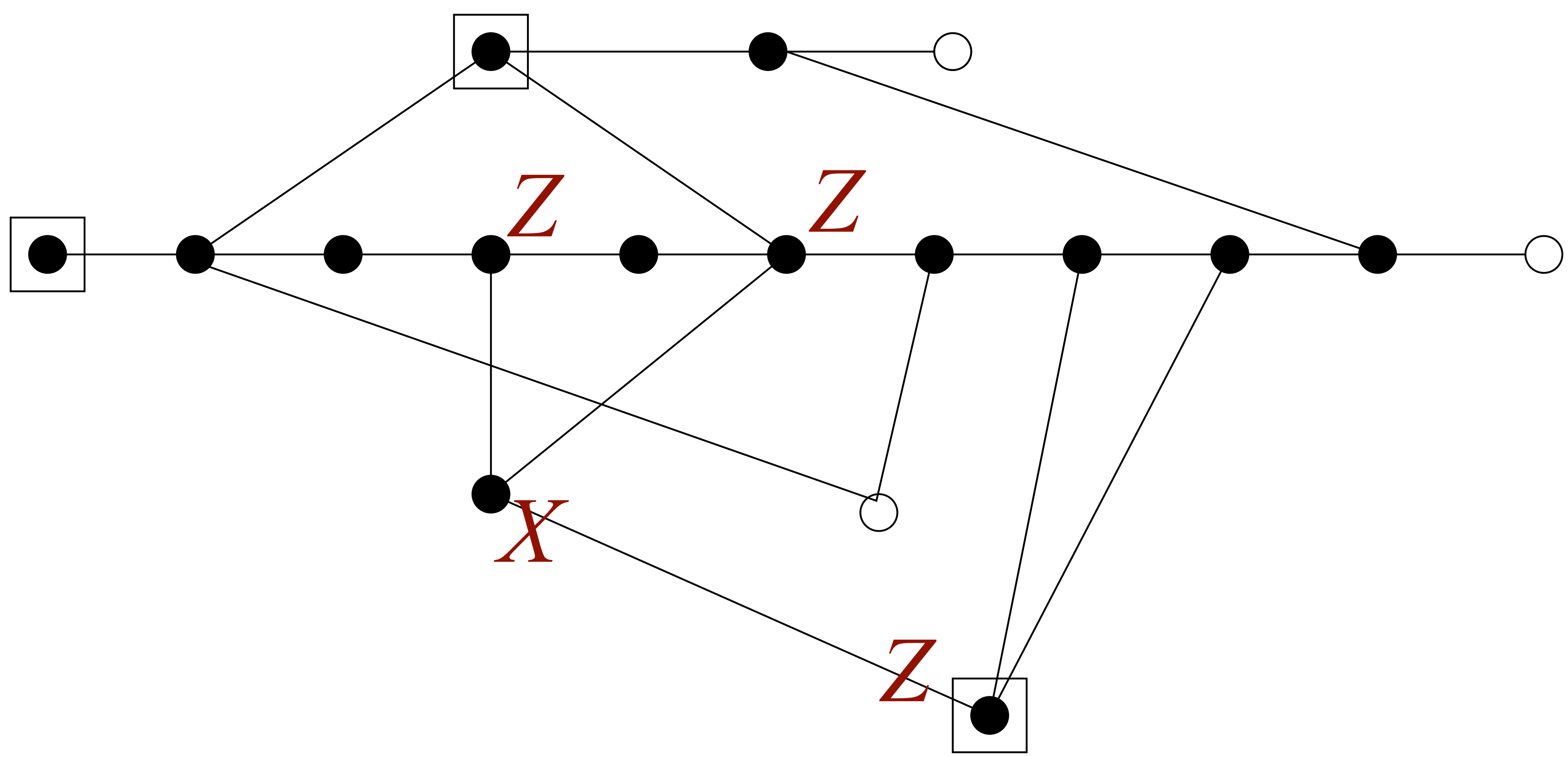

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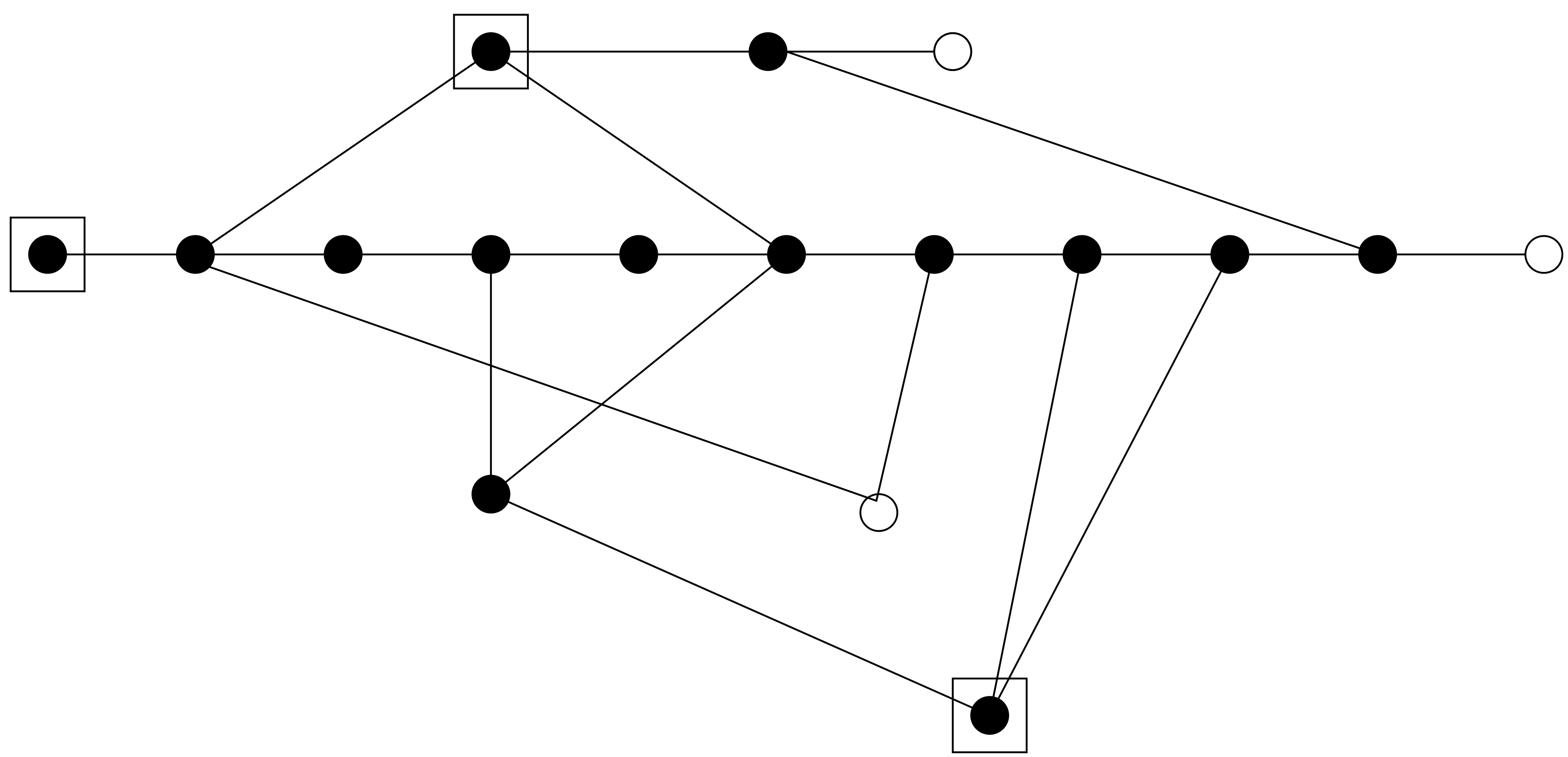


Classical Simulation

Corollary. Any MBQC pattern with only Pauli measurements can be efficiently simulated using Classical Computing.

Quantum Pattern

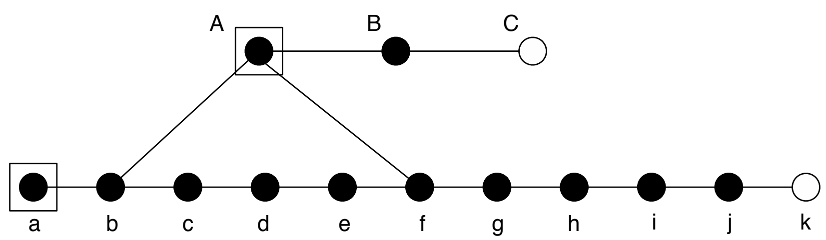
Fixing ↓ **Angles**

Classical Pattern

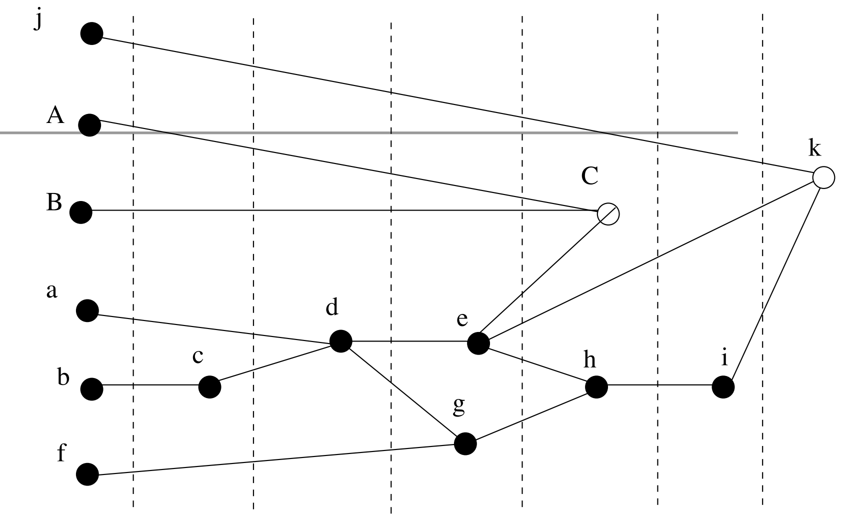
Model checking for a class of quantum protocols using PRISM

S. J. Gay, R. Nagarajan and N. Papanikolaou.

Parallelisation



Entanglement Graph



Execution Graph

Signal Shifting

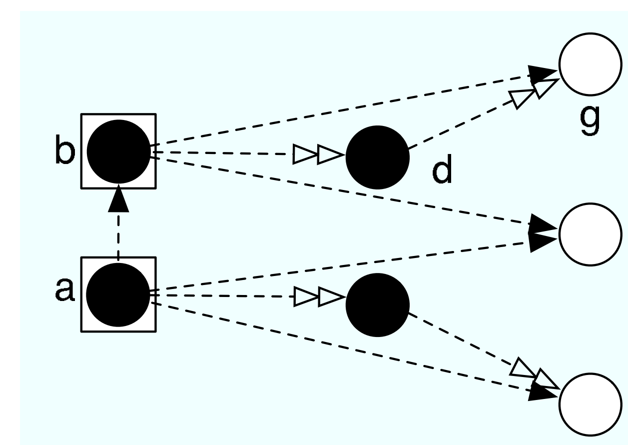
$$
\begin{array}{lcl}
{}_t[M_i^\alpha]^s & \Rightarrow & S_i^t\,[M_i^\alpha]^s \\
X_j^s S_i^t & \Rightarrow & S_i^t X_j^{s[t+s_i/s_i]} \\
Z_j^s S_i^t & \Rightarrow & S_i^t Z_j^{s[t+s_i/s_i]} \\
{}_t[M_j^\alpha]^s S_i^r & \Rightarrow & S_i^r\ {}_{t[r+s_i/s_i]}[M_j^\alpha]^{s[r+s_i/s_i]} \\
S_i^s S_j^t & \Rightarrow & S_j^t S_i^{s[t+s_j/s_j]}
\end{array}
$$

Reducing Depth

Depth of a pattern is the length of the longest feed-forward chain

Standardisation and Signal Shifting reduce depth.

$$Z_g^{s_b} X_g^{s_d} Z_f^{s_b} Z_f^{s_a} Z_e^{s_a} X_e^{s_c} [M_d^{\delta}]^{s_b} [M_c^{\gamma}]^{s_a} \, {}_{s_a}[M_b^{\beta}] M_a^{\alpha} E_G$$



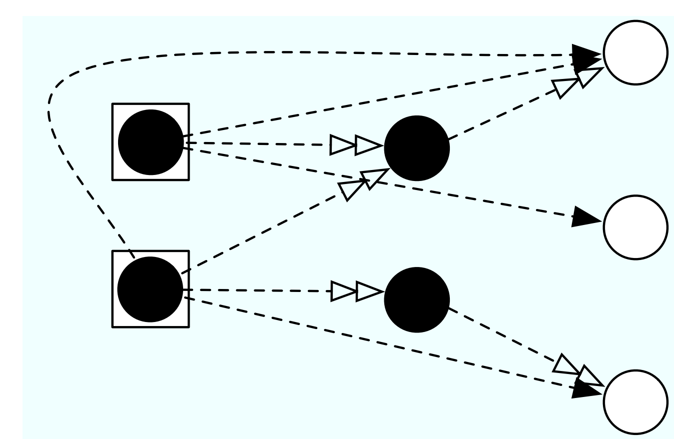
$$Z_g^{s_b} X_g^{s_d} Z_f^{s_b} Z_f^{s_a} Z_e^{s_a} X_e^{s_c} [M_d^{\delta}]^{s_b} [M_c^{\gamma}]^{s_a} \boxed{{}_{s_a}[M_b^{\beta}]} M_a^{\alpha} E_G$$

$$\Rightarrow \quad Z_g^{s_b} X_g^{s_d} Z_f^{s_b} Z_f^{s_a} Z_e^{s_a} X_e^{s_c} \boxed{[M_d^{\delta}]^{s_b} S_b^{s_a}} [M_c^{\gamma}]^{s_a} M_b^{\beta} M_a^{\alpha} E_G$$

$$\Rightarrow \quad Z_g^{s_b} X_g^{s_d} \boxed{Z_f^{s_b} S_b^{s_a}} Z_f^{s_a} Z_e^{s_a} X_e^{s_c} [M_d^{\delta}]^{s_b+s_a} [M_c^{\gamma}]^{s_a} M_b^{\beta} M_a^{\alpha} E_G$$

$$\Rightarrow \quad \boxed{Z_g^{s_b} S_b^{s_a}} X_g^{s_d} Z_f^{s_b+s_a} Z_f^{s_a} Z_e^{s_a} X_e^{s_c} [M_d^{\delta}]^{s_b+s_a} [M_c^{\gamma}]^{s_a} M_b^{\beta} M_a^{\alpha} E_G$$

$$\Rightarrow \quad Z_g^{s_b+s_a} X_g^{s_d} Z_f^{s_b} Z_e^{s_a} X_e^{s_c} [M_d^{\delta}]^{s_b+s_a} [M_c^{\gamma}]^{s_a} M_b^{\beta} M_a^{\alpha} E_G$$



Depth Complexity

All the models for QC are equivalent in computational power.

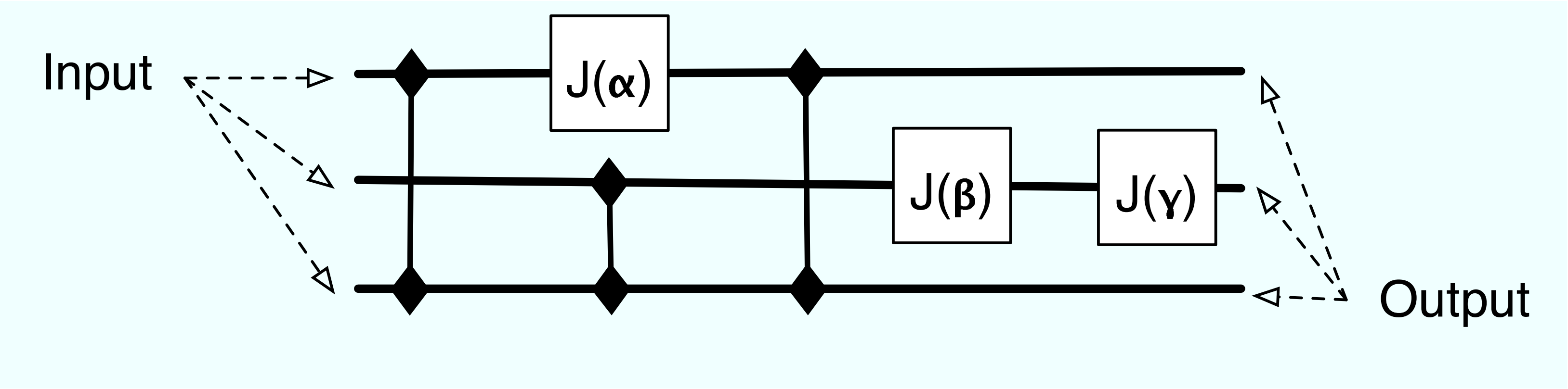
Theorem. There exists a logarithmic separation in depth complexity between MBQC and circuit model.

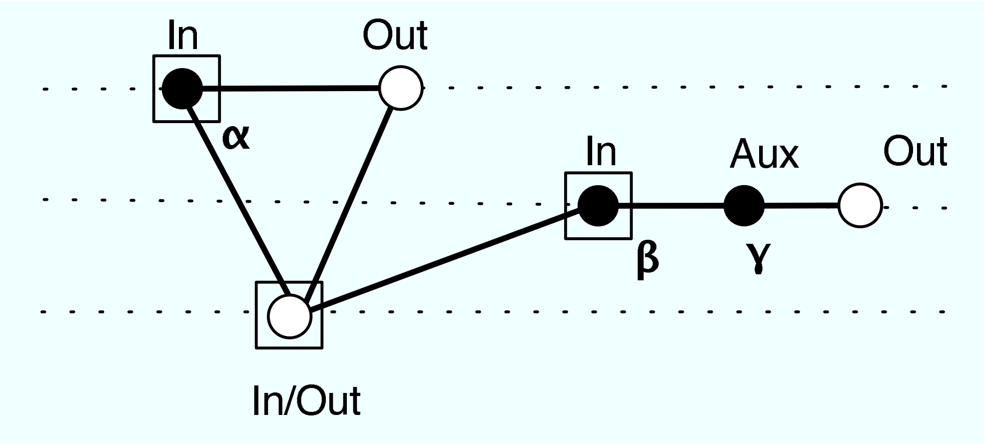
Parity function: MQC needs 1 quantum layer and $O(\log n)$ classical layers whereas in the circuit model the quantum depth is $\Omega(\log n)$

A. Broadbent and E. Kashefi, MBQC07

Automated Parallelising Scheme

Theorem. Forward and backward translation between circuit model and MQC can only decrease the depth.



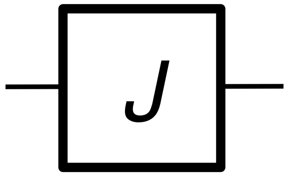


Characterisation

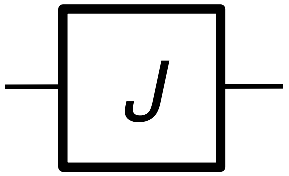
Theorem. A pattern has depth $d+2$ if and only if on any influencing path we obtain $P^* N^{i \leq d} P^*$ after applying the following rewriting rule:

$$N\, P_1^*\, \alpha_1 \beta_1\, P_2^*\, \alpha_2 \beta_2 \cdots P_k^*\, N \begin{cases} NN & if \quad \forall P_i^* \neq X(XY)^* \\ N & otherwise \end{cases}$$

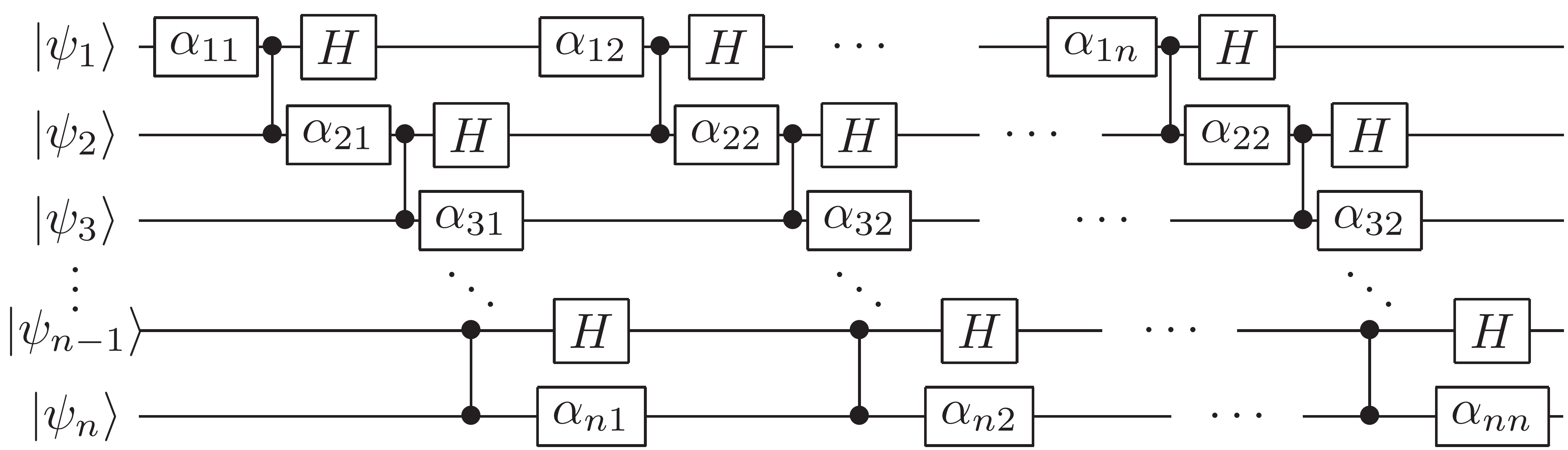
The Magical Clifford Sequence



$$(H)^{odd}(H^{i}(H)^{odd})^{*}$$



Example



Can be parallelised to a pattern with depth 2

Determinism

A pattern is deterministic if all the branches are the same.

How to obtain global determinism via local controls

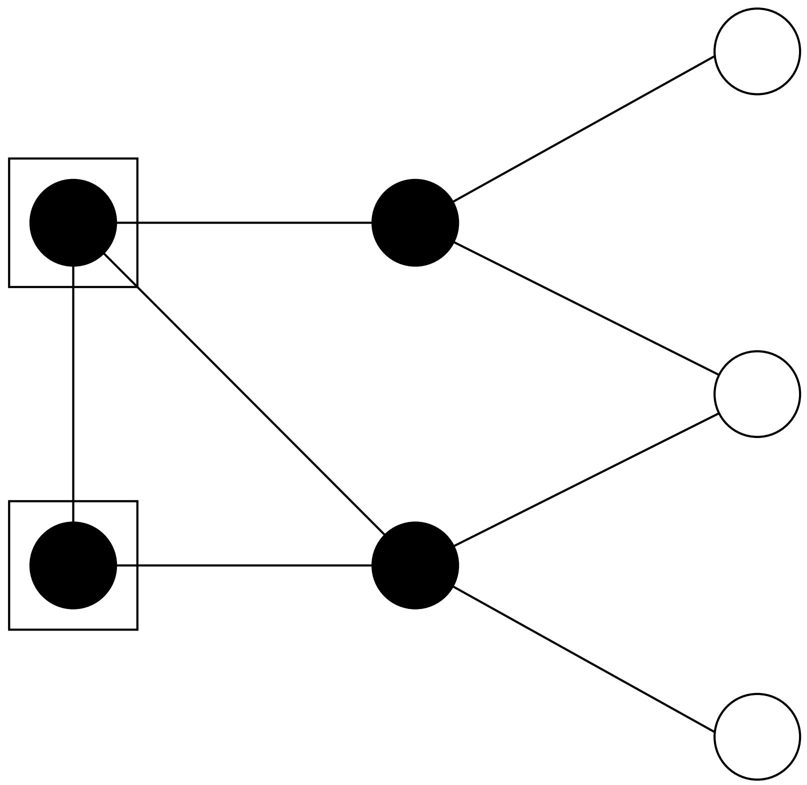
A necessary and sufficient condition for determinism
based on geometry of entanglement

Flow

Definition. An entanglement graph (G, I, O) has flow if there exists a map $f : O^c \to I^c$ and a partial order $\preceq$ over qubits

— (i) $x \sim f(x)$
— (ii) $x \preceq f(x)$
— (iii) for all $y \sim f(x)$, we have $x \preceq y$

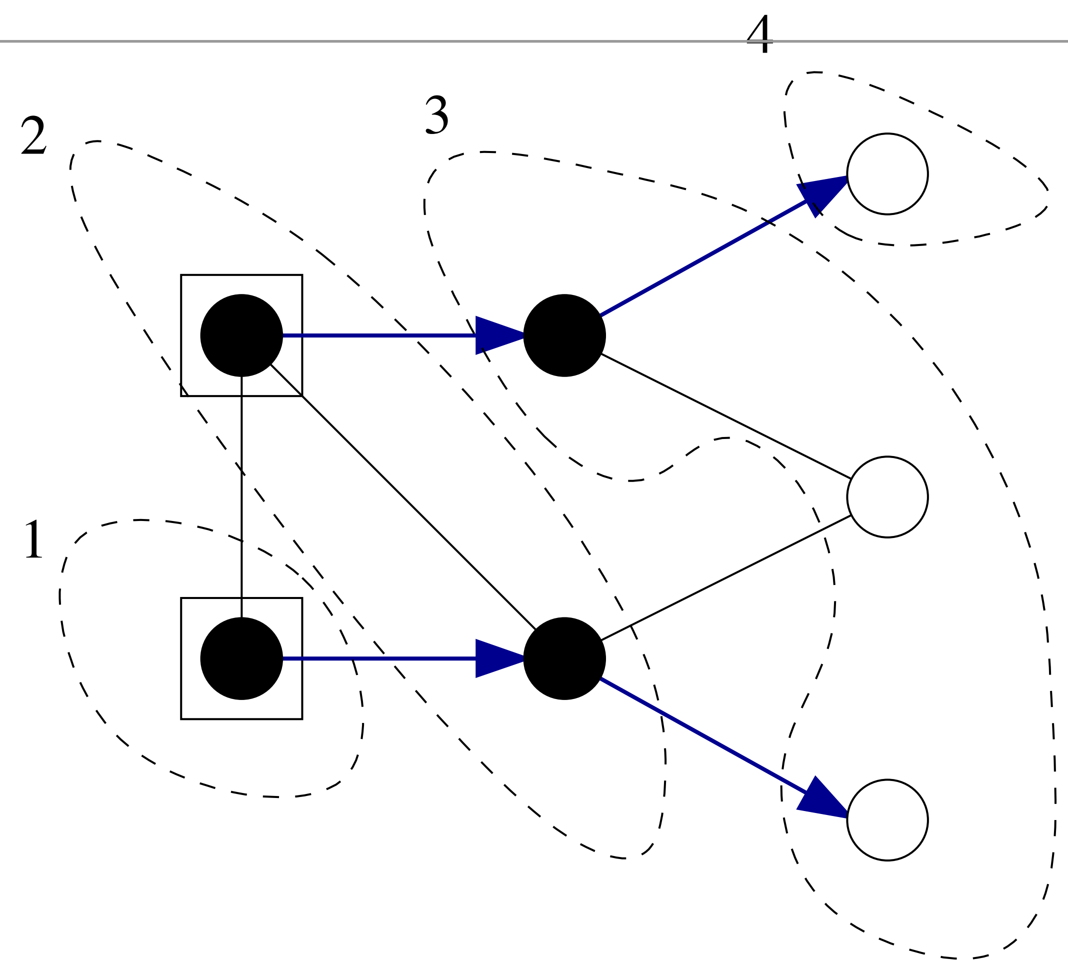
Flow



Find

- a qubits to qubits assignment
- a matching partial order

Flow

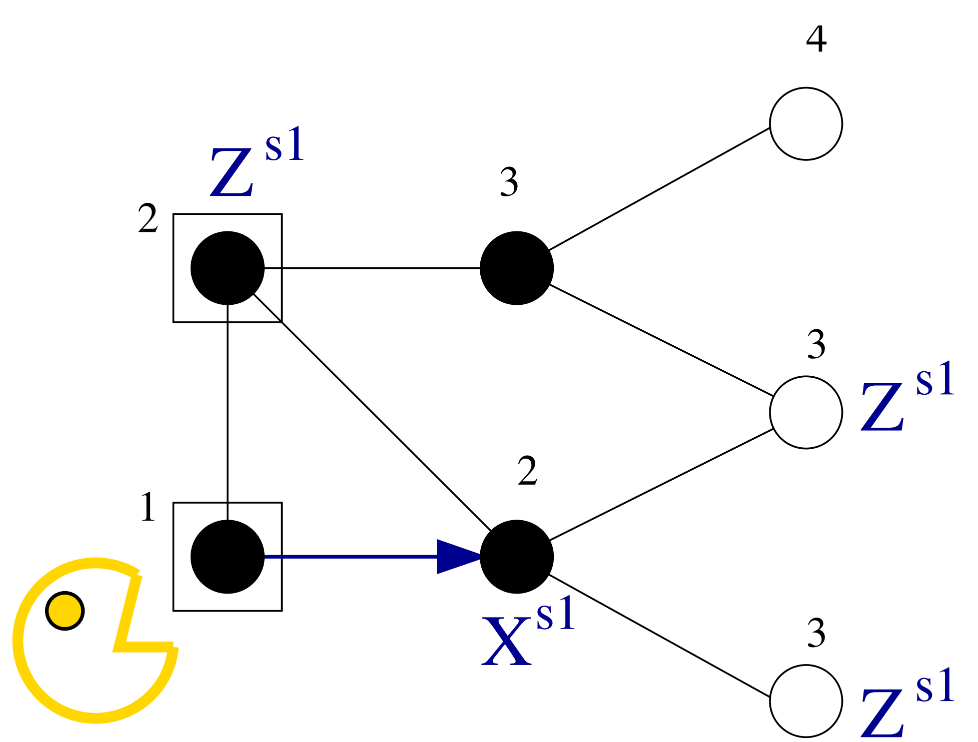


Find

- a qubits to qubits assignment
- a matching partial order

Constructive Determinism

Theorem. A pattern is uniformly and step-wise deterministic iff its graph has a flow.



$$\prod_{i\in O^c}(X^{s_i}_{f(i)}\prod_{k\in N_G(f(i))\smallsetminus\{i\}} Z^{s_i}_k M^{\alpha_i}_i) E_G$$