Probabilistic Semantics and Program Analysis

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joint work with
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The Monty Hall Problem

- The game show proceeds as follows: First the contestant is invited to pick one of three doors (behind one is the prize) but the door is not yet opened.
- Instead, the host – legendary Monty Hall – opens one of the other doors which is empty.
- After that the contestant is given a last chance to stick with his/her door or to switch to the other closed one.
- Note that the host (knowing where the prize is) has always at least one door he can open.

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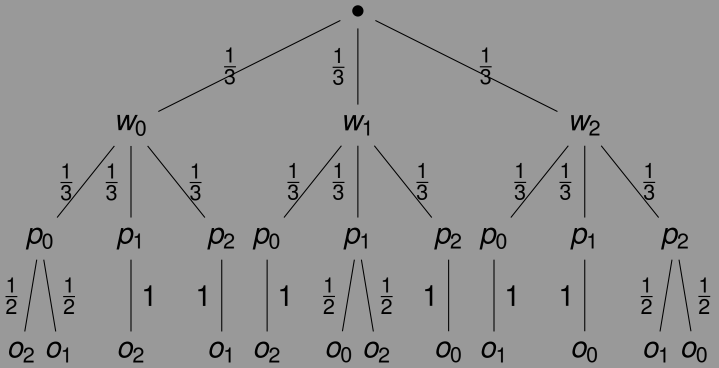
Optimal Strategy: To Switch or not to Switch



w_i = win behind i p_i = pick door i o_i = Monty opens door i

Double Factorial

$[m := 2]^1;$
while $[n > 1]^2$ **do**
    $[m := m \times n]^3;$
    $[n := n - 1]^4$
od
$[\textbf{stop}]^5$

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Quantitative Models

Certainty, Possibility, Probability

Certainty — Determinism
Model: Definite Value
e.g. $2 \in \mathbb{N}$

Possibility — Non-Determinism
Model: Set of Values
e.g. $\{2, 4, 6, 8, 10\} \in \mathcal{P}(\mathbb{N})$

Probability — Probabilistic Non-Determinism
Model: Distribution (Measure)
e.g. $(0, 0, \frac{1}{5}, 0, \frac{1}{5}, 0, \ldots) \in \mathcal{V}(\mathbb{N})$

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Structures: Power Sets

Given a finite set S (of states) we can construct the power set $\mathcal{P}(S)$ of S easily as:

$$\mathcal{P}(S) = \{X \mid X \subseteq S\}$$

Ordered by inclusion "$\subseteq$" this is *the* example of a lattice/order.

It can also be seen as the set of functions from S into a two element set, thus $\mathcal{P}(S) = 2^S$:

$$\mathcal{P}(S) = \{\chi : S \rightarrow \{0, 1\}\}$$

A priori, no major problems when S is (un)countable infinite.

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Structures: Vector Spaces

Vector Spaces = Abelian Additive Group + Quantities

Given a finite set S we can construct the (free) vector space $\mathcal{V}(S)$ of S as a tuple space:

$$\mathcal{V}(S) = \{\langle s, x_s \rangle \mid s \in S, x_s \in \mathbb{K}\} = \{(x_s)_{s \in S} \mid x_s \in \mathbb{K}\}$$

As function spaces $\mathcal{V}(S)$ and $\mathcal{P}(S)$ are not so different:

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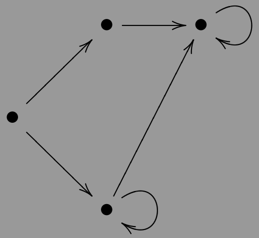
Hilbert Spaces

A complex vector space $\mathcal{H}$ is called an Inner Product Space (or (Pre-)Hilbert Space) if there is a complex valued function $\langle ., . \rangle$ on $\mathcal{H} \times \mathcal{H}$ that satisfies ($\forall x, y, z \in \mathcal{H}$ and $\forall \alpha \in \mathbb{C}$):

1. $\langle x, x \rangle \geq 0$
2. $\langle x, x \rangle = 0 \iff x = o$
3. $\langle \alpha x, y \rangle = \alpha \langle x, y \rangle$
4. $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$
5. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

The function $\langle ., . \rangle$ is called an inner product on $\mathcal{H}$. If the induced norm topology $\|x\| = \sqrt{\langle x, x \rangle}$ is complete it is Hilbert space.

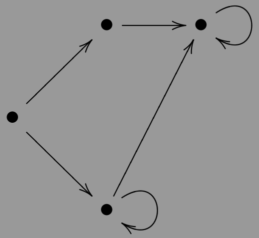
Transition Relations and Operators



$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Automata = Relation = Graph

Transition Relations and Operators



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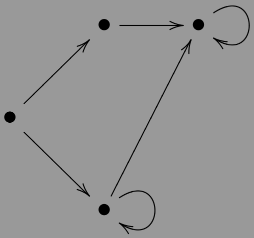

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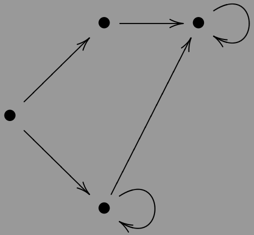

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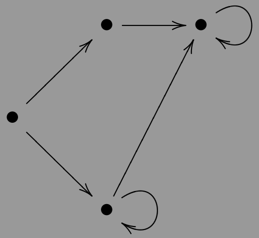
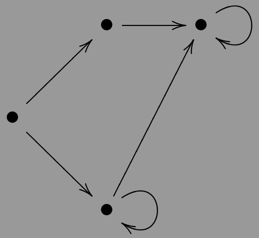


$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Automata = Relation = Graph = Matrix

Transition Relations and Operators



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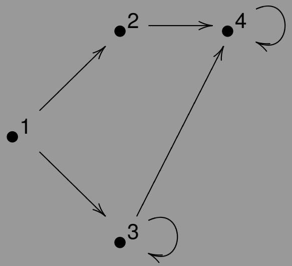
Automata = Relation = Graph = Operator

Linear Dynamical Systems

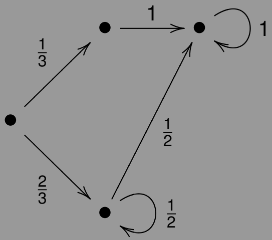
Adjacency Matrix: Dynamical behaviour



$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{T}$$

Linear Dynamical Systems

Stochastic Matrix: Probabilistic behaviour



$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{T}$$

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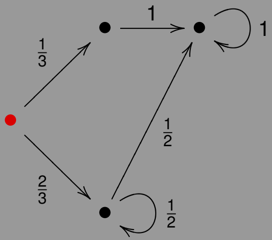

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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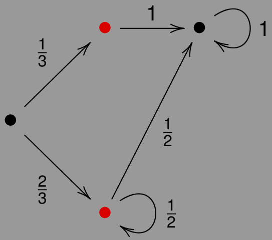

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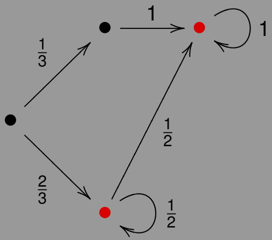

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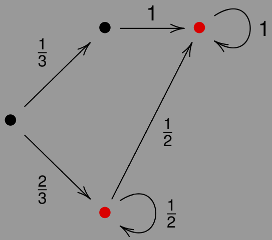

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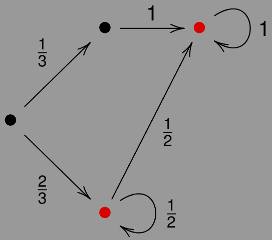

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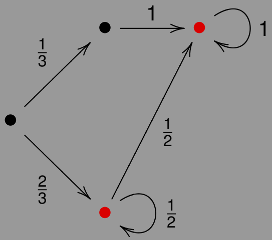

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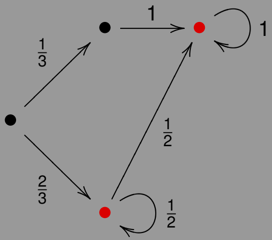

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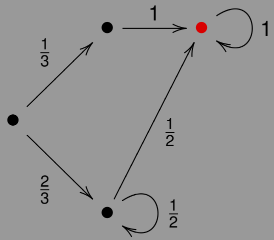

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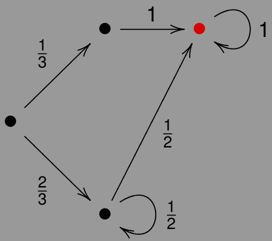
Stochastic Matrix: Probabilistic behaviour



$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^t \cdot \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}^{\infty} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}^t$$

Possibilistic vs probabilistic models: Different infinite behaviour

Analysis Problems and Issues

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1$;
while $[n > 1]^2$ **do**
    $[m := m \times n]^3$;
    $[n := n - 1]^4$
od
$[\textbf{stop}]^5$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := 1]^1;$ $\qquad \triangleright\ P(m = 1), P(m = 2), \ldots — P(n = 1), \ldots$
while $[n > 1]^2$ **do**
$\qquad [m := m \times n]^3;$
$\qquad [n := n - 1]^4$
od
$[\textbf{stop}]^5$

Concrete Probabilities

Probabilistic Problem I: Guards and Conditionals

$[m := \color{red}{1}]^1;$
while $[n > 1]^2$ **do**
    $[m := m \times n]^3;$
    $[n := n - 1]^4$
od
$[\textbf{stop}]^5$

$\triangleright\ (p_1, p_2, p_3, \ldots) — (q_1, q_2, \ldots)$

Concrete Probabilities

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▷ $(p_1, p_2, p_3, \ldots)$ — $(\frac{1}{2}, \frac{1}{2}, \ldots)$
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Concrete Probabilities

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Concrete Probabilities

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Concrete Probabilities

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Concrete Probabilities

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Concrete Probabilities

Perhaps better this way?

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Concrete Probabilities

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Concrete Probabilities

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$\triangleright$
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Concrete Probabilities

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Concrete Probabilities

Correct? How to justify this?

Probabilistic Problem I: Guards and Conditionals

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Concrete Probabilities

Probabilistic Problem II: Abstract Evaluation

$[m := 1]^1;$
while $[n > 1]^2$ **do**
    $[m := m \times n]^3;$
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od
$[\textbf{stop}]^5$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

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3: $\quad [m := m \times n]^3$;
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$\triangleright\ P(m = 2k), P(m \neq 2k)$ — $P(n = 1), \ldots$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

$[m := 1]^1$;
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$\triangleright\ (p_e, p_o)$ — $(q_1, q_2, \ldots)$

Abstract Probabilities

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$\triangleright$ (p_e, p_o) — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

Abstract Probabilities

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- (p_e, p_o) — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
  - $(0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

Abstract Probabilities

Probabilistic Problem II: Abstract Evaluation

$[m := \textcolor{red}{1}]^1;$
while $[n > 1]^2$ **do**
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- (p_e, p_o) — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
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Abstract Probabilities

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Abstract Probabilities

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Abstract Probabilities

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Abstract Probabilities

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Abstract Probabilities

Correct?

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Abstract Probabilities

How to justify this?

Probabilistic Problem III: Relational Dependency

Given an (input) distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$ for n one would expect an (output) distribution $(\frac{2}{3}, \frac{1}{3})$ for $even(m)$ and $odd(m)$.

For every pair (m, n) we can write the probabilities to observe it as $P(m = i \wedge n = j) = P(m = i)P(n = j)$ – assume perhaps that n does not change.

In fact, we have the following joint probability distribution:

	$n = 1$	$n = 2$	$n = 3$
$even(m)$	0	$\frac{1}{3}$	$\frac{1}{3}$
$odd(m)$	$\frac{1}{3}$	0	0

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For every pair (m, n) we can write the probabilities to observe it as $P(m = i \wedge n = j) = P(m = i)P(n = j)$ – assume perhaps that n does not change.

The available data thus suggest this probability distribution:

	$n = 1$	$n = 2$	$n = 3$
$even(m)$	$\frac{1}{3} \cdot \frac{2}{3}$	$\frac{1}{3} \cdot \frac{2}{3}$	$\frac{1}{3} \cdot \frac{2}{3}$
$odd(m)$	$\frac{1}{3} \cdot \frac{1}{3}$	$\frac{1}{3} \cdot \frac{1}{3}$	$\frac{1}{3} \cdot \frac{1}{3}$

Probabilistic Problem III: Relational Dependency

Given an (input) distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$ for n one would expect an (output) distribution $(\frac{2}{3}, \frac{1}{3})$ for $even(m)$ and $odd(m)$.

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The available data thus suggest this probability distribution:

	$n = 1$	$n = 2$	$n = 3$
$even(m)$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$
$odd(m)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

Probabilistic Problem III: Relational Dependency

Given an (input) distribution $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$ for n one would expect an (output) distribution $(\frac{2}{3}, \frac{1}{3})$ for $even(m)$ and $odd(m)$.

For every pair (m, n) we can write the probabilities to observe it as $P(m = i \wedge n = j) = P(m = i)P(n = j)$ – assume perhaps that n does not change.

In fact, we have the following joint probability distribution:

	$n = 1$	$n = 2$	$n = 3$
$even(m)$	0	$\frac{1}{3}$	$\frac{1}{3}$
$odd(m)$	$\frac{1}{3}$	0	0

Problems in Probabilistic Program Analysis

1: $[m := 1]^1;$ $\quad\triangleright\ (p_e, p_o)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

2: **while** $[n > 1]^2$ **do** $\quad\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

3: $\quad [m := m \times n]^3;$ $\quad\triangleright\ (0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$

4: $\quad [n := n - 1]^4$ $\quad\triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$

5: **od**

6: $[\textbf{stop}]^5$ $\quad\triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$

Splitting: How to distribute information along branches?

Transforming: How computing changes the information?

Joining: How to combine information along branches?

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while $[n > 1]^2$ **do**	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
$\quad [m := m \times n]^3;$	$\triangleright\ (0, 1)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
$\quad [n := n - 1]^4$	$\triangleright\ (1, 0)$ — $(0, \frac{1}{3}, \frac{1}{3}, \ldots)$
od	
[**stop**]^5	$\triangleright\ (0, 1)$ — $(\frac{1}{3}, 0, 0, \ldots)$

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5: **od**

6: $[\textbf{stop}]^5$ — $\triangleright$

$$(0, 1) — (\tfrac{1}{3}, 0, 0, \ldots)$$
$$(1, 0) — (\tfrac{1}{3}, 0, 0, \ldots)$$
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Splitting: How to distribute information along branches?

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A Probabilistic Language

pWhile – Syntax I

Full programs contain optional variable declarations:

$$
\begin{array}{lcl}
P & ::= & \textbf{begin } S \textbf{ end} \\
 & | & \textbf{var } D \textbf{ begin } S \textbf{ end}
\end{array}
$$

Declarations are of the form:

$$
\begin{array}{lcl}
r & ::= & \textbf{bool} \\
 & | & \textbf{int} \\
 & | & \{\, c_1, \ldots, c_n \,\} \\
 & | & \{\, c_1 \,..\, c_n \,\} \\
D & ::= & v : r \\
 & | & v : r ; D
\end{array}
$$

with c_i (integer) constants and r denoting ranges.

pWhile – Syntax I

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\end{array}
$$

with c_i (integer) constants and r denoting ranges.

pWhile – Syntax II

The syntax of statements S is as follows:

$$
\begin{array}{rcl}
S & ::= & \textbf{stop} \\
 & | & \textbf{skip} \\
 & | & v := a \\
 & | & v \mathrel{?{=}} r \\
 & | & S_1 ;\ S_2 \\
 & | & \textbf{choose}\ p_1 : S_1\ \textbf{or}\ p_2 : S_2\ \textbf{ro} \\
 & | & \textbf{if}\ b\ \textbf{then}\ S_1\ \textbf{else}\ S_2\ \textbf{fi} \\
 & | & \textbf{while}\ b\ \textbf{do}\ S\ \textbf{od}
\end{array}
$$

Where the p_i are constants, representing choice probabilities.

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$$
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& | & [\textbf{skip}]^{\ell} \\
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& | & S_1 ; \ \ S_2 \\
& | & \textbf{choose}^{\ell} \ p_1 : S_1 \ \textbf{or} \ p_2 : S_2 \ \textbf{ro} \\
& | & \textbf{if} \ [b]^{\ell} \ \textbf{then} \ S_1 \ \textbf{else} \ S_2 \ \textbf{fi} \\
& | & \textbf{while} \ [b]^{\ell} \ \textbf{do} \ S \ \textbf{od}
\end{array}
$$

Where the p_i are constants, representing choice probabilities.

Evaluation of Expressions

$$\sigma \ni \textbf{State} = \textbf{Var} \to \textbf{Z} \uplus \textbf{T}$$

To illustrate approach consider only finite sub-range of **Z**.

Evaluation $\mathcal{E}$ of expressions e in state σ:

$$\begin{aligned} \mathcal{E}(n)\sigma &= n \\ \mathcal{E}(v)\sigma &= \sigma(v) \\ \mathcal{E}(a_1 \odot a_2)\sigma &= \mathcal{E}(a_1)\sigma \odot \mathcal{E}(a_2)\sigma \end{aligned}$$

$$\begin{aligned} \mathcal{E}(\textbf{true})\sigma &= \textbf{tt} \\ \mathcal{E}(\textbf{false})\sigma &= \textbf{ff} \\ \mathcal{E}(\textbf{not}\ b)\sigma &= \neg\mathcal{E}(b)\sigma \\ \ldots &= \ldots \end{aligned}$$

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\ldots &= \ldots
\end{aligned}$$

pWhile – SOS Semantics I

R0 $\langle \textbf{skip}, \sigma\rangle \Rightarrow_1 \langle \textbf{stop}, \sigma\rangle$

R1 $\langle \textbf{stop}, \sigma\rangle \Rightarrow_1 \langle \textbf{stop}, \sigma\rangle$

R2 $\langle \mathrm{v} := e, \sigma\rangle \Rightarrow_1 \langle \textbf{stop}, \sigma[v \mapsto \mathcal{E}(e)\sigma]\rangle$

R3 $\langle \mathrm{v} \mathrel{?{=}} r, \sigma\rangle \Rightarrow_{\frac{1}{|r|}} \langle \textbf{stop}, \sigma[v \mapsto r_i \in r]\rangle$

R4$_1$ $$\frac{\langle S_1, \sigma\rangle \Rightarrow_p \langle S_1', \sigma'\rangle}{\langle S_1; S_2, \sigma\rangle \Rightarrow_p \langle S_1'; S_2, \sigma'\rangle}$$

R4$_2$ $$\frac{\langle S_1, \sigma\rangle \Rightarrow_p \langle \textbf{stop}, \sigma'\rangle}{\langle S_1; S_2, \sigma\rangle \Rightarrow_p \langle S_2, \sigma'\rangle}$$

pWhile – SOS Semantics II

R5$_1$	$\langle \textbf{choose } p_1 : S_1 \textbf{ or } p_2 : S_2, \sigma\rangle \Rightarrow_{p_1} \langle S_1, \sigma\rangle$	
R5$_2$	$\langle \textbf{choose } p_1 : S_1 \textbf{ or } p_2 : S_2, \sigma\rangle \Rightarrow_{p_2} \langle S_2, \sigma\rangle$	
R6$_1$	$\langle \textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2, \sigma\rangle \Rightarrow_1 \langle S_1, \sigma\rangle$	if $\mathcal{E}(b)\sigma = \textbf{tt}$
R6$_2$	$\langle \textbf{if } b \textbf{ then } S_1 \textbf{ else } S_2, \sigma\rangle \Rightarrow_1 \langle S_2, \sigma\rangle$	if $\mathcal{E}(b)\sigma = \textbf{ff}$
R7$_1$	$\langle \textbf{while } b \textbf{ do } S, \sigma\rangle \Rightarrow_1 \langle S;\ \textbf{while } b \textbf{ do } S, \sigma\rangle$	if $\mathcal{E}(b)\sigma = \textbf{tt}$
R7$_2$	$\langle \textbf{while } b \textbf{ do } S, \sigma\rangle \Rightarrow_1 \langle \textbf{stop}, \sigma\rangle$	if $\mathcal{E}(b)\sigma = \textbf{ff}$

Factorial

```
var
  m : {0..2};
  n : {0..2};

begin
m := 1;
while (n>1) do
  m := m*n;
  n := n-1;
od;
stop; # looping
end
```

Monty Hall - Stick

```
var
  d :{0,1,2}; g :{0,1,2}; o :{0,1,2};
begin
d ?= {0,1,2}; # Pick winning door
g ?= {0,1,2}; # Pick guessed door
o ?= {0,1,2}; # Open empty door
while ((o == g) || (o == d)) do
  o := (o+1)%3; od;
# Stick with guess
stop; # looping
end
```

Monty Hall - Switch

```
var
  d :{0,1,2}; g :{0,1,2}; o :{0,1,2};
begin
d ?= {0,1,2}; # Pick winning door
g ?= {0,1,2}; # Pick guessed door
o ?= {0,1,2}; # Open empty door
while ((o == g) || (o == d)) do
  o := (o+1)%3; od;
g := (g+1)%3; # Switch guess
while (g == o) do
  g := (g+1)%3; od;
stop; # looping
end
```

Unstructured Linear Operator Semantics (LOS)

Consider any enumeration of all configurations: $C_1, C_2, C_3, \ldots$

$$(\mathbf{T})_{ij} = \begin{cases} p & \text{if } C_i \Rightarrow_p C_j \\ 0 & \text{otherwise} \end{cases}$$

This is the generator of a Discrete Time Markov Chain (DTMC).

Transitions are implemented then as (row) vector/matrix multiplication or operator application:

$$\mathbf{d}_n \cdot \mathbf{T} = \sum_i (\mathbf{d}_n)_i \cdot \mathbf{T}_{ij} = \mathbf{d}_{n+1}$$

Gives next step "front" of possible successor configurations.

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Gives next step “front” of possible successor configurations.

Example Program

Let us investigate the possible transitions of the following labelled WHILE program (with $\mathbf{x} \in \{0, 1\}$):

$$
\begin{array}{l}
\textbf{if } [\mathbf{x} = 0]^a \textbf{ then} \\
\quad [\mathbf{x} := 0]^b; \\
\textbf{else} \\
\quad [\mathbf{x} := 1]^c; \\
\textbf{fi}; \\
[\textbf{stop}]^d
\end{array}
$$

Example DTMC

$$
\begin{array}{ll}
\langle x = 0, [\mathbf{x} = 0]^a \rangle & \ldots \\
\langle x = 0, [\mathbf{x} := 0]^b \rangle & \ldots \\
\langle x = 0, [\mathbf{x} := 1]^c \rangle & \ldots \\
\langle x = 0, [\mathbf{stop}]^d \rangle & \ldots \\
\langle x = 1, [\mathbf{x} = 0]^a \rangle & \ldots \\
\langle x = 1, [\mathbf{x} := 0]^b \rangle & \ldots \\
\langle x = 1, [\mathbf{x} := 1]^c \rangle & \ldots \\
\langle x = 1, [\mathbf{stop}]^d \rangle & \ldots
\end{array}
\quad
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

Tensor Product

Multi Variable State

The problem we first consider is how to describe distributions over the cartesian product in order to represent the probabilities that two or more variables have certain values.

As we have $\mathcal{D}(S) \subseteq \mathcal{V}(S)$ we investigate $\mathcal{V}(S \times S)$. In order to understand the relation between $\mathcal{V}(S)$ and $\mathcal{V}(S \times S)$ and in general $\mathcal{V}(S^n)$ we need to consider the tensor product.

Essential for the further treatment is the fact (more later) that

$$\mathcal{V}(S \times S) = \mathcal{V}(S) \otimes \mathcal{V}(S)$$

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Kronecker Product

Given a $n \times m$ **matrix A** and a $k \times l$ matrix **B**:

$$\mathbf{A} = \begin{pmatrix} a_{11} & \ldots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \ldots & a_{nm} \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} b_{11} & \ldots & b_{1l} \\ \vdots & \ddots & \vdots \\ b_{k1} & \ldots & b_{kl} \end{pmatrix}$$

The **tensor** or **Kronecker product A** $\otimes$ **B** is a $nk \times ml$ matrix:

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{11}\mathbf{B} & \ldots & a_{1m}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{n1}\mathbf{B} & \ldots & a_{nm}\mathbf{B} \end{pmatrix}$$

Special cases are **square matrices** ($n = m$ and $k = l$) and **vectors** (row $n = k = 1$, column $m = l = 1$).

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Abstract Tensor Product

The (algebraic) tensor product of vector spaces $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n$ is given by a vector space $\bigotimes_{i=1}^n \mathcal{V}_i$ and a map $p = \otimes_{i=1}^n \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n; \bigotimes_{i=1}^n \mathcal{V}_i)$ such that if $\mathcal{W}$ is any vector space and $f \in \mathcal{L}(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_n; \mathcal{W})$ then there exists a unique map $h : \bigotimes_{i=1}^n \mathcal{V}_i \to \mathcal{W}$ satisfying $f = h \circ p$.

$$
\begin{tikzcd}
\mathcal{V}_1 \times \mathcal{V}_2 \times \ldots \times \mathcal{V}_n \arrow[r, "f"] \arrow[d, "p"'] & \mathcal{W} \\
\mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \ldots \otimes \mathcal{V}_n \arrow[ur, "h"'] &
\end{tikzcd}
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\end{tikzcd}$$

Tensor Product Properties

The tensor product of n linear operators $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_n$ is associative (but in general not commutative) and has e.g. the following properties:

1. $(\mathbf{A}_1 \otimes \ldots \otimes \mathbf{A}_n) \cdot (\mathbf{B}_1 \otimes \ldots \otimes \mathbf{B}_n) = \mathbf{A}_1 \cdot \mathbf{B}_1 \otimes \ldots \otimes \mathbf{A}_n \cdot \mathbf{B}_n$
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Tensor Product Base

Every vector space has an algebraic base $\{\mathbf{e}_i\}$

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots$$

This allows to specify vectors via coordinates $\mathbf{x} = (x_1, x_2, \ldots)$. Base vectors are in this context simply of the form

$$\mathbf{e}_i = (e_{i1}, e_{i2}, \ldots) \quad \text{with } e_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{otherwise} \end{cases}$$

The tensor product space $\mathcal{V} \otimes \mathcal{W}$ can be seen as generated by (formal) tensors of the form $\mathbf{v}_i \otimes \mathbf{w}_j$ with in $\mathbf{v}_i \in \mathcal{V}$ and $\mathbf{w} \in \mathcal{W}$ base vectors.

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Combining Distributions

$[m := 1]^1;$
$\textbf{while } [n > 1]^2 \textbf{ do}$
$\quad [m := m \times n]^3;$
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$\textbf{od}$
$[\textbf{stop}]^5$

Concrete Probabilities

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Expressing Correlation, Entanglement, etc.

Some joint probability distributions can be expressed as tensor product of two (independent) probability distributions $\mathbf{d}_1$ and $\mathbf{d}_2$:

$$\begin{pmatrix} \frac{2}{9} & \frac{2}{9} & \frac{2}{9} \\ \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \end{pmatrix} = (\frac{2}{3}, \frac{1}{3})^t \otimes (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$$

However, in general we can express any joint probability distribution as a linear combination of distributions.

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & 0 \end{pmatrix} = \frac{1}{3}(\mathbf{e}_2 \otimes \mathbf{f}_1) + \frac{1}{3}(\mathbf{e}_1 \otimes \mathbf{f}_2) + \frac{1}{3}(\mathbf{e}_1 \otimes \mathbf{f}_3)$$

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Linear Operator Semantics

Classical and Probabilistic State

We have (always) a finite number v of variables.

Classical state $\sigma \in$ **State** given by:

$$\sigma \in \textbf{State} = (\textbf{Var} \rightarrow \textbf{Value}) = \textbf{Value}^v$$

For each variable we assume also a finite range of values.

Probabilistic state **d** of a single variable is a distribution over possible values of the variable.

$$\mathbf{d} \in \mathcal{V}(\textbf{Value}) = \{ (x_c)_{c \in \textbf{Value}} \mid x_i \in \mathbb{R} \}$$

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States and Tensor Products

For finite ranges we can represent distributions over cartesian product as an element in the tensor product in $\mathcal{V}(\mathbf{Value})^{\otimes v}$.

Probabilistic state $\mathbf{d}$ of a all variables together

$$\begin{aligned} \mathbf{d} &\in \mathcal{V}(\mathbf{Var} \to \mathbf{Value}) = \\ &= \mathcal{V}(\mathbf{Value}_1 \times \mathbf{Value}_2 \times \ldots \times \mathbf{Value}_v) = \\ &= \mathcal{V}(\mathbf{Value}_1) \otimes \mathcal{V}(\mathbf{Value}_2) \otimes \ldots \otimes \mathcal{V}(\mathbf{Value}_v) \end{aligned}$$

For infinite value ranges we would need to consider measures. Product measures exist, for example, by Fubini's Theorem.

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States and Tensor Products

For finite ranges we can represent distributions over cartesian product as an element in the tensor product in $\mathcal{V}(\textbf{Value})^{\otimes v}$.

Probabilistic state **d** of a all variables together

$$
\begin{aligned}
\mathbf{d} &\in \mathcal{V}(\textbf{Var} \to \textbf{Value}) = \\
&= \mathcal{V}(\textbf{Value}_1 \times \textbf{Value}_2 \times \ldots \times \textbf{Value}_v) = \\
&= \mathcal{V}(\textbf{Value}_1) \otimes \mathcal{V}(\textbf{Value}_2) \otimes \ldots \otimes \mathcal{V}(\textbf{Value}_v)
\end{aligned}
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For infinite value ranges we would need to consider measures. Product measures exist, for example, by Fubini's Theorem.

Probabilistic Control Flow

Consider the following (labelled) program:

1: **while** $[\mathbf{z} < 100]^1$ **do**
2: **choose**^{2} $\frac{1}{3} : [\mathbf{x} := 3]^3$ **or** $\frac{2}{3} : [\mathbf{x} := 1]^4$ **ro**
3: **od**
4: $[\mathbf{stop}]^5$

Its probabilistic control flow is given by:

$$flow(P) = \{\langle 1, 1, \underline{2}\rangle, \langle 1, 1, 5\rangle, \langle 2, \frac{1}{3}, 3\rangle, \langle 2, \frac{2}{3}, 4\rangle, \langle 3, 1, 1\rangle, \langle 4, 1, 1\rangle\}.$$

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$$

Init — First Statement

$$\begin{aligned}
init([\textbf{skip}]^\ell) &= \ell \\
init([\textbf{stop}]^\ell) &= \ell \\
init([\mathtt{v} := e]^\ell) &= \ell \\
init([\mathtt{v} \mathbin{?=} e]^\ell) &= \ell \\
init(S_1; S_2) &= init(S_1) \\
init(\textbf{choose}^\ell\ p_1 : S_1 \textbf{ or } p_2 : S_2) &= \ell \\
init(\textbf{if } [b]^\ell \textbf{ then } S_1 \textbf{ else } S_2) &= \ell \\
init(\textbf{while } [b]^\ell \textbf{ do } S) &= \ell
\end{aligned}$$

Final — Last Statements

$$\begin{array}{rcl}
\mathit{final}([\textbf{skip}]^\ell) & = & \{\ell\} \\
\mathit{final}([\textbf{stop}]^\ell) & = & \{\ell\} \\
\mathit{final}([\mathtt{v} \mathrel{:=} e]^\ell) & = & \{\ell\} \\
\mathit{final}([\mathtt{v} \mathrel{?=} e]^\ell) & = & \{\ell\} \\
\mathit{final}(S_1; S_2) & = & \mathit{final}(S_2) \\
\mathit{final}(\textbf{choose}^\ell\ p_1 : S_1 \textbf{ or } p_2 : S_2) & = & \mathit{final}(S_1) \cup \mathit{final}(S_2) \\
\mathit{final}(\textbf{if } [b]^\ell \textbf{ then } S_1 \textbf{ else } S_2) & = & \mathit{final}(S_1) \cup \mathit{final}(S_2) \\
\mathit{final}(\textbf{while } [b]^\ell \textbf{ do } S) & = & \{\ell\}
\end{array}$$

Flow I — Control Transfer

$$
\begin{array}{rcl}
\mathit{flow}([\mathbf{skip}]^{\ell}) & = & \emptyset \\
\mathit{flow}([\mathbf{stop}]^{\ell}) & = & \{\langle \ell, 1, \ell \rangle\} \\
\mathit{flow}([\mathtt{v} := e]^{\ell}) & = & \emptyset \\
\mathit{flow}([\mathtt{v} \,?\!\!= e]^{\ell}) & = & \emptyset \\
\mathit{flow}(S_1 ; S_2) & = & \mathit{flow}(S_1) \cup \mathit{flow}(S_2) \cup \\
 & \cup & \{(\ell, 1, \mathit{init}(S_2)) \mid \ell \in \mathit{final}(S_1)\}
\end{array}
$$

Flow ||— Control Transfer

$$
\begin{array}{rcl}
\mathit{flow}(\textbf{choose}^{\ell}\ p_1 : S_1\ \textbf{or}\ p_2 : S_2) & = & \mathit{flow}(S_1) \cup \mathit{flow}(S_2) \cup \\
 & \cup & \{(\ell, p_1, \mathit{init}(S_1)), (\ell, p_2, \mathit{init}(S_2))\} \\
\mathit{flow}(\textbf{if}\ [b]^{\ell}\ \textbf{then}\ S_1\ \textbf{else}\ S_2) & = & \mathit{flow}(S_1) \cup \mathit{flow}(S_2) \cup \\
 & \cup & \{(\ell, 1, \underline{\mathit{init}(S_1)}), (\ell, 1, \mathit{init}(S_2))\} \\
\mathit{flow}(\textbf{while}\ [b]^{\ell}\ \textbf{do}\ S) & = & \mathit{flow}(S) \cup \\
 & \cup & \{(\ell, 1, \underline{\mathit{init}(S)})\} \\
 & \cup & \{(\ell', 1, \ell) \mid \ell' \in \mathit{final}(S)\}
\end{array}
$$

Collecting Semantics

The collecting semantics of a program P is given by:

$$\mathbf{T}(P) = \sum_{\langle i, p_{ij}, j\rangle \in \mathcal{F}(P)} p_{ij} \cdot \mathbf{T}(\ell_i, \ell_j)$$

i.e. as a linear operator on $\mathcal{V}(\mathbf{Value})^{\otimes v} \otimes \mathcal{V}(\mathbf{Lab})$.

Local effects $\mathbf{T}(\ell_i, \ell_j)$: Data Update $\mathbf{N}$ + Control Step $\mathbf{M}$

$$\mathbf{T}(\ell_i, \ell_j) = \mathbf{N}_i \otimes \mathbf{M}_{ij} = \mathbf{N}_{i1} \otimes \mathbf{N}_{i2} \otimes \ldots \otimes \mathbf{N}_{iv} \otimes \mathbf{M}_{ij}$$

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Local Transfer Operators

$$
\begin{array}{rcll}
\mathbf{T}(\ell_1,\ell_2) & = & \mathbf{I} \otimes \mathbf{E}(\ell_1,\ell_2) & \text{for } [\mathbf{skip}]^{\ell_1} \\
\mathbf{T}(\ell,\ell) & = & \mathbf{I} \otimes \mathbf{E}(\ell,\ell) & \text{for } [\mathbf{stop}]^{\ell} \\
\mathbf{T}(\ell_1,\ell_2) & = & \mathbf{U}(v \leftarrow e) \otimes \mathbf{E}(\ell_1,\ell_2) & \text{for } [\mathtt{v} := e]^{\ell_1} \\
\mathbf{T}(\ell_1,\ell_2) & = & \left(\frac{1}{|r|}\sum_{c\in r}\mathbf{U}(v := c)\right) \otimes \mathbf{E}(\ell_1,\ell_2) & \text{for } [\mathtt{v} \,?\!= r]^{\ell_1} \\
\mathbf{T}(\ell,\ell_k) & = & \mathbf{I} \otimes \mathbf{E}(\ell,\ell_k) & \text{for } [\mathbf{choose}]^{\ell} \\
\mathbf{T}(\ell,\underline{\ell_t}) & = & \mathbf{P}(b = \mathbf{tt}) \otimes \mathbf{E}(\ell,\ell_t) & \text{for } [b]^{\ell} \\
\mathbf{T}(\ell,\ell_f) & = & \mathbf{P}(b = \mathbf{ff}) \otimes \mathbf{E}(\ell,\ell_f) & \text{for } [b]^{\ell}
\end{array}
$$

Trivial Operators

Matrix Units – Represent a single transition

$$(\mathbf{E}(m,n))_{ij} = \begin{cases} 1 & \text{if } m = i \wedge n = j \\ 0 & \text{otherwise.} \end{cases}$$

Identity – Represents "no change" transition

$$(\mathbf{I})_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Tests Operators and Filters

Select a certain value $c \in \textbf{Value}$:

$$(\mathbf{P}(c))_{ij} = \begin{cases} 1 & \text{if } i = c = j \\ 0 & \text{otherwise.} \end{cases}$$

Select a certain classical state $\sigma \in \textbf{State}$:

$$\mathbf{P}(\sigma) = \bigotimes_{i=1}^{v} \mathbf{P}(\sigma(\mathrm{v}_i))$$

Select states where expression $e = a \mid b$ evaluates to c:

$$\mathbf{P}(e = c) = \sum_{\mathcal{E}(e)\sigma = c} \mathbf{P}(\sigma)$$

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Selection via Projections

Filtering out *relevant* configurations, i.e. only those which fulfil a certain condition. Use diagonal matrix $\mathbf{P}$:

$$(\mathbf{P})_{ii} = \begin{cases} 1 & \text{if condition holds for } C_i \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \end{pmatrix}^t \cdot \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 \\ C_2 \\ C_3 \\ 0 \\ C_5 \\ 0 \end{pmatrix}^t$$

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Splitting Distributions

$[m := 1]^1;$
while $[n > 1]^2$ **do**
$\quad [m := m \times n]^3;$
$\quad [n := n - 1]^4$
od
$[\textbf{stop}]^5$

- $\triangleright\ (p_e, p_o) \otimes (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$
  - $\triangleright\ (0, 1) \otimes (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

$$((0,1) \otimes (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots))(\mathbf{I} \otimes \mathbf{P}(> 1)) = (0,1) \otimes (0, \frac{1}{3}, \frac{1}{3}, \ldots)$$

$$((0,1) \otimes (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots))(\mathbf{I} \otimes \mathbf{P}(\leq 1)) = (0,1) \otimes (\frac{1}{3}, 0, 0, \ldots)$$

Splitting Distributions

1: $[m := 1]^1$; $\triangleright\ (p_e, p_o) \otimes (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

2: **while** $[n > 1]^2$ **do** $\triangleright\ (0, 1) \otimes (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots)$

3: $[m := m \times n]^3$; $\triangleright\ (0, 1) \otimes (0, \frac{1}{3}, \frac{1}{3}, \ldots)$

4: $[n := n - 1]^4$

5: **od**

6: $[\textbf{stop}]^5$ $\triangleright\ (0, 1) \otimes (\frac{1}{3}, 0, 0, \ldots)$

$$((0,1) \otimes (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots))(\mathbf{I} \otimes \mathbf{P}(> 1)) = (0,1) \otimes (0, \frac{1}{3}, \frac{1}{3}, \ldots)$$

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$\quad [n := n - 1]^4$

od

$[\textbf{stop}]^5$ $\quad \triangleright\ (0, 1) \otimes (\frac{1}{3}, 0, 0, \ldots)$

$$\mathbf{P}(n > 1) = \mathbf{I} \otimes \begin{pmatrix} 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \mathbf{P}(n \leq 1) = \mathbf{I} \otimes \begin{pmatrix} 1 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ 0 & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

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Update Operators

For all initial values change to constant $c \in \textbf{Value}$:

$$(\mathbf{U}(c))_{ij} = \begin{cases} 1 & \text{if } j = c \\ 0 & \text{otherwise.} \end{cases}$$

Set value of the kth variable $\mathrm{v}_k \in \textbf{Var}$ to constant $c \in \textbf{Value}$:

$$\mathbf{U}(\mathrm{v}_k \leftarrow c) = \left(\bigotimes_{i=1}^{k-1} \mathbf{I}\right) \otimes \mathbf{U}(c) \otimes \left(\bigotimes_{i=k+1}^{v} \mathbf{I}\right)$$

Set value of variable $\mathrm{v}_k \in \textbf{Var}$ to value given by $e = a \mid b$:

$$\mathbf{U}(\mathrm{v}_k \leftarrow e) = \sum_c \mathbf{P}(e = c)\mathbf{U}(\mathrm{v}_k \leftarrow c)$$

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LOS Example

if $[\mathbf{x} = 0]^a$ **then**
    $[\mathbf{x} := 0]^b$;
else
    $[\mathbf{x} := 1]^c$;
fi;
$[\mathbf{stop}]^d$

$$
\begin{aligned}
\mathbf{T}(P) \quad = \quad & \mathbf{P}(\mathbf{x} = 0) \otimes \mathbf{E}(a, b) + \\
+ \quad & \mathbf{P}(\mathbf{x} \neq 0) \otimes \mathbf{E}(a, c) + \\
+ \quad & \mathbf{U}(\mathbf{x} \leftarrow 0) \otimes \mathbf{E}(b, d) + \\
+ \quad & \mathbf{U}(\mathbf{x} \leftarrow 1) \otimes \mathbf{E}(c, d) + \\
+ \quad & \mathbf{I} \otimes \mathbf{E}(d, d)
\end{aligned}
$$

LOS Example

$$
\begin{array}{l}
\textbf{if } [\mathbf{x} = 0]^a \textbf{ then} \\
\quad [\mathbf{x} := 0]^b; \\
\textbf{else} \\
\quad [\mathbf{x} := 1]^c; \\
\textbf{fi}; \\
[\textbf{stop}]^d
\end{array}
$$

$$
\begin{array}{rcl}
\mathbf{T}(P) & = & \mathbf{P}(\mathbf{x} = 0) \otimes \mathbf{E}(a, b) + \\
& + & \mathbf{P}(\mathbf{x} \neq 0) \otimes \mathbf{E}(a, c) + \\
& + & \mathbf{U}(\mathbf{x} \leftarrow 0) \otimes \mathbf{E}(b, d) + \\
& + & \mathbf{U}(\mathbf{x} \leftarrow 1) \otimes \mathbf{E}(c, d) + \\
& + & \mathbf{I} \otimes \mathbf{E}(d, d)
\end{array}
$$

LOS Example

$$
\begin{aligned}
\mathbf{T}(P) &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbf{I}_4\right) \cdot (\mathbf{I}_2 \otimes \mathbf{E}(a,b)) + \\
&+ \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{I}_4\right) \cdot (\mathbf{I}_2 \otimes \mathbf{E}(a,c)) + \\
&+ \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \mathbf{E}(b,d)\right) + \\
&+ \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \mathbf{E}(c,d)\right) + \\
&+ (\mathbf{I} \otimes \mathbf{E}(d,d))
\end{aligned}
$$

LOS Example

$$
\begin{aligned}
\mathbf{T}(P) = & \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) \\
+ & \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right) \cdot \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) \\
+ & \left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) \\
+ & \left(\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right) \\
+ & \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)
\end{aligned}
$$

LOS Example

Not really surprisingly we get the same LOS operator as previously "by hand".

$$\begin{array}{ll}
\langle x=0, [\mathbf{x}=0]^a\rangle & \ldots \\
\langle x=0, [\mathbf{x}:=0]^b\rangle & \ldots \\
\langle x=0, [\mathbf{x}:=1]^c\rangle & \ldots \\
\langle x=0, [\mathbf{stop}]^d\rangle & \ldots \\
\langle x=1, [\mathbf{x}=0]^a\rangle & \ldots \\
\langle x=1, [\mathbf{x}:=0]^b\rangle & \ldots \\
\langle x=1, [\mathbf{x}:=1]^c\rangle & \ldots \\
\langle x=1, [\mathbf{stop}]^d\rangle & \ldots
\end{array}
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} = \mathbf{T}(P)$$

Monty Hall – Switch H_w

```
var
  d :{0,1,2};   g :{0,1,2};   o :{0,1,2};
begin
[d ?= {0,1,2}]^1;
[g ?= {0,1,2}]^2;
[o ?= {0,1,2}]^3;
while [((o == g)||(o == d))]^4 do
  [o := (o+1)%3]^5;
od;
[g := (g+1)%3]^6;
while [(g == o)]^7 do
  [g := (g+1)%3]^8;
od;
[stop]^9;
end
```

Monty Hall – Blocks and Flow

$$
\begin{aligned}
&\mathit{blocks}(H_w) = \\
&= \{[\texttt{d ?= \{ 0, 1, 2 \}}]^1, [\texttt{g ?= \{ 0, 1, 2 \}}]^2, \\
&\quad [\texttt{o ?= \{ 0, 1, 2 \}}]^3, [\texttt{((o == g)||(o == d))}]^4, \\
&\quad [\texttt{o := ((o + 1) \% 3)}]^5, [\texttt{g := ((g + 1) \% 3)}]^6, \\
&\quad [\texttt{(g == o)}]^7, [\texttt{g := ((g + 1) \% 3)}]^8, [\texttt{stop}]^9\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{flow}(H_w) = \\
&= \{(1,1,2),(2,1,3),(3,1,4),(4,1,\underline{5}),(5,1,4),(4,1,6), \\
&\quad (6,1,7),(7,1,\underline{8}),(8,1,7),(7,1,9),(9,1,9)\}
\end{aligned}
$$

Monty Hall – Blocks and Flow

$$
\begin{aligned}
&\mathit{blocks}(H_w) = \\
&= \{[\texttt{d ?= \{ 0, 1, 2 \}}]^1, [\texttt{g ?= \{ 0, 1, 2 \}}]^2, \\
&\quad [\texttt{o ?= \{ 0, 1, 2 \}}]^3, [\texttt{((o == g)||(o == d))}]^4, \\
&\quad [\texttt{o := ((o + 1) \% 3)}]^5, [\texttt{g := ((g + 1) \% 3)}]^6, \\
&\quad [\texttt{(g == o)}]^7, [\texttt{g := ((g + 1) \% 3)}]^8, [\texttt{stop}]^9\}
\end{aligned}
$$

$$
\begin{aligned}
&\mathit{flow}(H_w) = \\
&= \{(1,1,2),(2,1,3),(3,1,4),(4,1,\underline{5}),(5,1,4),(4,1,6), \\
&\quad (6,1,7),(7,1,\underline{8}),(8,1,7),(7,1,9),(9,1,9)\}
\end{aligned}
$$

Monty Hall – Stick H_t

$$
\begin{aligned}
\mathbf{T}(H_t) \quad = \quad & \frac{1}{3}(\mathbf{U}(\mathtt{d} \leftarrow 0) + \mathbf{U}(\mathtt{d} \leftarrow 1) + \mathbf{U}(\mathtt{d} \leftarrow 2)) \otimes \mathbf{E}(1,2) + \\
& \frac{1}{3}(\mathbf{U}(\mathtt{g} \leftarrow 0) + \mathbf{U}(\mathtt{g} \leftarrow 1) + \mathbf{U}(\mathtt{g} \leftarrow 2)) \otimes \mathbf{E}(2,3) + \\
& \frac{1}{3}(\mathbf{U}(\mathtt{o} \leftarrow 0) + \mathbf{U}(\mathtt{o} \leftarrow 1) + \mathbf{U}(\mathtt{o} \leftarrow 2)) \otimes \mathbf{E}(3,4) + \\
& \mathbf{P}((\mathtt{o} == \mathtt{g}) || (\mathtt{o} == \mathtt{d}) = \mathbf{tt}) \otimes \mathbf{E}(4,5) + \\
& \mathbf{P}((\mathtt{o} == \mathtt{g}) || (\mathtt{o} == \mathtt{d}) = \mathbf{ff}) \otimes \mathbf{E}(4,6) + \\
& \mathbf{I} \otimes \mathbf{E}(6,6)
\end{aligned}
$$

Monty Hall – Switch H_w

$$
\begin{aligned}
\mathbf{T}(H_w) \quad = \quad & \frac{1}{3}(\mathbf{U}(\mathtt{d} \leftarrow 0) + \mathbf{U}(\mathtt{d} \leftarrow 1) + \mathbf{U}(\mathtt{d} \leftarrow 2)) \otimes \mathbf{E}(1,2) + \\
& \frac{1}{3}(\mathbf{U}(\mathtt{g} \leftarrow 0) + \mathbf{U}(\mathtt{g} \leftarrow 1) + \mathbf{U}(\mathtt{g} \leftarrow 2)) \otimes \mathbf{E}(2,3) + \\
& \frac{1}{3}(\mathbf{U}(\mathtt{o} \leftarrow 0) + \mathbf{U}(\mathtt{o} \leftarrow 1) + \mathbf{U}(\mathtt{o} \leftarrow 2)) \otimes \mathbf{E}(3,4) + \\
& \mathbf{P}((\mathtt{o} == \mathtt{g}) || (\mathtt{o} == \mathtt{d}) = \mathbf{tt}) \otimes \mathbf{E}(4,5) + \\
& \mathbf{P}((\mathtt{o} == \mathtt{g}) || (\mathtt{o} == \mathtt{d}) = \mathbf{ff}) \otimes \mathbf{E}(4,6) + \\
& \mathbf{U}(\mathtt{g} \leftarrow (g + 1)\%3) \otimes \mathbf{E}(6,7) + \\
& \mathbf{P}((\mathtt{g} == \mathtt{o}) = \mathbf{tt}) \otimes \mathbf{E}(7,8) + \\
& \mathbf{P}((\mathtt{g} == \mathtt{o}) = \mathbf{ff}) \otimes \mathbf{E}(7,9) + \\
& \mathbf{U}(\mathtt{g} \leftarrow (g + 1)\%3) \otimes \mathbf{E}(6,7) + \mathbf{I} \otimes \mathbf{E}(9,9)
\end{aligned}
$$

Monty Hall – Enumeration

$$
\begin{array}{lll}
1 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 0) \\
2 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 1) \\
3 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 2) \\
4 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 0) \\
5 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 1) \\
6 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 2) \\
7 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 0) \\
8 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 1) \\
9 & \ldots & (\mathrm{d} \mapsto 0, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 2) \\
10 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 0) \\
11 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 1) \\
12 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 2) \\
13 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 0) \\
14 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 1)
\end{array}
$$

$$
\begin{array}{lll}
15 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 2) \\
16 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 0) \\
17 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 1) \\
18 & \ldots & (\mathrm{d} \mapsto 1, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 2) \\
19 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 0) \\
20 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 1) \\
21 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 0, \mathrm{o} \mapsto 2) \\
22 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 0) \\
23 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 1) \\
24 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 1, \mathrm{o} \mapsto 2) \\
25 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 0) \\
26 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 1) \\
27 & \ldots & (\mathrm{d} \mapsto 2, \mathrm{g} \mapsto 2, \mathrm{o} \mapsto 2)
\end{array}
$$

Monty Hall – $\mathbf{T}(1,2)$

$$\mathbf{T}(1,2) = \begin{pmatrix}
\frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} \\
\frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} \\
\frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3}
\end{pmatrix} \otimes \mathbf{E}(1,2)$$

Monty Hall – $\mathbf{T}(2,3)$

$$
\mathbf{T}(2,3) = \begin{pmatrix}
\frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3} & \cdot & \cdot & \frac{1}{3}
\end{pmatrix} \otimes \mathbf{E}(2,3)
$$

Monty Hall – $\mathbf{T}(3,4)$

$$\mathbf{T}(3,4) = \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix} \otimes \mathbf{E}(3,4)$$

Monty Hall – $\mathbf{T}(4,\underline{5})$ and $\mathbf{T}(4,6)$

$$\mathbf{T}(4,\underline{5}) = \mathrm{diag}\left(100110101110010011101011001\right) \otimes \mathbf{E}(4,5)$$

$$\mathbf{T}(4,6) = \mathrm{diag}\left(011001010001101100010100110\right) \otimes \mathbf{E}(4,6)$$

Monty Hall – $\mathbf{T}(5,4)$

$$\mathbf{T}(5,4) = \begin{pmatrix}
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot
\end{pmatrix} \otimes \mathbf{E}(5,4)$$

Monty Hall – $\mathbf{T}(6,7)$

$$
\mathbf{T}(6,7) = \begin{pmatrix}
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
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\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix} \otimes \mathbf{E}(6,7)
$$

Monty Hall – $\mathbf{T}(7, \underline{8})$ and $\mathbf{T}(7, 9)$

$$\mathbf{T}(7,\underline{8}) = \mathrm{diag}\left(100010001100010001100010001\right) \otimes \mathbf{E}(7,8)$$

$$\mathbf{T}(7,9) = \mathrm{diag}\left(011101110011101110011101110\right) \otimes \mathbf{E}(7,9)$$

Monty Hall – $\mathbf{T}(8,7)$

$$\mathbf{T}(8,7) = \left(\begin{array}{ccccccccccccccccccccccccccc}
. & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
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. & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\
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. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & .
\end{array}\right) \otimes \mathbf{E}(8,7)$$

Monty Hall – $\mathbf{T}(H_s)$ and $\mathbf{T}(H_s)$

$$
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Probabilistic Program Analysis

Problems and Solutions

A general approach towards problems and attempts to solve them could be described as follows:

- If the problem is to difficult
  - formulate a simplified version,
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Possible aims of Static Program Analysis:

- Safe Approximations:
  Correct under all circumstances.
- Good/Close Estimates:
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Semantical Abstraction

Consider a Concrete Domain $\mathcal{C}$ and an Abstract Domain $\mathcal{D}$:

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\mathbf{A}} & \mathcal{D} \\
\Big\downarrow \mathbf{T} & & \Big\downarrow \mathbf{T}^{\#} \\
\mathcal{C} & \xrightarrow[\mathbf{A}]{} & \mathcal{D}
\end{array}$$

With an abstraction $\mathbf{A} : \mathbf{C} \rightarrow \mathbf{D}$ and a concretisation $\mathbf{G} : \mathbf{D} \rightarrow \mathbf{C}$:

$$\mathbf{T}^{\#} = \mathbf{GTA}$$

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Abstract Interpretation: $(\mathbf{A}, \mathbf{G})$ form a Galois Connection.

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Probabilistic Abst.Int.: $(\mathbf{A}, \mathbf{G})$ Moore-Penrose Pseudo-Inverse.

Galois Connections

Definition

Let $\mathcal{C} = (\mathcal{C}, \leq)$ and $\mathcal{D} = (\mathcal{D}, \sqsubseteq)$ be two partially ordered set. If there are two functions $\alpha : \mathcal{C} \rightarrow \mathcal{D}$ and $\gamma : \mathcal{D} \rightarrow \mathcal{C}$ such that for all $c \in \mathcal{C}$ and all $d \in \mathcal{D}$:

$$c \leq_{\mathcal{C}} \gamma(d) \text{ iff } \alpha(c) \sqsubseteq d,$$

then $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ form a Galois connection.

Moore-Penrose Pseudo-Inverse I

Definition

Let $\mathcal{C}$ and $\mathcal{D}$ be two Hilbert spaces and $\mathbf{A} : \mathcal{C} \rightarrow \mathcal{D}$ a bounded linear map. A bounded linear map $\mathbf{A}^\dagger = \mathbf{G} : \mathcal{D} \rightarrow \mathcal{C}$ is the Moore-Penrose pseudo-inverse of $\mathbf{A}$ iff

(i) $\mathbf{A} \circ \mathbf{G} = \mathbf{P}_A$,

(ii) $\mathbf{G} \circ \mathbf{A} = \mathbf{P}_G$,

where $\mathbf{P}_A$ and $\mathbf{P}_G$ denote orthogonal projections onto the ranges of $\mathbf{A}$ and $\mathbf{G}$.

(Orthogonal) Projections – Idempotents

On <u>finite</u> dimensional vector (Hilbert) spaces we have an inner product $\langle ., . \rangle$. This allows us to define an adjoint via:

$$\langle \mathbf{A}(x), y \rangle = \langle x, \mathbf{A}^*(y) \rangle$$

- An operator $\mathbf{A}$ is self-adjoint if $\mathbf{A} = \mathbf{A}^*$.
- An operator $\mathbf{A}$ is positive, i.e. $\mathbf{A} \sqsupseteq 0$, if there exists an operator $\mathbf{B}$ such that $\mathbf{A} = \mathbf{B}^*\mathbf{B}$.
- An (orthogonal) projection is a self-adjoint $\mathbf{E}$ with $\mathbf{E}\mathbf{E} = \mathbf{E}$.

Projections identify (closed) sub-spaces $Y_{\mathbf{E}} = \{\mathbf{E}x \mid x \in \mathcal{V}\}$.

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Moore-Penrose Pseudo-Inverse II

Definition

An operator $\mathbf{A} \in \mathcal{B}(\mathcal{H})$ is Moore-Penrose invertible if there exists an element $\mathbf{G} \in \mathcal{B}(\mathcal{H})$ such that:

(i) $\mathbf{AGA} = \mathbf{A}$,

(ii) $\mathbf{GAG} = \mathbf{G}$,

(iii) $(\mathbf{AG})^* = \mathbf{AG}$,

(iv) $(\mathbf{GA})^* = \mathbf{GA}$.

If it exists $\mathbf{G} = \mathbf{A}^\dagger$ is called Moore-Penrose pseudo-inverse.

Galois Connection II

Definition

Let $\mathcal{C} = (\mathcal{C}, \leq_\mathcal{C})$ and $\mathcal{D} = (\mathcal{D}, \leq_\mathcal{D})$ be two partially ordered sets with two order-preserving functions $\alpha : \mathcal{C} \mapsto \mathcal{D}$ and $\gamma : \mathcal{D} \mapsto \mathcal{C}$. Then $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ form a Galois connection iff

(i) $\alpha \circ \gamma$ is reductive i.e. $\forall d \in \mathcal{D}$, $\alpha \circ \gamma(d) \leq_\mathcal{D} d$,

(ii) $\gamma \circ \alpha$ is extensive i.e. $\forall c \in \mathcal{C}$, $c \leq_\mathcal{C} \gamma \circ \alpha(c)$.

Proposition

Let $(\mathcal{C}, \alpha, \gamma, \mathcal{D})$ be a Galois connection. Then α and γ are quasi-inverse, i.e.

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Examples of Abstractions

Parity Abstraction operator on $\mathcal{V}(\{1, \ldots, n\})$ (with n even):

$$\mathbf{A}_p = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix} \qquad \mathbf{A}_p^\dagger = \begin{pmatrix} \frac{2}{n} & 0 & \frac{2}{n} & 0 & \ldots & 0 \\ 0 & \frac{2}{n} & 0 & \frac{2}{n} & \ldots & \frac{2}{n} \end{pmatrix}$$

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Abstractions and Analysis: Monty Hall

Initial configuration is 162 or 243 dimensional (last: dim $= 6/9$)

$$x_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \end{pmatrix}$$

Final configurations of the same dimension, non-zero entries:

$$\text{for } H_t \left\{ \begin{array}{rcl} x_{12} & = & 0.074074 \\ x_{18} & = & 0.037037 \\ x_{36} & = & 0.11111 \\ x_{48} & = & 0.11111 \\ x_{72} & = & 0.11111 \\ x_{78} & = & 0.037037 \\ x_{90} & = & 0.074074 \\ x_{96} & = & 0.11111 \\ x_{120} & = & 0.11111 \\ x_{132} & = & 0.11111 \\ x_{150} & = & 0.074074 \\ x_{156} & = & 0.037037 \end{array} \right. \qquad \text{for } H_w \left\{ \begin{array}{rcl} x_{18} & = & 0.11111 \\ x_{27} & = & 0.11111 \\ x_{54} & = & 0.037037 \\ x_{72} & = & 0.074074 \\ x_{108} & = & 0.074074 \\ x_{117} & = & 0.11111 \\ x_{135} & = & 0.11111 \\ x_{144} & = & 0.037037 \\ x_{180} & = & 0.037037 \\ x_{198} & = & 0.074074 \\ x_{225} & = & 0.11111 \\ x_{234} & = & 0.11111 \end{array} \right.$$

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Extracting results: Monty Hall

Consider the terminal configurations

$$x_t = \lim_{n\to\infty} (\mathbf{T}(H_t))^n x_0 \quad \text{and} \quad x_w = \lim_{n\to\infty} (\mathbf{T}(H_w))^n x_0$$

Abstract relevant information with $\mathbf{A} = \mathbf{I}$ and $\mathbf{A}_f = (1, 1, \ldots, 1)^t$:

$$\vec{x}_t \cdot (\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{A}_f \otimes \mathbf{A}_f) =$$
$$\begin{pmatrix} 0.11 & 0.11 & 0.11 & 0.11 & 0.11 & 0.11 & 0.11 & 0.11 & 0.11 \end{pmatrix}$$

$$x_w \cdot (\mathbf{I} \otimes \mathbf{I} \otimes \mathbf{A}_f \otimes \mathbf{A}_f) =$$
$$\begin{pmatrix} 0.22 & 0.04 & 0.07 & 0.07 & 0.22 & 0.04 & 0.04 & 0.07 & 0.22 \end{pmatrix}$$

Monty Hall: Optimal Strategy

With further abstraction

$$\mathbf{A}_w^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

we get

$$x_t \cdot (\mathbf{A}_w \otimes \mathbf{A}_f \otimes \mathbf{A}_f) = \begin{pmatrix} 0.33333 & 0.66667 \end{pmatrix}$$

and

$$x_w \cdot (\mathbf{A}_w \otimes \mathbf{A}_f \otimes \mathbf{A}_f) = \begin{pmatrix} 0.66667 & 0.33333 \end{pmatrix}$$

Example: Function Approximation

Concrete and abstract domain are step-functions on $[a, b]$.

The set of (real-valued) step-function $\mathcal{T}_n$ is based on the sub-division of the interval into n sub-intervals.

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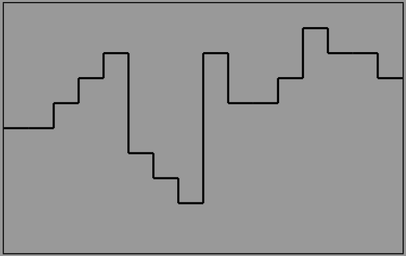

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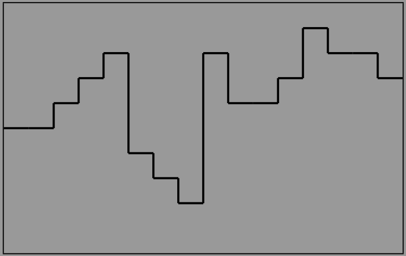

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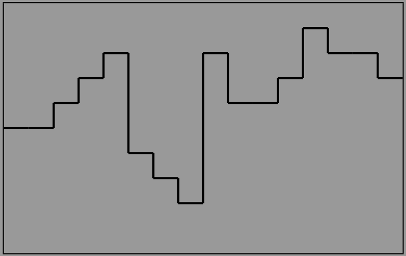

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$$\begin{pmatrix} 5 & 5 & 6 & 7 & 8 & 4 & 3 & 2 & 8 & 6 & 6 & 7 & 9 & 8 & 8 & 7 \end{pmatrix}$$

Example: Abstractions

$$\mathbf{A}_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Example: Abstractions

$$\mathbf{G}_8 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

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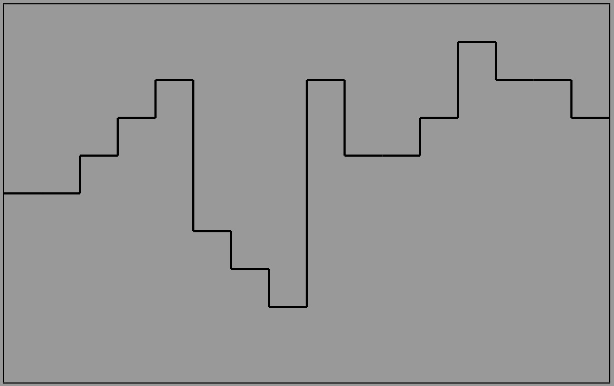

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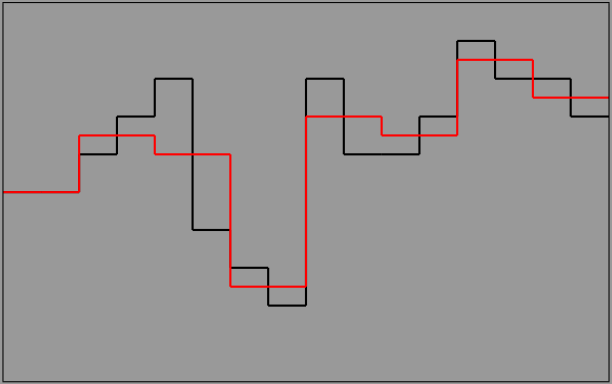

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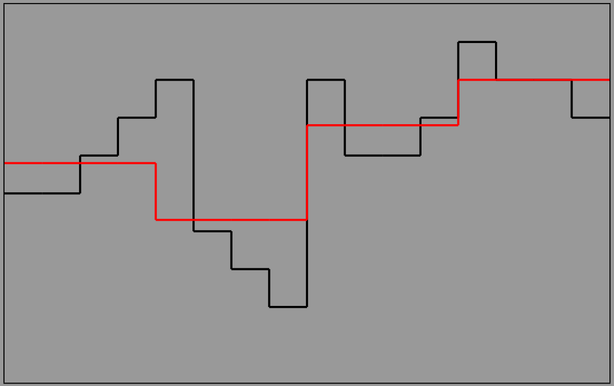

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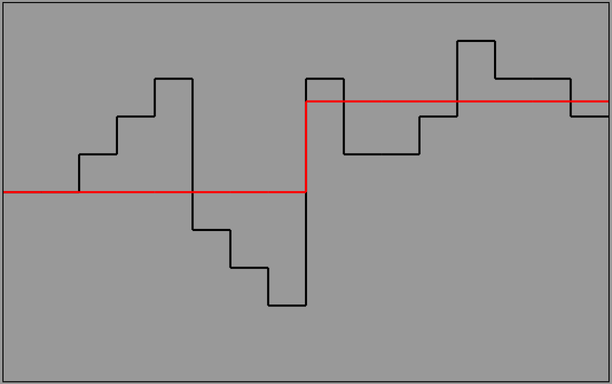

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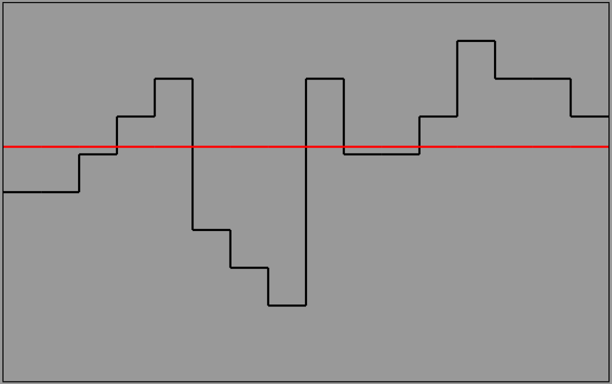

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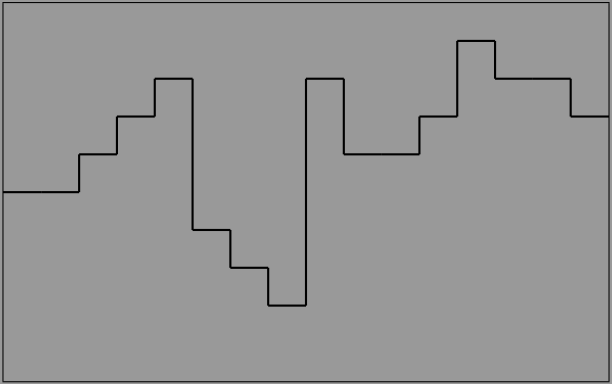

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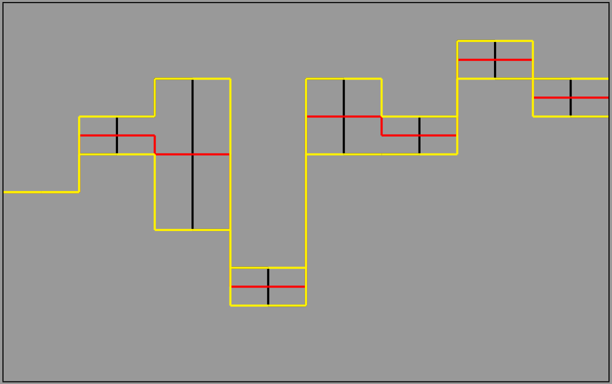

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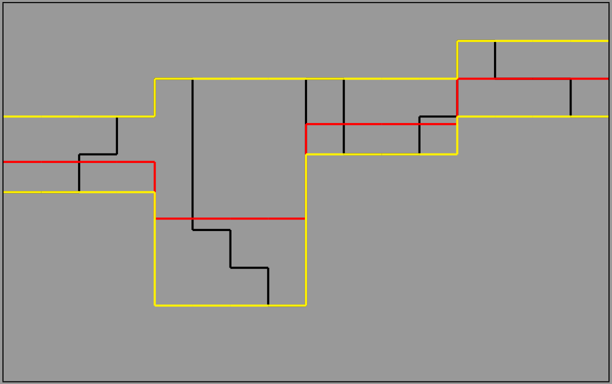

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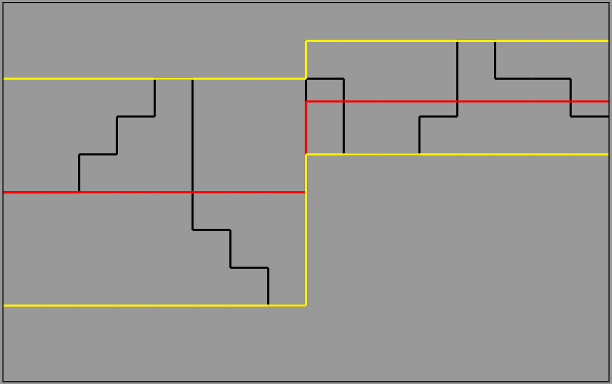

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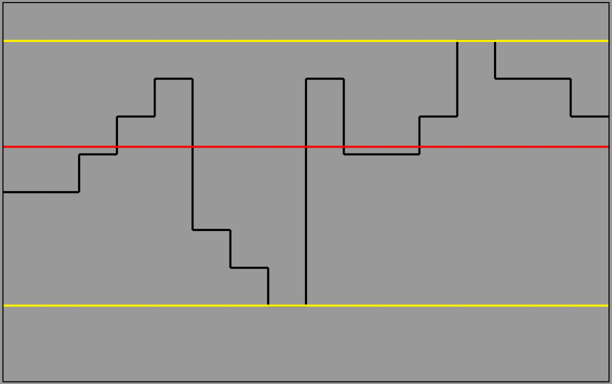

Estimate - Conference Budget

Determine the “cover” registration fee for 130 delegates.

	Costs	Overest.	Estimate	Underest.
Proceedings	3.345 £	4.000 £	3.500 £	3.000 £
Dinner	5.672 £	6.000 £	5.500 £	5.000 £
Excursion	1.813 £	2.000 £	2.000 £	1.000 £
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Least Square Approximation

Given a linear equation

$$x\mathbf{A} = b$$

it has either (i) a (unique) solution $\bar{x}$, or (ii) the residual

$$r_x = b - x\mathbf{A}$$

is non-zero for all x.

The (unique) least-square solution $\bar{x}$, i.e. for which the residual $\|b - \bar{x}\mathbf{A}\|$ is minimal, can be obtained using the Moore-Penrose pseudo-inverse:

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Quantified Precision: Example

The vector space framework allows us to determine the "quality" of an abstraction or approximation via the Euclidian distance between the concrete and abstract function.

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For the previous example we get:

$$\begin{aligned} \|f - f\mathbf{A}_8\mathbf{G}_8\| &= 3.5355 \\ \|f - f\mathbf{A}_4\mathbf{G}_4\| &= 5.3151 \\ \|f - f\mathbf{A}_2\mathbf{G}_2\| &= 5.9896 \\ \|f - f\mathbf{A}_1\mathbf{G}_1\| &= 7.6444 \end{aligned}$$

Abstract LOS Semantics

Moore-Penrose Pseudo-Inverse of a Tensor Product is simply

$$(\mathbf{A}_1 \otimes \mathbf{A}_2 \otimes \ldots \otimes \mathbf{A}_n)^\dagger = \mathbf{A}_1^\dagger \otimes \mathbf{A}_2^\dagger \otimes \ldots \otimes \mathbf{A}_n^\dagger$$

Via linearity we can construct $\mathbf{T}^\#$ in the same way as $\mathbf{T}$, i.e

$$\mathbf{T}^\#(P) = \sum_{\langle i,p_{ij},j\rangle \in \mathcal{F}(P)} p_{ij} \cdot \mathbf{T}^\#(\ell_i, \ell_j)$$

with local abstraction of individual variables:

$$\mathbf{T}^\#(\ell_i, \ell_j) = (\mathbf{A}_1^\dagger \mathbf{N}_{i1} \mathbf{A}_1) \otimes (\mathbf{A}_2^\dagger \mathbf{N}_{i2} \mathbf{A}_2) \otimes \ldots \otimes (\mathbf{A}_v^\dagger \mathbf{N}_{iv} \mathbf{A}_v) \otimes \mathbf{M}_{ij}$$

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Proof Argument

$$
\begin{aligned}
\mathbf{T}^{\#} &= \mathbf{A}^{\dagger}\mathbf{T}\mathbf{A} \\
&= \mathbf{A}^{\dagger}\Big(\sum_{i,j}\mathbf{T}(i,j)\Big)\mathbf{A} \\
&= \sum_{i,j}\mathbf{A}^{\dagger}\mathbf{T}(i,j)\mathbf{A} \\
&= \sum_{i,j}\Big(\bigotimes_{k}\mathbf{A}_k\otimes\mathbf{I}\Big)^{\dagger}\mathbf{T}(i,j)\Big(\bigotimes_{k}\mathbf{A}_k\otimes\mathbf{I}\Big) \\
&= \sum_{i,j}\Big(\bigotimes_{k}\mathbf{A}_k\otimes\mathbf{I}\Big)^{\dagger}\Big(\bigotimes_{k}\mathbf{N}_{ik}\otimes\mathbf{M}_{ij}\Big)\Big(\bigotimes_{k}\mathbf{A}_k\otimes\mathbf{I}\Big) \\
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Example: Factorial

$[m \leftarrow 1]^1$;
while $[n > 1]^2$ **do**
    $[m \leftarrow m \times n]^3$;
    $[n \leftarrow n - 1]^4$
od
$[\textbf{stop}]^5$

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4: & \quad [n \leftarrow n - 1]^4 \\
5: & \textbf{od} \\
6: & [\textbf{stop}]^5
\end{array}
$$

$$
\begin{array}{rcl}
\mathbf{T} & = & \mathbf{U}(m \leftarrow 1) \otimes \mathbf{E}(1,2) \\
& + & \mathbf{P}(n > 1) \otimes \mathbf{E}(2,3) \\
& + & \mathbf{P}(n \leq 1) \otimes \mathbf{E}(2,5) \\
& + & \mathbf{U}(m \leftarrow m \times n) \otimes \mathbf{E}(3,4) \\
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The abstract versions of the local filters and updates, e.g. $\mathbf{P}^{\#}(n > 1), \mathbf{U}^{\#}(m \leftarrow m \times n), \mathbf{U}^{\#}(n \leftarrow n - 1)$ etc. justify our previous ad hoc analysis.

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Abstract Semantics

Abstraction: $\mathbf{A} = \mathbf{A}_p \otimes \mathbf{I}$, i.e. m abstract (parity) but n concrete.

$$
\begin{aligned}
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&+ \mathbf{P}^{\#}(n > 1) \otimes \mathbf{E}(2,3) \\
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Abstract Semantics

$$\mathbf{U}^{\#}(m \leftarrow 1) =$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & \dots & 1 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{U}^{\#}(n \leftarrow n-1) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{P}^{\#}(n > 1)) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{P}^{\#}(n \leq 1) =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Abstract Semantics

$$\mathbf{U}^{\#}(m \leftarrow m \times n) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} +$$

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Implementation

Implementation of concrete and abstract semantics of Factorial using **octave**. Ranges: $n \in \{1, 2, \max\}$ and $m \in \{1, 2, \max!\}$.

n	$\dim(\mathbf{T}(F))$	$\dim(\mathbf{T}^{\#}(F))$
2	45	30
3	140	40
4	625	50
5	3630	60
6	25235	70
7	201640	80
8	1814445	90
9	18144050	100

Using uniform initial distributions $\mathbf{d}_0$ for n and m.

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Scaleablity

The abstract probabilities for m being **even** or **odd** when we execute the abstract program for various maximal n values are:

n	**even**	**odd**
10	0.81818	0.18182
100	0.98019	0.019802
1000	0.99800	0.0019980
10000	0.99980	0.00019998

Some Further Applications

Probabilistic Pointer Analysis [APLAS07]

The typical result of a probabilistic pointer analysis is a so-called points-to matrix: records for every program point the probability that a pointer refers to particular (other) variable.

There are problems and short-comings with common approaches, such as:

- Branching probabilities obtained by profiling.
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Points-To Tensor

$$\textbf{if } [(z_0 \bmod 2 = 0)]^1 \textbf{ then } [x \leftarrow \&z_1]^2;\ [y \leftarrow \&z_2]^3$$
$$\textbf{else } [x \leftarrow \&z_2]^4;\ [y \leftarrow \&z_1]^5 \textbf{ end if};\ [\textbf{stop}]^6$$

Points-To Matrix

	$\&x$	$\&y$	$\&z_0$	$\&z_1$	$\&z_2$
x	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$
y	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$

Points-To Tensor

$$\frac{1}{2} \cdot (0, 0, 0, 1, 0) \otimes (0, 0, 0, 0, 1) + \frac{1}{2} \cdot (0, 0, 0, 0, 1) \otimes (0, 0, 0, 1, 0)$$

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$$(0,0,0,\frac{1}{2},\frac{1}{2}) — (0,0,0,\frac{1}{2},\frac{1}{2}).$$

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Probabilistic Bisimulation [CONCUR'03]

Given the operator representation $\mathbf{M}(X)$ and $\mathbf{M}(Y)$ of two PTS's X and Y, then X and Y are (probabilistically) bisimilar iff there exist classification matrices $\mathbf{K}_X$ and $\mathbf{K}_Y$ such that:

$$\mathbf{K}_X^\dagger \mathbf{M}(X) \mathbf{K}_X = \mathbf{K}_Y^\dagger \mathbf{M}(Y) \mathbf{K}_Y$$

$\mathbf{K}$ is a classification $m \times n$-matrix iff $\mathbf{K}_{ij} = \begin{cases} 1 & \text{for one } j \\ 0 & \text{otherwise.} \end{cases}$

From "perfect" to "as good as it gets": approximate bisimulation

$$\varepsilon = inf_{\mathbf{K}_X, \mathbf{K}_Y} \| \mathbf{K}_X^\dagger \mathbf{M}(X) \mathbf{K}_X - \mathbf{K}_Y^\dagger \mathbf{M}(Y) \mathbf{K}_Y \|$$

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$$(\bigoplus_\alpha \mathbf{K}_X^\dagger)\mathbf{M}(X)(\bigoplus_\alpha \mathbf{K}_X) = (\bigoplus_\alpha \mathbf{K}_Y^\dagger)\mathbf{M}(Y)(\bigoplus_\alpha \mathbf{K}_Y)$$

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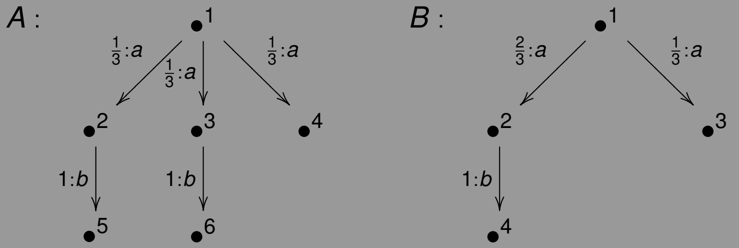
Bisimulation Example



Bisimulation Example

$$\mathbf{M}(A) = \mathbf{M}_a(A) \oplus \mathbf{M}_b(A) =$$

$$= \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Bisimulation Example

$$\mathbf{M}(B) = \mathbf{M}_a(B) \oplus \mathbf{M}_b(B) =$$

$$= \begin{pmatrix} 0 & \frac{2}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Bisimulation Example

$$
\mathbf{K}_A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathbf{K}_A^\dagger = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}
$$

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$$\begin{aligned} \mathbf{K}_A^\dagger \mathbf{M}_a(A) \mathbf{K}_A &= \mathbf{M}_a(B) \\ \mathbf{K}_A^\dagger \mathbf{M}_b(A) \mathbf{K}_A &= \mathbf{M}_b(B) \end{aligned}$$

Order on Projection Operators

Define a partial order on self-adjoint operators and projections as follows: $\mathbf{H} \sqsubseteq \mathbf{K}$ iff $\mathbf{K} - \mathbf{H}$ is positive. Alternatively, order projections by inclusion of their images: $\mathbf{E} \sqsubseteq \mathbf{F}$ iff $Y_{\mathbf{E}} \subseteq Y_{\mathbf{F}}$.

The orthogonal projections (on a Hilbert space $\mathcal{H}$) form a complete ortho-lattice (Birkhoff, von Neumann).

The range of the intersection $\mathbf{E} \sqcap \mathbf{F}$ is to the closure of the intersection of the image spaces of $\mathbf{E}$ and $\mathbf{F}$. The union $\mathbf{E} \sqcup \mathbf{F}$ corresponds to the union of the images.

The (ortho)complement $\mathbf{E}^{\perp}$ of a projection $\mathbf{E}$ is the projection into $Y_{\mathbf{E}}^{\perp}$, i.e. $\langle x, y \rangle = 0$ for all $x \in Y_{\mathbf{E}}$ and $y \in Y_{\mathbf{E}}^{\perp}$ s.t. $Y_{\mathbf{E}} + Y_{\mathbf{E}}^{\perp} = \mathcal{H}$ and $Y_{\mathbf{E}} \cap Y_{\mathbf{E}}^{\perp} = \{o\}$.

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Lattice of PAI's [LOPSTR'05, QAPL'08]

Associate to every PAI $(\mathbf{A}, \mathbf{G})$ a projection (similar to uco):

$$\mathbf{E} = \mathbf{A}\mathbf{G} = \mathbf{A}\mathbf{A}^\dagger.$$

A way to construct $\mathbf{E} \sqcap \mathbf{F}$ and (by de Morgan's law) also $\mathbf{E} \sqcup \mathbf{F} = (\mathbf{E}^\perp \sqcap \mathbf{F}^\perp)^\perp$ is via an infinite approximation (Halmos):

$$\mathbf{E} \sqcap \mathbf{F} = \lim_{n\rightarrow\infty} (\mathbf{EFE})^n = \mathbf{E}(\mathbf{E}+\mathbf{F})^\dagger\mathbf{F} + \mathbf{F}(\mathbf{E}+\mathbf{F})^\dagger\mathbf{E}$$

Only for commuting projections we have:

$$\mathbf{E} \sqcup \mathbf{F} = \mathbf{E} + \mathbf{F} - \mathbf{EF} \quad \text{and} \quad \mathbf{E} \sqcap \mathbf{F} = \mathbf{EF}.$$

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