Yet another alternative definition

Franck van Breugel, Claudio Hermida, Michael Makkai and James Worrell. An Accessible Approach to Behavioural Pseudometrics.

In *Proceedings of 32nd International Colloquium on Automata, Languages and Programming*, *Lecture Notes in Computer Science*, Lisbon, July 2005. Springer-Verlag.

Main results of BHMW

If the category $\mathbb{C}$ is accessible and complete and the functor $\mathcal{F} : \mathbb{C} \to \mathbb{C}$ is accessible, then a terminal $\mathcal{F}$-coalgebra exists.

The category $\mathbb{PM}et$ is accessible (and complete).

Many endofunctors on $\mathbb{PM}et$ that are used in semantics are accessible.

A terminal $\mathcal{K}$-coalgebra exists and induces a behavioural pseudometric for PTSs.

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Definitions

An infinite cardinal κ is *regular* if κ cannot be expressed as the sum of at most κ smaller cardinals.

A category is *accessible* if it is κ-accessible for some infinite regular cardinal κ.

A category is *κ-accessible* if it has κ-filtered colimits and it has a small set of κ-representable objects such that each object is a κ-filtered colimit of these objects.

A category $\mathbb{C}$ is *κ-filtered* if $\mathbb{C}$ is nonempty, for each collection $(C_i)_{i\in I}$ of less than κ objects in $\mathbb{C}$ there exist an object C and a collection $(f_i : C_i \longrightarrow C)_{i\in I}$ of morphisms in $\mathbb{C}$, and for each collection $(f_i : C \longrightarrow D)_{i\in I}$ of less than κ morphisms in $\mathbb{C}$ there exists a morphism $g : D \longrightarrow E$ in $\mathbb{C}$ such that $g \circ f_i = g \circ f_j$ for all $i, j \in I$.

An object C is *κ-representable* if $\hom(C, -)$ preserves κ-filtered colimits.

A functor is *accessible* if it is κ-accessible for some infinite regular cardinal κ.

A functor $\mathcal{F} : C \longrightarrow D$ is *κ-accessible* if the categories C and D are κ-accessible and the functor $\mathcal{F}$ preserves κ-filtered colimits.

...

Let $\mathbb{C}$ be a category and let $\mathcal{F} : \mathbb{C} \to \mathbb{C}$ be a functor.

Definition

An $\mathcal{F}$-*coalgebra* consists of an object C in $\mathbb{C}$ and a morphism $f : C \to \mathcal{F}(C)$ in $\mathbb{C}$.

$$C \xrightarrow{\quad f \quad} \mathcal{F}(C)$$

Coalgebras

Let $\mathbb{C}$ be a category and let $\mathcal{F} : \mathbb{C} \to \mathbb{C}$ be a functor.

Definition

An *$\mathcal{F}$-homomorphism* from $\mathcal{F}$-coalgebra $\langle C, f \rangle$ to $\mathcal{F}$-coalgebra $\langle D, g \rangle$ is a morphism $h : C \to D$ in $\mathbb{C}$ such that $g \circ h = \mathcal{F}(h) \circ f$.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "h"'] & \mathcal{F}(C) \arrow[d, "\mathcal{F}(h)"] \\ D \arrow[r, "g"'] & \mathcal{F}(D) \end{tikzcd}$$

Proposition

The $\mathcal{F}$-coalgebras and $\mathcal{F}$-homomorphisms form a category.

Systems are coalgebras

Many different types of system can be represented as $\mathcal{F}$-coalgebras for a suitable functor $\mathcal{F}$.

$$C \xrightarrow{\quad f \quad} \mathcal{F}(C)$$

states transitions

PTSs are coalgebras

A PTS $\langle S, T \rangle$ can be represented by a $\mathcal{K}$-coalgebra $\langle S, t \rangle$.

$$S \xrightarrow{\quad t \quad} \mathcal{K}(S)$$

The Kantorovich functor $\mathcal{K}$ maps the 1-bounded pseudometric space S to the set of the Borel probability measures on S endowed with the Kantorovich metric.

The function $t : S \to \mathcal{K}(S)$ is defined by

$$t(s)(A) = \sum_{s' \in A} T(s, s').$$

A terminal coalgebra

Given a PTS $\langle S, T \rangle$, consider the corresponding $\mathcal{K}$-coalgebra

$$S \xrightarrow{\quad t \quad} \mathcal{K}(S)$$

Proposition

A terminal $\mathcal{K}$-coalgebra exists.

The proof exploits the theory of accessible categories.

Relating the logical and coalgebraic approach

$$
\begin{tikzcd}
S \arrow[r, "t"] \arrow[d, "!"'] & \mathcal{K}(S) \arrow[d, "\mathcal{K}(!)"] \\
X \arrow[r, equal] & \mathcal{K}(X)
\end{tikzcd}
$$

Recall that d_S is the behavioural pseudometric defined in terms of a logic.

Theorem

$d_S(s_1, s_2) = d_X(!(s_1), !(s_2))$.

Overview of Part III: the 21$^{\text{st}}$ century

- Desharnais, Gupta, Jagadeesan and Panangaden showed how Tarski's fixed point theorem can be used to define behavioural pseudometrics.
- Van Breugel, Sharma and Worrell related the logical and ordered approach.
- Van Breugel, Hermida, Makkai and Worrell showed how the theory of coalgebras can be used to define behavioural pseudometrics and related the logical and coalgebraic appproach.
- Van Breugel, Sharma and Worrell presented an approximation algorithm.

Other work in the 21^{st} century

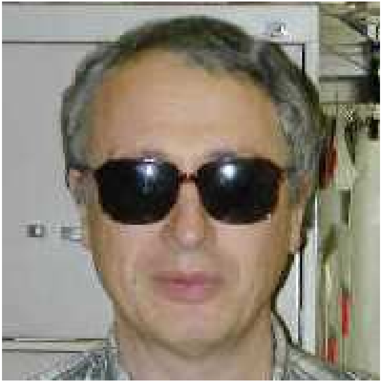
- Behavioural pseudometrics for other types of system have been introduced (timed, hybrid, ...)
- Behavioural pseudometrics have been characterized in terms of tests.
- Behavioural pseudometrics have been characterized in terms of games.
- ...

The metric on Borel probability measures . . .

. . . was proposed by Kantorovich

but has also been named after others including . . .

Wasserstein



and ...

Hutchinson

The Hutchinson metric plays a key role in the theory of fractals.

One of the chapters of Benoit Mandelbrot's book *The Fractal Geometry of Nature* is entitled "How Long is the Coast of Britain?"

Mandelbrot answers his own question with the apparent absurd claim that "the coast in infinitely long."

Mark Tansey's painting "Coastline Measure" translates the title of that chapter into visual form.

Mark Tansey

Coastline Measure