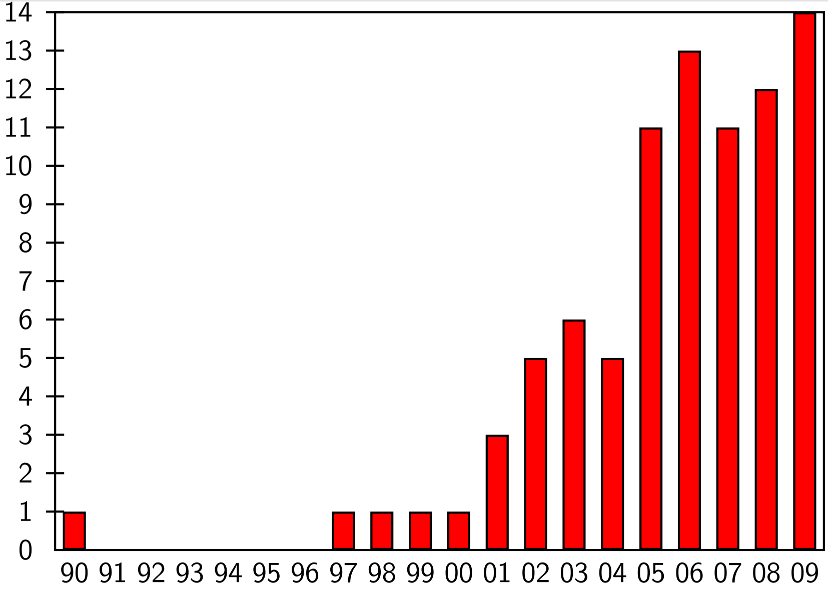
Overview of Part II: the rest of the nineties

Desharnais, Gupta, Jagadeesan and Panangaden generalized the behavioural pseudometric of Giacalone, Jou and Smolka to

- all PTSs and
- labelled Markov processes.

Approximate number of publications



An alternative definition

Josée Desharnais, Vineet Gupta, Radha Jagadeesan and Prakash Panangaden. The metric analogue of weak bisimulation for probabilistic processes.

In *Proceedings of 17th Annual IEEE Symposium on Logic in Computer Science*, pages 413–422, Copenhagen, July 2002. IEEE.

The key ingredients

Tarski

Kantorovich

Tarski's fixed point theorem

Theorem

Let X be a complete lattice. Let $f : X \to X$ be a monotone function. The set of fixed points of f forms a complete lattice. In particular, f has a least fixed point $\text{lfp}(f)$. This least fixed point of f is also the least pre-fixed point of f, that is, $f(\text{lfp}(f)) \sqsubseteq \text{lfp}(f)$.

A. Tarski. A lattice-theoretic fixed point theorem and its applications.

Pacific Journal of Mathematics, 5(2):285 309, June 1955.

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A complete lattice

Definition

Let X be a set. The set $D(X)$ is defined by

$$D(X) = \{\, d \in X \times X \to [0,1] \mid d \text{ is a 1-bounded pseudometric} \,\}.$$

The relation $\sqsubseteq$ is defined by

$$d_1 \sqsubseteq d_2 \text{ if } d_1(x_1, x_2) \leq d_2(x_1, x_2) \text{ for all } x_1,\ x_2 \in S.$$

Proposition

$\langle D(X), \sqsubseteq \rangle$ is a complete lattice.

Kantorovich metric

Definition

Let X be a set and let d_X be a 1-bounded pseudometric on X. Let μ_1 and μ_2 be Borel probability measures on X.

$$d(\mu_1, \mu_2) = \sup \left\{ \int_X f \, d\mu_1 - \int_X f \, d\mu_2 \;\middle|\; f \in \langle X, d_X \rangle \nrightarrow [0,1] \right\}.$$

L. Kantorovich. On the transfer of masses (in Russian).

Doklady Akademii Nauk, 37(2):227 229, 1942.

Related to Roberto's "transfer of masses."

A monotone function

Let $\langle S, T\rangle$ be a PTS. Let S be finite.

Definition

The function $\Delta_S : D(S) \to D(S)$ is defined by

$$\Delta_S(d)(s_1, s_2) = \max\left\{ \sum_{s \in S} f(s) \times (T(s_1, s) - T(s_2, s)) \;\middle|\; f \in \langle S, d\rangle \twoheadrightarrow [0,1] \right\}$$

Proposition

Δ_S is monotone.

Corollary

Δ_S has a least fixed point $\mathrm{lfp}(\Delta_S)$.

Relating the logical and ordered approach

Recall that d_S is the behavioural pseudometric defined in terms of a logic.

Theorem

$d_S = \text{lfp}(\Delta_S)$.

Tarski's fixed point theorem

Theorem

Let X be a complete lattice. Let $f : X \to X$ be a monotone function. The set of fixed points of f forms a complete lattice. In particular, f has a least fixed point: $\mathrm{lfp}(f)$. This least fixed point of f is also the least pre-fixed point of f, that is, $f(\mathrm{lfp}(f)) \sqsubseteq \mathrm{lfp}(f)$.

If you can read this, then you are sitting in one of the first few rows.

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Corollary

d_S is the smallest distance function d such that

$$\Delta_S(d) \sqsubseteq d.$$

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Corollary

$d_S(s_1, s_2) \leq \epsilon$
iff
$\exists d : d$ *is a 1-bounded pseudometric* $\wedge$
$\Delta_S(d) \sqsubseteq d \wedge d(s_1, s_2) \leq \epsilon$.

Theorem

The first order theory over reals is decidable.

A. Tarski. A decision method for elementary algebra and geometry. University of California Press, Berkeley, 1951.

Tarski's decision procedure

Theorem

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Corollary

$d_S(s_1, s_2) \leq \epsilon$ is decidable
iff
$\exists d : d$ is a 1-bounded pseudometric $\wedge$
$\Delta_S(d) \sqsubseteq d \wedge d(s_1, s_2) \leq \epsilon$
can be expressed in the first order theory over reals.

Expressing in the first order theory over reals

$\exists d : d$ is a 1-bounded pseudometric $\wedge \Delta_S(d) \sqsubseteq d \wedge d(s_1, s_2) \leq \epsilon$

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$\exists d : \ldots \forall \ldots \sqsubseteq d \ldots$

Kantorovich-Rubinstein duality theorem

Theorem

Let X be a compact metric space. Let μ_1 and μ_2 be Borel probability measures on X.

$$\sup \left\{ \int_X f \; d\mu_1 - \int_X f \; d\mu_2 \;\middle|\; f \in X \to [0,1] \right\}$$
$$= \quad \inf \left\{ \int_{X^2} d_X \; d\mu \;\middle|\; \mu \in \mu_1 \otimes \mu_2 \right\}.$$

L.V. Kantorovich and G.Sh. Rubinstein. On the space of completely additive functions (in Russian).

Vestnik Leningradskogo Universiteta, 3(2):52 59, 1958.

Related to Roberto's "transfer of masses."

Expressing in the first order theory over reals

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$\exists d : \ldots \exists \sqsubseteq \leq d \ldots$

Approximating the behavioural pseudometric

Corollary

$d_S(s_1, s_2) \leq \epsilon$ is decidable.

Hence, we can use binary search to approximate $d_S(s_1, s_2)$.

Franck van Breugel, Babita Sharma, and James Worrell.
Approximating a behavioural pseudometric without discount.

In H. Seidl, editor, *Proceedings of the 10th International Conference on Foundations of Software Science and Computation Structures*, volume 4423 of *Lecture Notes in Computer Science*, pages 123–137, Braga, March 2007. Springer-Verlag.

Biggest system