Summary of Part I: back to 1990

Giacalone, Jou and Smolka

- advocated the use of pseudometrics instead of equivalence relations to compare the behaviour of states of systems with approximate quantitative data,
- introduced a pseudometric for deterministic PTSs, and
- proposed nonexpansiveness as a quantitative generalization of congruence.

Papers on behavioural pseudometrics in 1991

Papers on behavioural pseudometrics in 1992

Papers on behavioural pseudometrics in 1993

Papers on behavioural pseudometrics in 1994

Papers on behavioural pseudometrics in 1995

Papers on behavioural pseudometrics in 1996

Vineet Gupta, Thomas A. Henzinger, and Radha Jagadeesan. Robust timed automata.

In O. Maler, editor, *Proceedings of the International Workshop on Hybrid and Real-Time Systems*, volume 1201 of *Lecture Notes in Computer Science*, pages 331–345, Grenoble, March 1997. Springer-Verlag.

Timed systems

Papers on behavioural pseudometrics in 1998

Mireille Broucke. Regularity of solutions and homotopic equivalence for hybrid systems.

In *Proceedings of the 37th IEEE Conference on Decision and Control*, volume 4, pages 4283-4288, Tampa, December 1998. IEEE.

Hybrid systems

Papers on behavioural pseudometrics in 1999

Josée Desharnais, Vineet Gupta, Radha Jagadeesan, and Prakash Panangaden. Metrics for labeled Markov systems.

In J.C.M. Baeten and S. Mauw, editors, *Proceedings of 10th International Conference on Concurrency Theory*, volume 1664 of *Lecture Notes in Computer Science*, pages 258-273, Eindhoven, August 1999. Springer-Verlag.

Probabilistic systems

Probabilistic transition system used by GJS

Definition

A *probabilistic transition system* (PTS) is a tuple $\langle S, \mathcal{A}, T \rangle$ consisting of

- a set S of states,
- a set $\mathcal{A}$ of actions, and
- a function $T : S \times A \times S \to [0, 1]$ such that for all $s \in S$,

$$\sum_{a \in \mathcal{A} \wedge s' \in S} T(s, a, s') \in \{0, 1\}.$$

This is a generative model.

Probabilistic transition system used by DGJP

Definition

A *probabilistic transition system* (PTS) is a tuple $\langle S, \mathcal{A}, (T_a)_{a \in \mathcal{A}} \rangle$ consisting of

- a set S of states,
- a set $\mathcal{A}$ of actions, and
- for each $a \in \mathcal{A}$, a function $T_a : S \times S \to [0, 1]$ such that for all $s \in S$,

$$\sum_{s' \in S} T_a(s, s') \in \{0, 1\}.$$

This is a reactive model.

Probabilistic transition system

The generative model and the reactive model coincide in case there is one (= no) action.

Definition

A *probabilistic transition system* (PTS) is a tuple $\langle S, T \rangle$ consisting of

- a set S of states,
- a function $T : S \times S \to [0, 1]$ such that for all $s \in S$,

$$\sum_{s' \in S} T(s, s') \in \{0, 1\}.$$

Logical characterization of probabilistic bisimilarity

Prakash already presented

Definition

The logic $\mathcal{L}$ is defined by

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \Diamond_q \varphi$$

where $q \in [0,1] \cap \mathbb{Q}$.

and

Theorem

$s_1 \sim s_2$ iff s_1 and s_2 satisfy the same formulae.

The interpretation of formulae can be formalized as

Definition

The function $[\![\cdot]\!] : \mathcal{L} \to S \to \mathbb{B}$ is defined by

$$\begin{array}{rcl} [\![\top]\!](s) & = & \top \\ [\![\varphi_1 \wedge \varphi_2]\!](s) & = & [\![\varphi_1]\!](s) \wedge [\![\varphi_2]\!](s) \\ [\![\Diamond_q \varphi]\!](s) & = & \sum\{\, T(s, s') \mid s' \in S \wedge [\![\varphi]\!](s') \,\} > q \end{array}$$

The theorem on the previous slide can be reformulated as

Theorem

$s_1 \sim s_2$ iff $\forall \varphi \in \mathcal{L} : [\![\varphi]\!](s_1) = [\![\varphi]\!](s_2)$.

From equivalence relation to 1-bounded pseudometric

We restrict our attention to

Definition

A pseudometric $d_S : S \times S \to \mathbb{R}$ is *1-bounded* if for all $s_1, s_2 \in S$,

$$d_S(s_1, s_2) \leq 1.$$

When generalizing from equivalence relations to 1-bounded pseudometrics, we go from $\mathbb{B}$ to $[0, 1]$.
Hence,

Theorem

$s_1 \sim s_2$ iff $\forall \varphi \in \mathcal{L} : [\![\varphi]\!](s_1) = [\![\varphi]\!](s_2)$.

suggests

Definition

$d_S(s_1, s_2) = \sup_{\varphi \in \mathcal{L}} [\![\varphi]\!](s_1) - [\![\varphi]\!](s_2)$.

A real-valued interpretation

Definition

The logic $\mathcal{L}$ is defined by

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \varphi \ominus q \mid \neg\varphi$$

where $q \in [0,1] \cap \mathbb{Q}$.

Definition

The function $[\![\cdot]\!] : \mathcal{L} \to S \to [0,1]$ is defined by

$$\begin{array}{rcl} [\![\top]\!](s) & = & 1 \\ [\![\varphi_1 \wedge \varphi_2]\!](s) & = & \\ [\![\Diamond\varphi]\!](s) & = & \\ [\![\varphi \ominus q]\!](s) & = & \\ [\![\neg\varphi]\!](s) & = & \end{array}$$

A real-valued interpretation

Definition

The logic $\mathcal{L}$ is defined by

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \varphi \ominus q \mid \neg\varphi$$

where $q \in [0,1] \cap \mathbb{Q}$.

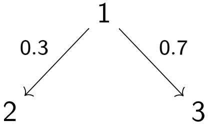
Definition

The function $[\![\cdot]\!] : \mathcal{L} \to S \to [0,1]$ is defined by

$$\begin{array}{rcl} [\![\top]\!](s) & = & 1 \\ [\![\varphi_1 \wedge \varphi_2]\!](s) & = & [\![\varphi_1]\!](s) \min [\![\varphi_2]\!](s) \\ [\![\Diamond\varphi]\!](s) & = & \\ [\![\varphi \ominus q]\!](s) & = & \\ [\![\neg\varphi]\!](s) & = & \end{array}$$

A real-valued interpretation

$[\![\varphi]\!](s) \in [0,1]$: "the probability that φ holds in s"



$$[\![\Diamond\varphi]\!](1) = 0.3 \times [\![\varphi]\!](2) + 0.7 \times [\![\varphi]\!](3)$$

A real-valued interpretation

Definition

The logic $\mathcal{L}$ is defined by

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \varphi \ominus q \mid \neg\varphi$$

where $q \in [0,1] \cap \mathbb{Q}$.

Definition

The function $[\![\cdot]\!] : \mathcal{L} \to S \to [0,1]$ is defined by

$$\begin{aligned}
[\![\top]\!](s) &= 1 \\
[\![\varphi_1 \wedge \varphi_2]\!](s) &= [\![\varphi_1]\!](s) \min [\![\varphi_2]\!](s) \\
[\![\Diamond\varphi]\!](s) &= \sum\{\, T(s,s') \times [\![\varphi]\!](s') \mid s' \in S \,\} \\
[\![\varphi \ominus q]\!](s) &= \\
[\![\neg\varphi]\!](s) &=
\end{aligned}$$

A real-valued interpretation

Definition

The logic $\mathcal{L}$ is defined by

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \varphi \ominus q \mid \neg\varphi$$

where $q \in [0,1] \cap \mathbb{Q}$.

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The function $[\![\cdot]\!] : \mathcal{L} \to S \to [0,1]$ is defined by

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[\![\Diamond\varphi]\!](s) &= \sum\{\, T(s,s') \times [\![\varphi]\!](s') \mid s' \in S \,\} \\
[\![\varphi \ominus q]\!](s) &= \max\{[\![\varphi]\!](s) - q, 0\} \\
[\![\neg\varphi]\!](s) &=
\end{aligned}$$

A real-valued interpretation

Definition

The logic $\mathcal{L}$ is defined by

$$\varphi ::= \top \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \varphi \ominus q \mid \neg\varphi$$

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Definition

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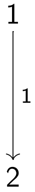
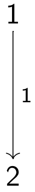
Example



Which formula distinguishes 1 and 2 the most?

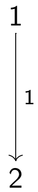
Example



Which formula distinguishes 1 and 2 the most?

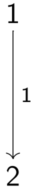
$$
\begin{aligned}
d_S(1,2) &= [\![\Diamond\top]\!](1) - [\![\Diamond\top]\!](2) \\
&= 1 \times [\![\top]\!](2) - 0 \\
&= 1.
\end{aligned}
$$

Example



Which formula distinguishes 1 and 2 the most?

Example

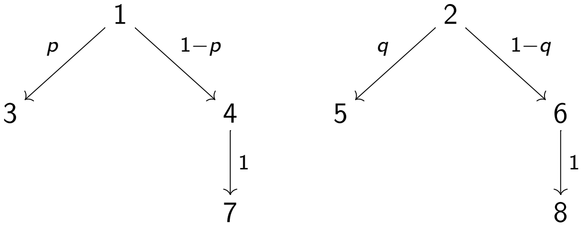


Which formula distinguishes 1 and 2 the most?

$$
\begin{aligned}
d_S(2,1) &= [\![\neg\Diamond\top]\!](2) - [\![\neg\Diamond\top]\!](1) \\
&= (1 - [\![\Diamond\top]\!](2)) - (1 - [\![\Diamond\top]\!](1)) \\
&= [\![\Diamond\top]\!](1) - [\![\Diamond\top]\!](2) \\
&= 1.
\end{aligned}
$$

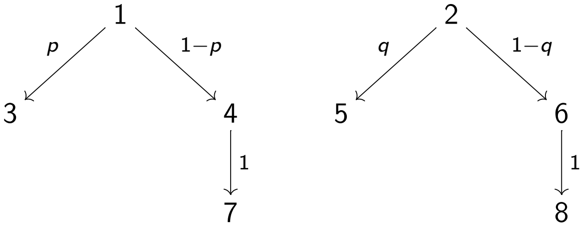
The formula $\neg\Diamond\top$ is abbreviated to $\surd$.

Example



Assume that $p \geq q$. Which formula distinguishes 1 and 2 the most?

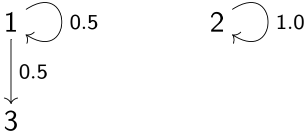
Example



Assume that $p \geq q$. Which formula distinguishes 1 and 2 the most?

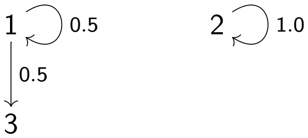
$$
\begin{aligned}
d_S(1,2) &= [\![\Diamond\surd]\!](1) - [\![\Diamond\surd]\!](2) \\
&= (p \times [\![\surd]\!](3) + (1-p) \times [\![\surd]\!](4)) \\
&\quad -(q \times [\![\surd]\!](5) + (1-q) \times [\![\surd]\!](6)) \\
&= (p \times 1 + (1-p) \times 0) - (q \times 1 + (1-q) \times 0) \\
&= p - q.
\end{aligned}
$$

Example



Which formula distinguishes 1 and 2 the most?

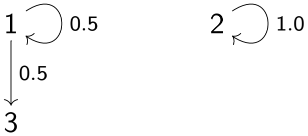
Example



Which formula distinguishes 1 and 2 the most?

$$
\begin{aligned}
& [\![\Diamond\surd]\!](1) - [\![\Diamond\surd]\!](2) \\
= \quad & (0.5 \times [\![\surd]\!](1) + 0.5 \times [\![\surd]\!](3)) - 1 \times [\![\surd]\!](2) \\
= \quad & 0.5.
\end{aligned}
$$

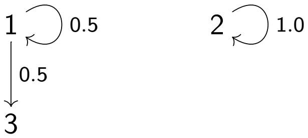
Example



Which formula distinguishes 1 and 2 the most?

$$
\begin{aligned}
&\llbracket \Diamond(\surd \vee \Diamond\surd)\rrbracket(1) - \llbracket \Diamond(\surd \vee \Diamond\surd)\rrbracket(2) \\
&\quad = \quad (0.5 \times \llbracket \surd \vee \Diamond\surd \rrbracket(1) + 0.5 \times \llbracket \surd \vee \Diamond\surd \rrbracket(3)) - 1 \times \llbracket \surd \vee \Diamond\surd \rrbracket(2) \\
&\quad = \quad (0.5 \times (\llbracket \surd \rrbracket(1) \max \llbracket \Diamond\surd \rrbracket(1)) + 0.5 \times (\llbracket \surd \rrbracket(3) \max \llbracket \Diamond\surd \rrbracket(3))) \\
&\quad -1 \times (\llbracket \surd \rrbracket(2) \max \llbracket \Diamond\surd \rrbracket(2)) \\
&\quad = \quad (0.5 \times (0 \max 0.5) + 0.5 \times (1 \max 0)) - 1 \times (0 \max 0) \\
&\quad = \quad 0.75.
\end{aligned}
$$

Example



Definition

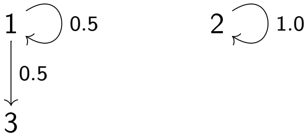
For $n \in \mathbb{N}$, the formula φ_n is defined by

$$\varphi_n = \begin{cases} \Diamond\surd & \text{if } n = 0 \\ \Diamond(\surd \vee \varphi_{n-1}) & \text{otherwise.} \end{cases}$$

Proposition

For all $n \in \mathbb{N}$, $[\![\varphi_n]\!](1) = 1 - 2^{-(n+1)}$ and $[\![\varphi_n]\!](2) = 0$.

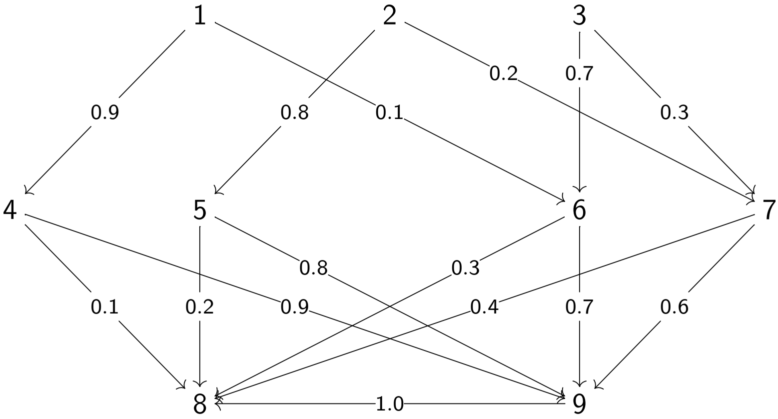
Example



$$
\begin{aligned}
d_S(1,2) &= \sup_{n \in \mathbb{N}} [\![\varphi_n]\!](1) - [\![\varphi_n]\!](2) \\
&= \sup_{n \in \mathbb{N}} (1 - 2^{-(n+1)}) - 0 \\
&= 1.
\end{aligned}
$$

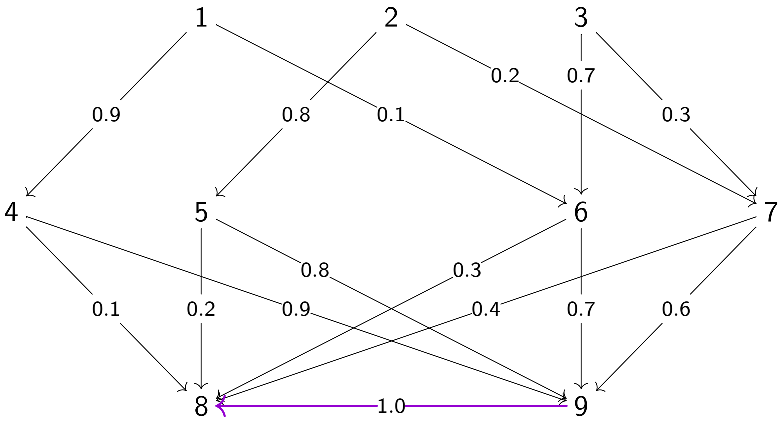
Counter-example revisited

$[\![\neg\surd]\!](8) = 0$ and $[\![\neg\surd]\!](9) = 1$



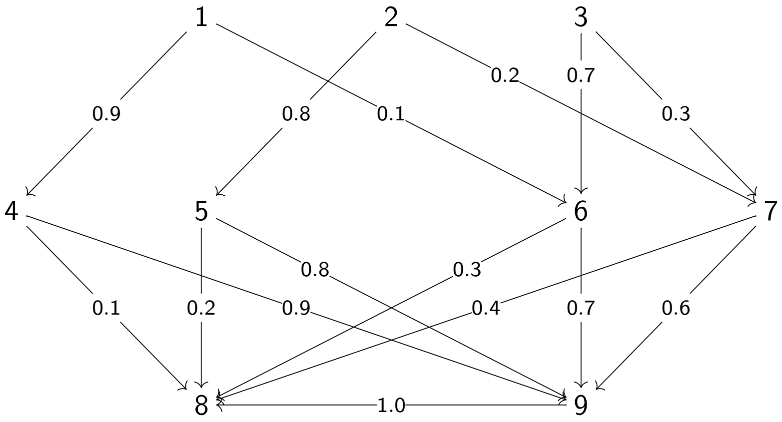
Counter-example revisited

$[\![\neg\surd]\!](8) = 0$ and $[\![\neg\surd]\!](9) = 1$



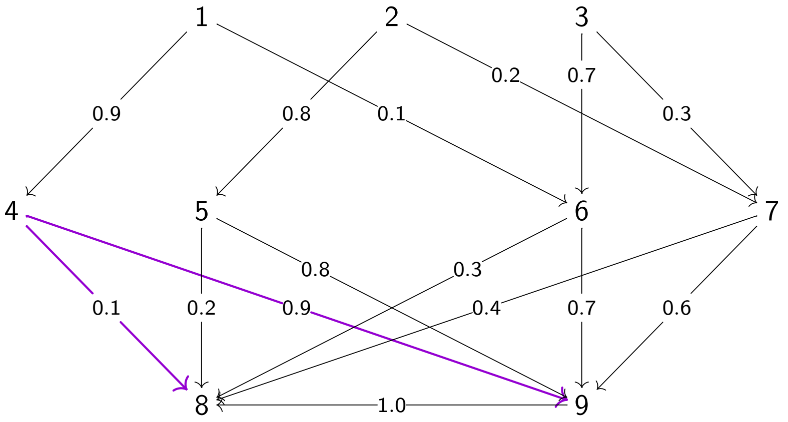
Counter-example revisited

$[\![\Diamond\neg\surd]\!](4) = 0.9$, $[\![\Diamond\neg\surd]\!](5) = 0.8$, $[\![\Diamond\neg\surd]\!](6) = 0.7$ and $[\![\Diamond\neg\surd]\!](7) = 0.6$



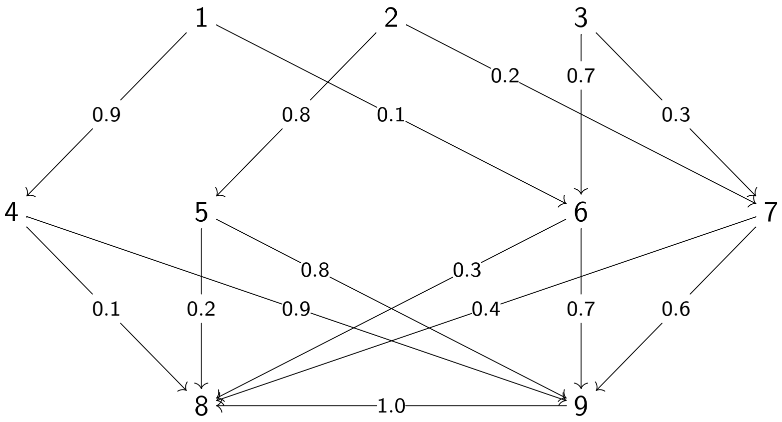
Counter-example revisited

$[\![\Diamond\neg\surd]\!](4) = 0.9$, $[\![\Diamond\neg\surd]\!](5) = 0.8$, $[\![\Diamond\neg\surd]\!](6) = 0.7$ and $[\![\Diamond\neg\surd]\!](7) = 0.6$



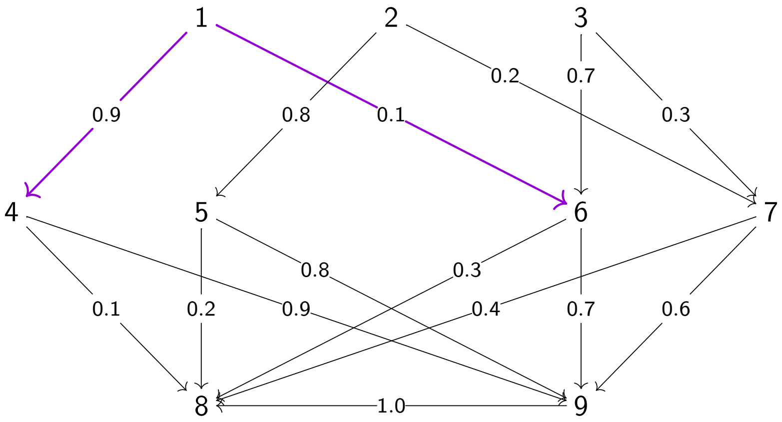
Counter-example revisited

$[\![\Diamond\Diamond\neg\surd]\!](1) = 0.88$, $[\![\Diamond\Diamond\neg\surd]\!](2) = 0.76$ and $[\![\Diamond\Diamond\neg\surd]\!](3) = 0.67$



Counter-example revisited

$$[\![\Diamond\Diamond\neg\surd]\!](1) = 0.88,\ [\![\Diamond\Diamond\neg\surd]\!](2) = 0.76 \text{ and } [\![\Diamond\Diamond\neg\surd]\!](3) = 0.67$$



Counter-Example (revisited)

According to the definition of GJS

$$\begin{aligned} d_S(1,2) &\leq 0.1 \\ d_S(2,3) &\leq 0.1 \\ d_S(1,3) &\geq 0.25 \end{aligned}$$

According to the definition of DGJP

$$\begin{aligned} d_S(1,2) &= 0.12 \\ d_S(2,3) &= 0.09 \\ d_S(1,3) &= 0.21 \end{aligned}$$

Some properties of d_S

DGJP proved

Proposition

d_S is a pseudometric.

and

Theorem

$s_1 \sim s_2$ iff $d_S(s_1, s_2) = 0$.

and also

Proposition

$\oplus$ and $\|$ are nonexpansive.

Markov process

Definition

A *Markov process* is a tuple $\langle S, \Sigma, \tau \rangle$ consisting of

- a set S of states,
- a σ-algebra Σ on S, and
- a function $\tau : S \times \Sigma \to [0, 1]$ such that
  - for all $s \in S$, $\tau(s, \cdot)$ is a measure and
  - for all $A \in \Sigma$, $\tau(\cdot, A)$ is a measurable function.

$$[\![\Diamond\varphi]\!](s) = \int_S [\![\varphi]\!] d\tau(s).$$

Overview of Part II: the rest of the nineties

Desharnais, Gupta, Jagadeesan and Panangaden generalized the behavioural pseudometric of Giacalone, Jou and Smolka to

- all PTSs and
- labelled Markov processes.

Are there other ways to define behavioural pseudometrics?