Timed Process Algebra

Part I

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Process algebra

Has ordering of actions.

But we want quantative analysis.

Time domain

Discrete vs. dense.

Linear vs. branching.

Least element.

Here,

- natural numbers,
- non-negative reals.

Timing syntax

Time-stamped vs. two-phase.

Axioms in two-phase.

But operational semantics and predicates for dense time work better time-stamped.

Timing semantics

Absolute vs. relative ($\rightarrow$ parametric).

Time-determinism: $\sigma.a + \sigma.\sigma.\sigma.b$.

- No delay of 2 possible: strong time-determinism,
- Choice by delay of 1: time-nondeterminism,
- Choice by delay of 2, not by delay of 1: weak time-determinism.

Timing semantics (cont.)

Duration of actions.

Urgent actions vs. multi-actions.

The book

PAwT, with Kees Middelburg, Springer 2002.

Add skip / explicit termination:

- Action prefix
- Focus on relative, dense

MPA$_{drt}$

- $\underline{\underline{\delta}}$ undelayable deadlock
- $\underline{\underline{\epsilon}}$ undelayable termination
- $\underline{\underline{a}}.$ current time slice action prefix
- $\sigma.$ next time slice prefix (‘tick’)
- $+$ alternative composition (choice)

Axioms

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

$$x + x = x$$

$$x + \underline{\underline{\delta}} = x$$

$$\sigma.x + \sigma.y = \sigma.(x + y)$$

Operational rules

$$
{}^{1}\frac{}{\underline{\underline{\epsilon}}\downarrow}
\qquad
{}^{2}\frac{}{\underline{\underline{a}}.x \xrightarrow{a} x}
\qquad
{}^{3}\frac{}{\sigma.x \overset{1}{\longmapsto} x}
$$

$$
{}^{4}\frac{x\downarrow}{x+y\downarrow}
\qquad
{}^{5}\frac{y\downarrow}{x+y\downarrow}
\qquad
{}^{6}\frac{x \xrightarrow{a} x'}{x+y \xrightarrow{a} x'}
\qquad
{}^{7}\frac{y \xrightarrow{a} y'}{x+y \xrightarrow{a} y'}
$$

$$
{}^{8}\frac{x \overset{1}{\longmapsto} x',\ y \overset{1}{\longmapsto} y'}{x+y \overset{1}{\longmapsto} x'+y'}
\qquad
{}^{9}\frac{x \overset{1}{\longmapsto} x',\ y \overset{1}{\not\longmapsto}}{x+y \overset{1}{\longmapsto} x'}
\qquad
{}^{10}\frac{x \overset{1}{\not\longmapsto},\ y \overset{1}{\longmapsto} y'}{x+y \overset{1}{\longmapsto} y'}
$$

$$a \in \mathsf{A}$$

Bisimulation

A bisimulation R is a symmetric, binary relation on processes:

$\forall p, q : pRq \Rightarrow$

- $p \downarrow \quad \Rightarrow \quad q \downarrow$
- $p \xrightarrow{a} p' \quad \Rightarrow \quad \exists q' : q \xrightarrow{a} q' \;\&\; p' R q'$
- $p \overset{1}{\longmapsto} p' \quad \Rightarrow \quad \exists q' : q \overset{1}{\longmapsto} q' \;\&\; p' R q'$

$p \leftrightarrows q \quad \Leftrightarrow \quad \exists$ bisimulation R

Theorems

Bisimilarity is a congruence relation for the operators.

MPA_{drt} is a sound and complete axiomatization of the model of bisimulation equivalence classes of closed terms.

Generalized time step

Generalize σ to σ^n.

$\sigma^0.x = x$

$\sigma^{n+1}.x = \sigma.\sigma^n.x$

Generalized time step as basic operator

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

$$x + x = x$$

$$x + \underline{\underline{\delta}} = x$$

$$\sigma^0 . x = x$$

$$\sigma^n . \sigma^m . x = \sigma^{n+m} . x$$

$$\sigma^n . x + \sigma^n . y = \sigma^n . (x + y)$$

Operational rules

Either $\xrightarrow{a}$, $\downarrow$, $\xmapsto{1}$ as before,

or $\xrightarrow{a}$, $\downarrow$, $\xmapsto{n}$.

Both do not generalize to dense time easily.

Axiomatically, generalization is straightforward.

MPT_{srt}

- $\tilde{\tilde{\delta}}$ undelayable deadlock
- $\tilde{\tilde{\epsilon}}$ undelayable termination
- $\tilde{\tilde{a}}.$ urgent action prefix
- $\sigma^t.$ relative time delay prefix
- $+$ alternative composition (choice)

Axioms

$$x + y = y + x$$

$$(x + y) + z = x + (y + z)$$

$$x + x = x$$

$$x + \tilde{\tilde{\delta}} = x$$

$$\sigma^0 . x = x$$

$$\sigma^t . \sigma^s . x = \sigma^{t+s} . x$$

$$\sigma^t . x + \sigma^t . y = \sigma^t . (x + y)$$

Operational rules

Can use $\xrightarrow{a}$, $\downarrow$, $\overset{t}{\longmapsto}$.

But will have problems further on. Therefore go to time-stamped:

- $x \xrightarrow{a}_t x'$: after delay of t, x can execute a and thereby evolve to x'
- $x \downarrow_t$: x can terminate after delay of t
- $\Delta_t(x)$: x can delay for time t

Operational rules

$$ {}^{1}\frac{}{\tilde{\tilde{\epsilon}} \downarrow_0} \qquad {}^{2}\frac{}{\tilde{\tilde{a}}.x \xrightarrow{a}_0 x} \qquad {}^{3}\frac{}{\Delta_0(x)} $$

$$ {}^{4}\frac{x \downarrow_t}{x + y \downarrow_t} \qquad {}^{5}\frac{y \downarrow_t}{x + y \downarrow_t} \qquad {}^{6}\frac{x \xrightarrow{a}_t x'}{x + y \xrightarrow{a}_t x'} \qquad {}^{7}\frac{y \xrightarrow{a}_t y'}{x + y \xrightarrow{a}_t y'} $$

$$ {}^{8}\frac{\Delta_t(x)}{\Delta_t(x + y)} \qquad {}^{9}\frac{\Delta_t(y)}{\Delta_t(x + y)} $$

$$ {}^{10}\frac{x \downarrow_t}{\sigma^s.x \downarrow_{t+s}} \qquad {}^{11}\frac{x \xrightarrow{a}_t x'}{\sigma^s.x \xrightarrow{a}_{t+s} x'} \qquad {}^{12}\frac{t \leq s}{\Delta_t(\sigma^s.x)} \qquad {}^{13}\frac{\Delta_t(x)}{\Delta_{t+s}(\sigma^s.x)} $$

$$ a \in \mathsf{A}, t, s \geq 0 $$

Theorems

Congruence, soundness, completeness.

Uses elimination to **basic terms**:

- $\forall t \geq 0 \quad \sigma^t.\tilde{\tilde{\epsilon}} \in \mathcal{B}$
- $\forall t \geq 0 \quad \sigma^t.\tilde{\tilde{\delta}} \in \mathcal{B}$
- $\forall a \in \mathsf{A}, t \geq 0, x \in \mathcal{B} \quad \sigma^t.\tilde{\tilde{a}}.x \in \mathcal{B}$
- $\forall x, y \in \mathcal{B} \quad x + y \in \mathcal{B}$

Sequential composition

$$(x + y) \cdot z = x \cdot z + y \cdot z$$

$$(x \cdot y) \cdot z = x \cdot (y \cdot z)$$

$$\tilde{\tilde{a}}.x \cdot y = \tilde{\tilde{a}}.(x \cdot y)$$

$$\sigma^t.x \cdot y = \sigma^t.(x \cdot y)$$

$$\tilde{\tilde{\delta}} \cdot x = \tilde{\tilde{\delta}}$$

$$\tilde{\tilde{\epsilon}} \cdot x = x$$

$$x \cdot \tilde{\tilde{\epsilon}} = x$$

Operational rules

$$
{}_{1}\frac{x \downarrow_t,\ y \downarrow_s}{x \cdot y \downarrow_{t+s}}
\qquad
{}_{2}\frac{x \xrightarrow{a}_t x'}{x \cdot y \xrightarrow{a}_t x' \cdot y}
\qquad
{}_{3}\frac{x \downarrow_t,\ y \xrightarrow{a}_s y'}{x \cdot y \xrightarrow{a}_{t+s} y'}
$$

$$
{}_{4}\frac{\Delta_t(x)}{\Delta_t(x \cdot y)}
\qquad
{}_{5}\frac{x \downarrow_t,\ \Delta_s(y)}{\Delta_{t+s}(x \cdot y)}
$$

$$a \in \mathsf{A}, t, s \geq 0$$

The problem with two-phase semantics

Sequential composition in two phase, discrete time:

$$\frac{x \stackrel{1}{\longmapsto} x',\ (x \not\downarrow \vee y \not\stackrel{1}{\longmapsto})}{x \cdot y \stackrel{1}{\longmapsto} x' \cdot y}\,1 \qquad \frac{x \not\stackrel{1}{\longmapsto},\ x\downarrow,\ y \stackrel{1}{\longmapsto} y'}{x \cdot y \stackrel{1}{\longmapsto} y'}\,2 \qquad \frac{x \stackrel{1}{\longmapsto} x',\ x\downarrow,\ y \stackrel{1}{\longmapsto} y'}{x \cdot y \stackrel{1}{\longmapsto} x' \cdot y + y'}\,3$$

$$a \in \mathsf{A}$$

A delay t of $x \cdot y$ may be written in several ways as $r + s$, and

$x \stackrel{r}{\longmapsto} x_r,\ x_r\downarrow,\ y \stackrel{s}{\longmapsto} y_s$ leads to $x \cdot y \stackrel{t}{\longmapsto} y_s + \ldots$.

Theorems

Congruence,

soundness,

completeness,

elimination to basic terms,

conservative extension.

Parallel composition

$x \parallel y$ has interleaving and synchronisation.

Interleaving: x or y can execute an action now, urgent and independent. Left merge.

Synchronisation (communication merge):

- x and y can execute a communication action synchronously, by each executing their half (send and receive give communication);
- x and y can terminate now, synchronously;
- x and y can delay together.

AXIOMS

$x \parallel y = x \mathbin{\lfloor\!\lfloor} y + y \mathbin{\lfloor\!\lfloor} x + x \mid y$	$x \mid y = y \mid x$
	$\tilde{\tilde{\delta}} \mid x = \tilde{\tilde{\delta}}$
$\tilde{\tilde{\delta}} \mathbin{\lfloor\!\lfloor} x = \tilde{\tilde{\delta}}$	$\tilde{\tilde{\epsilon}} \mid \tilde{\tilde{\epsilon}} = \tilde{\tilde{\epsilon}}$
$\tilde{\tilde{\epsilon}} \mathbin{\lfloor\!\lfloor} x = \tilde{\tilde{\delta}}$	$\tilde{\tilde{a}}.x \mid \tilde{\tilde{\epsilon}} = \tilde{\tilde{\delta}}$
$\tilde{\tilde{a}}.x \mathbin{\lfloor\!\lfloor} y = \tilde{\tilde{a}}.(x \parallel y)$	$\tilde{\tilde{a}}.x \mid \tilde{\tilde{b}}.y = \tilde{\tilde{c}}.(x \parallel y)$ if $\gamma(a,b) = c$
$\sigma^u.x \mathbin{\lfloor\!\lfloor} y = \tilde{\tilde{\delta}}$	$\tilde{\tilde{a}}.x \mid \tilde{\tilde{b}}.y = \tilde{\tilde{\delta}}$ otherwise
$(x + y) \mathbin{\lfloor\!\lfloor} z = x \mathbin{\lfloor\!\lfloor} z + y \mathbin{\lfloor\!\lfloor} z$	$\tilde{\tilde{a}}.x \mid \sigma^u.y = \tilde{\tilde{\delta}}$
	$\sigma^u.x \mid \tilde{\tilde{\epsilon}} = \sigma^u.x$
	$\sigma^u.x \mid \sigma^u.y = \sigma^u.(x \parallel y)$
	$(x + y) \mid z = x \mid z + y \mid z$

$$a, b \in \mathsf{A},\ u > 0$$

Operational rules

$$1\;\frac{x \downarrow_t,\; y \downarrow_s}{x \parallel y \downarrow_{max(t,s)}} \qquad 2\;\frac{x \xrightarrow{a}_t x',\; \Delta_t(y)}{x \parallel y \xrightarrow{a}_t x' \parallel (t \gg y)} \qquad 3\;\frac{x \xrightarrow{a}_t x',\; y \downarrow_s,\; s \leq t}{x \parallel y \xrightarrow{a}_t x'}$$

$$4\;\frac{x \xrightarrow{a}_t x',\; y \xrightarrow{b}_t y',\; \gamma(a,b) = c}{x \parallel y \xrightarrow{c}_t x' \parallel y'} \qquad 5\;\frac{\Delta_t(y),\; y \xrightarrow{a}_t y'}{x \parallel y \xrightarrow{a}_t (t \gg x') \parallel y} \qquad 6\;\frac{y \downarrow_s,\; s \leq t,\; y \xrightarrow{a}_t y'}{x \parallel y \xrightarrow{a}_t y'}$$

$$7\;\frac{\Delta_t(x),\; \Delta_t(y)}{\Delta_t(x \parallel y)} \qquad 8\;\frac{\Delta_t(x),\; y \downarrow_s, s \leq t}{\Delta_t(x \parallel y)} \qquad 9\;\frac{x \downarrow_s,\; s \leq t,\; \Delta_t(y)}{\Delta_t(x \parallel y)}$$

$$a, b, c \in \mathsf{A}, t, s \geq 0$$

Additional axioms

$\partial_H(\tilde{\tilde{\delta}}) = \tilde{\tilde{\delta}}$		$0 \gg x = x$
$\partial_H(\tilde{\tilde{\epsilon}}) = \tilde{\tilde{\epsilon}}$		$t \gg \tilde{\tilde{\delta}} = \tilde{\tilde{\delta}}$
$\partial_H(\tilde{\tilde{a}}.x) = \tilde{\tilde{a}}.\partial_H(x)$	if $a \notin H$	$u \gg \tilde{\tilde{\epsilon}} = \tilde{\tilde{\delta}}$
$\partial_H(\tilde{\tilde{a}}.x) = \tilde{\tilde{\delta}}$	if $a \in H$	$u \gg \tilde{\tilde{a}}.x = \tilde{\tilde{\delta}}$
$\partial_H(\sigma^t.x) = \sigma^t.\partial_H(x)$		$t \gg \sigma^{t+s}.x = \sigma^s.x$
$\partial_H(x+y) = \partial_H(x) + \partial_H(y)$		$(t+s) \gg \sigma^s.x = t \gg x$
		$t \gg (x+y) = t \gg x + t \gg y$

$$a \in \mathsf{A}, H \subseteq \mathsf{A},\ t, s \geq 0, u > 0$$

Additional rules

Straightforward.

Congruence, soundness, completeness, elimination, conservativity.

Axioms of standard concurrency

$$x \parallel \tilde{\tilde{\epsilon}} = x$$

$$(x \parallel y) \parallel z = x \parallel (y \parallel z)$$

$$(x \mid y) \mid z = x \mid (y \mid z)$$

$$(x \mathbin{\lfloor\!\lfloor} y) \mathbin{\lfloor\!\lfloor} z = x \mathbin{\lfloor\!\lfloor} (y \parallel z)$$

$$(x \mid y) \mathbin{\lfloor\!\lfloor} z = x \mid (y \mathbin{\lfloor\!\lfloor} z)$$

Expansion theorem.

Handshaking communication

Only binary communication.

Processes send, receive and communicate data at **ports**.

Internal and external ports.

$s_i(d)$: the action of sending datum d at port i;

$r_i(d)$: the action of receiving datum d at port i;

$c_i(d)$: the action of communicating datum d at port i.

$\gamma(s_i(d), r_i(d)) = \gamma(r_i(d), s_i(d)) = c_i(d)$

Delayable actions

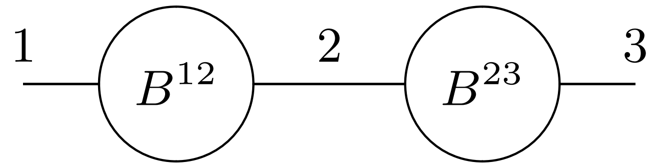
$a.x = \underline{\underline{a}}.x + \sigma.a.x$

$a.x \xrightarrow{a}_t x \qquad \Delta_t(a.x)$

Embedding of time-free theory.

Arbitrary delay prefix.

Communicating buffers



Time-free process algebra:

$$B^{ij} = \sum_{d \in D} r_i(d).s_j(d).B^{ij}$$

Using time-free calculations:

$$\tau_I \circ \partial_H (B^{12} \parallel B^{23}) = B_2^{13}$$

No delay buffers

At most one input per time slice, output with no delay.

$$C^{ij} = \sum_{d \in D} r_i(d).\underline{\underline{s_j(d)}}.\sigma.C^{ij}$$

or equivalently

$$C^{ij} = \sum_{d \in D} \underline{\underline{r_i(d)}}.\underline{\underline{s_j(d)}}.\sigma.C^{ij} + \sigma.C^{ij}$$

Time-free projection

$$\pi_{\mathsf{tf}}(C^{ij}) = B^{ij}$$

CALCULATION

$$\partial_H(C^{12} \parallel C^{23}) = \sum_{d \in D} r_1(d).\partial_H(\underline{\underline{s_2(d)}}.\sigma.C^{12} \parallel C^{23}) =$$

$$= r_1(d).\underline{\underline{c_2(d)}}.\partial_H(\sigma.C^{12} \parallel \underline{\underline{s_3(d)}}.\sigma.C^{23}) =$$

$$= r_1(d).\underline{\underline{c_2(d)}}.\underline{\underline{s_3(d)}}.\partial_H(\sigma.C^{12} \parallel \sigma.C^{23}) =$$

$$= r_1(d).\underline{\underline{c_2(d)}}.\underline{\underline{s_3(d)}}.\sigma.\partial_H(C^{12} \parallel C^{23}).$$

Abstraction from internal communications:

$$\tau_I \circ \partial_H(C^{12} \parallel C^{23}) = r_1(d).\underline{\underline{s_3(d)}}.\sigma.\tau_I \circ \partial_H(C^{12} \parallel C^{23}) = C^{13}$$

Unit delay buffers

At most one input per time slice, output with unit delay.

$$D^{ij} = \sum_{d \in D} r_i(d).\sigma.\underline{\underline{s_j(d)}}.D^{ij}$$

Time-free projection

$$\pi_{\mathsf{tf}}(D^{ij}) = B^{ij}$$

Calculation

$$X = \partial_H(D^{12} \parallel D^{23}) = \sum_{d \in D} r_1(d).\sigma\partial_H(\underline{\underline{s_2(d)}}.D^{12} \parallel D^{23}) =$$

$$= r_1(d).\sigma.\underline{\underline{c_2(d)}}.\partial_H(D^{12} \parallel \sigma.\underline{\underline{s_3(d)}}.D^{23}) = r_1(d).\sigma.\underline{\underline{c_2(d)}}.X_d$$

$$X_d = \partial_H(D^{12} \parallel \sigma.\underline{\underline{s_3(d)}}.D^{23}) =$$

$$= \sum_{e \in D} \underline{\underline{r_1(e)}}.\partial_H(\sigma.\underline{\underline{s_2(e)}}.D^{12} \parallel \sigma.\underline{\underline{s_3(d)}}.D^{23}) + \sigma.\partial_H(D^{12} \parallel \underline{\underline{s_3(d)}}.D^{23}) =$$

$$= \sum_{e \in D} \underline{\underline{r_1(e)}}.\sigma.\partial_H(\underline{\underline{s_2(e)}}.D^{12} \parallel \underline{\underline{s_3(d)}}.D^{23}) +$$

$$+\sigma.(\sum_{e \in D} \underline{\underline{r_1(e)}}.\partial_H(\sigma.\underline{\underline{s_2(e)}}.D^{12} \parallel \underline{\underline{s_3(d)}}.D^{23}) + \underline{\underline{s_3(d)}}.\partial_H(D^{12} \parallel D^{23})) =$$

$$
\begin{aligned}
&= \sum_{e\in D} \underline{\underline{r_i(e)}}.\sigma.\underline{\underline{s_3(d)}}.\partial_H(\underline{\underline{s_2(e)}}.D^{12} \parallel D^{23})+ \\
&\qquad +\sigma.(\sum_{e\in D} \underline{\underline{r_1(e)}}.\underline{\underline{s_3(d)}}.\partial_H(\sigma.\underline{\underline{s_2(e)}}.D^{12} \parallel D^{23}) + \underline{\underline{s_3(d)}}.X) = \\
&= \sum_{e\in D} \underline{\underline{r_1(e)}}.\sigma.\underline{\underline{s_3(d)}}.\underline{\underline{c_2(e)}}.\partial_H(D^{12} \parallel \sigma.\underline{\underline{s_3(e)}}.D^{23})+ \\
&\qquad +\sigma.(\sum_{e\in D} \underline{\underline{r_1(e)}}.\underline{\underline{s_3(d)}}.\sigma.\partial_H(\underline{\underline{s_2(e)}}.D^{12} \parallel D^{23}) + \underline{\underline{s_3(d)}}.X) = \\
&= \sum_{e\in D} \underline{\underline{r_1(e)}}.\sigma.\underline{\underline{s_3(d)}}.\underline{\underline{c_2(e)}}.X_e+ \\
&\qquad +\sigma.(\sum_{e\in D} \underline{\underline{r_1(e)}}.\underline{\underline{s_3(d)}}.\sigma.\underline{\underline{c_2(e)}}.\partial_H(D^{12} \parallel \sigma.\underline{\underline{s_3(e)}}.D^{23}) + \underline{\underline{s_3(d)}}.X) = \\
&= \sum_{e\in D} \underline{\underline{r_1(e)}}.\sigma.\underline{\underline{s_3(d)}}.\underline{\underline{c_2(e)}}.X_e + \sigma.(\sum_{e\in D} \underline{\underline{r_1(e)}}.\underline{\underline{s_3(d)}}.\sigma.\underline{\underline{c_2(e)}}.X_e + \underline{\underline{s_3(d)}}.X).
\end{aligned}
$$

Calculation (cont.)

After abstraction:

$$\tau_I(X) = \sum_{d \in D} r_1(d).\sigma.\tau_I(X_d)$$

$$\tau_I(X_d) = \sum_{e \in D} \underline{\underline{r_1(e)}}.\sigma.\underline{\underline{s_3(d)}}.\tau_I(X_e) +$$

$$+\sigma.(\sum_{e \in D} \underline{\underline{r_1(e)}}.\underline{\underline{s_3(d)}}.\sigma.\tau_I(X_e) + \underline{\underline{s_3(d)}}.\tau_I(X).$$

A two-place buffer with delay of two time units.

In accordance with time-free theory.

See picture.

Input-enabled buffer

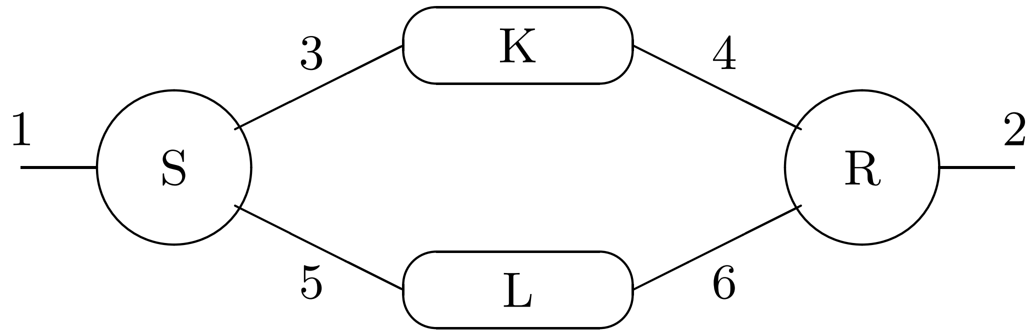
$$E^{ij} = \sum_{d \in D} r_i(d).\sigma.\left(\underline{\underline{s_j(d)}}.\underline{\underline{\epsilon}} \parallel E^{ij}\right)$$

Again, delay of 1 between input and output.

But input always possible, at most one per time slice.

Possible to have three elements inside in $\partial_H(E^{12} \parallel E^{23})$.

PAR PROTOCOL



$$S = S_0$$

$$S_b = \sum_{d \in D} r_1(d).\sigma^{t_1}.SF_{d,b}$$

$$SF_{d,b} = \underline{\underline{s_3(d,b)}}.\left(\sum_{k<t_1'} \sigma^k.\underline{\underline{r_5(ack)}}.S_{1-b} + \sigma^{t_1'}.SF_{d,b}\right)$$

SPECIFICATION

$$R = R_0$$

$$R_b = \sum_{d \in D} r_4(d,b).\sigma^{t_2}.\underline{\underline{s_2(d)}}.\sigma^{t_2'}.\underline{\underline{s_6(ack)}}.R_{1-b} + \sum_{d \in D} r_4(d,1-b).\sigma^{t_2'}.\underline{\underline{s_6(ack)}}.R_b$$

$$K = \sum_{f \in F} r_3(f).\left(\sigma^{t_3}.\underline{\underline{s_4(f)}}.K + \sum_{k \leq t_3} \sigma^k.\underline{\underline{error}}.K\right)$$

$$L = r_6(ack).\left(\sigma^{t_4}.\underline{\underline{s_5(ack)}}.L + \sum_{k \leq t_4} \sigma^k.\underline{\underline{error}}.L\right)$$

$$\partial_H(S \parallel K \parallel L \parallel R)$$

Expansion

$$S''_{d,b,t} = \sum_{k<t} \sigma^k.\underline{\underline{r_5(ack)}}.S_{1-b} + \sigma^t.\underline{\underline{s_3(d,b)}}.S''_{d,b}$$

$$\partial_H(S_b \parallel K \parallel L \parallel R_b) = \sum_{d \in D} r_1(d).\partial_H(S'_{d,b} \parallel K \parallel L \parallel R_b)$$

$$\partial_H(S'_{d,b} \parallel K \parallel L \parallel R_b) = \sigma^{t_1}.\underline{\underline{c_3(d,b)}}.\partial_H(S''_{d,b} \parallel K'_{d,b} \parallel L \parallel R_b)$$

$$\begin{aligned}\partial_H(S''_{d,b} \parallel K'_{d,b} \parallel L \parallel R_b) = \sigma^{t_3}.\underline{\underline{c_4(d,b)}}.\partial_H(S''_{d,b,t'_1-t_3} \parallel K \parallel L \parallel R'_{d,b})+ \\ + \sum_{k \leq t_3} \sigma^k.\underline{\underline{error}}.\partial_H(S''_{d,b,t'_1-k} \parallel K \parallel L \parallel R_b)\end{aligned}$$

$$\partial_H(S''_{d,b,t} \parallel K \parallel L \parallel R'_{d,b}) = \sigma^{t_2}.\underline{\underline{s_2(d)}}.\partial_H(S''_{d,b,t-t_2} \parallel K \parallel L \parallel R''_{1-b})$$

$$\partial_H(S''_{d,b,t} \parallel K \parallel L \parallel R''_{1-b}) = \sigma^{t'_2}.\underline{\underline{c_6(ack)}}.\partial_H(S''_{d,b,t-t'_2} \parallel K \parallel L' \parallel R_{1-b})$$

Expansion (cont.)

$$\partial_H(S''_{d,b,t} \parallel K \parallel L' \parallel R_{1-b}) = \sigma^{t_4}.\underline{\underline{c_5(ack)}}.\partial_H(S_{1-b} \parallel K \parallel L \parallel R_{1-b}) + \\ + \sum_{k \le t_4} \sigma^k.\underline{\underline{error}}.\partial_H(S''_{d,b,t-k} \parallel K \parallel L \parallel R_{1-b})$$

$$\partial_H(S''_{d,b,t} \parallel K \parallel L \parallel R_b) = \sigma^t.\underline{\underline{c_3(d,b)}}.\partial_H(S''_{d,b} \parallel K'_{d,b} \parallel L \parallel R_b)$$

$$\partial_H(S''_{d,b,t} \parallel K \parallel L \parallel R_{1-b}) = \sigma^t.\underline{\underline{c_3(d,b)}}.\partial_H(S''_{d,b} \parallel K'_{d,b} \parallel L \parallel R_{1-b})$$

$$\partial_H(S''_{d,b} \parallel K'_{d,b} \parallel L \parallel R_{1-b}) = \sigma^{t_3}.\underline{\underline{c_4(d,b)}}.\partial_H(S''_{d,b,t'_1-t_3} \parallel K \parallel L \parallel R''_{1-b}) + \\ + \sum_{k \le t_3} \sigma^k.\underline{\underline{error}}.\partial_H(S''_{d,b,t'_1-k} \parallel K \parallel L \parallel R_{1-b})$$

Conclusion: system without deadlocks if $t'_1 > t_2 + t'_2 + t_3 + t_4$.

Abstraction

Abstract from all actions except inputs (r_1) and outputs (s_2).

Functional correctness: first abstract from timing, then use time-free theory, yields a one-place buffer. Still, timing was necessary to reach this result!

Can also do abstraction in the timed theory, using branching laws. Obtain the following recursive specification for $\tau_I \circ \partial_H(S_b \parallel K \parallel L \parallel R_b)$:

Additional τ law

$$X = \sum_{d \in D} r_1(d).\sigma^{t_1}.Y_d$$

$$Y_d = \sigma^{t_3}.\underline{\underline{\tau}}.\sigma^{t_2}.\underline{\underline{s_2(d)}}.\sigma^{t_2'}.Z + \sum_{k \leq t_3} \sigma^k.\underline{\underline{\tau}}.\sigma^{t_1'-k}.Y_d,$$

$$Z = \sigma^{t_4}.\underline{\underline{\tau}}.X + \sum_{k \leq t_4} \sigma^k.\underline{\underline{\tau}}.\sigma^{t_1'-(t_2+t_2'+t_3+k)}.U$$

$$U = \sigma^{t_3}.\underline{\underline{\tau}}.\sigma^{t_2'}.V'' + \sum_{k \leq t_3} \sigma^k.\underline{\underline{\tau}}.\sigma^{t_1'-k}.U$$

$$V = \sigma^{t_4}.\underline{\underline{\tau}}.X + \sum_{k \leq t_4} \sigma^k.\underline{\underline{\tau}}.\sigma^{t_1'-(t_2'+t_3+k)}.U$$

Timing of each silent action is preserved, so timing of all choices is preserved.

Leads to formulate extra τ-law: $\underline{\underline{\tau}}.\sigma.x = \sigma.\underline{\underline{\tau}}.x.$

Formulation of semantical equivalence difficult.

Final result, using the extra law

Functional **and** timing behaviour of the PAR protocol is captured in the following.

$$A = \sum_{d \in D} r_1(d).\sigma^{t_1+t_2+t_3}.B_d$$

$$B_d = \underline{\underline{s_2(d)}}.\sigma^{t'_2}.C + \sigma^{t'_1}.B_d$$

$$C = \sigma^{t_4}.(A + \sigma^{t'_1-t_2-t_4}.D)$$

$$D = \sigma^{t_4}.(A + \sigma^{t'_1-t_4}.D)$$