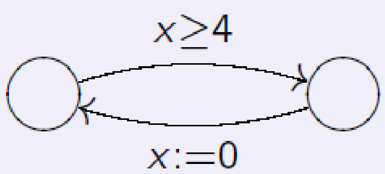
Metric Concurrency Theory

Giovanni Bacci, Giorgio Bacci
Kim G. Larsen, Radu Mardare

Aalborg University
DENMARK

Quantities & Metrics

Quantitative *Models*



Quantitative *Logics*

$$\mathrm{Pr}_{\leq .1}(\Diamond error)$$

Quantitative *Verification*

$$[\![\varphi]\!](s) = 0.89$$
$$d(s,t) = 0.09$$

Boolean world	**"Quantification"**
Trace equivalence $\equiv$	Linear distance d_L
Bisimilarity $\sim$	Branching distance d_B
$s \sim t$ implies $s \equiv t$	$d_L(s,t) \leq d_B(s,t)$
$s \models \varphi$ or $s \not\models \varphi$	$[\![\varphi]\!](s)$ is a quantity
$s \sim t$ iff $\forall \varphi : s \models \varphi \Leftrightarrow t \models \varphi$	$d_B(s,t) = \sup_\varphi d\left([\![\varphi]\!](s), [\![\varphi]\!](t)\right)$

Markov Chains & Probabilistic Bisimulation

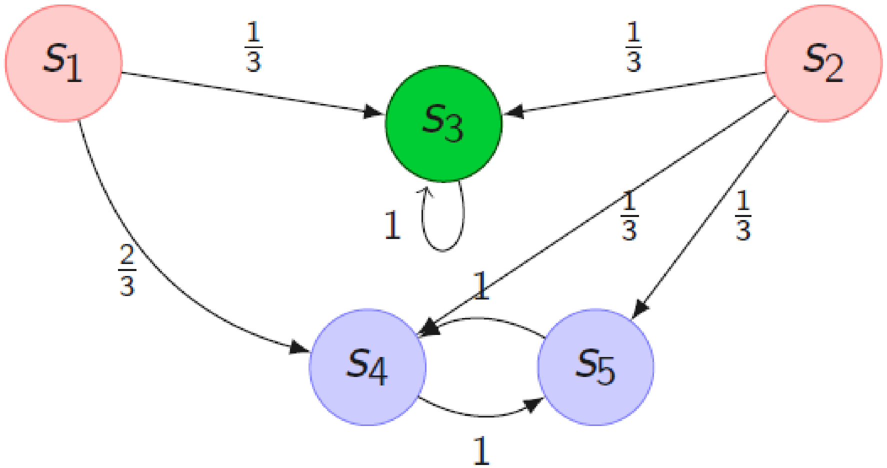
$$\mathcal{M} = (S, \Sigma, \pi, \ell)$$

S: finite set of states

Σ: set of labels

$\pi : S \times S \to [0,1]$

$\ell : S \to \Sigma$.



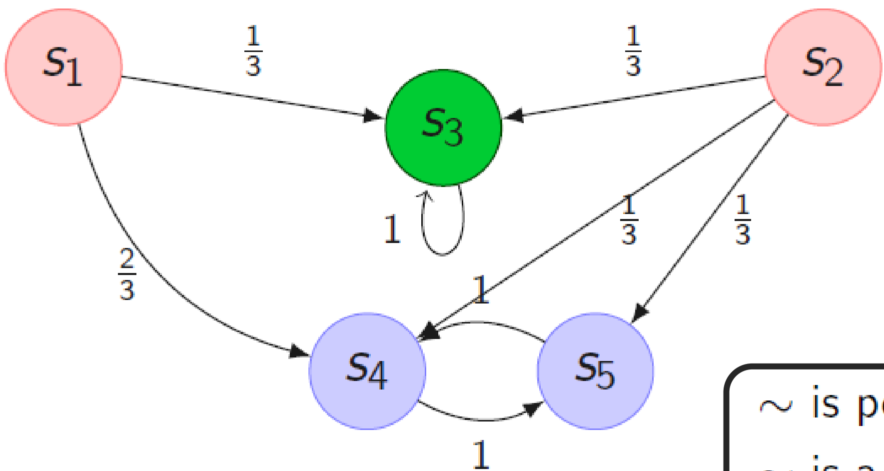
Markov Chains & Probabilistic Bisimulation

Probabilistic Bisimulation [LS]

Equivalence relation $R \subseteq S \times S$ such that

$$s\,R\,t \implies \begin{cases} \ell(s) = \ell(t) \\ \forall R-\text{equiv. class}\,C.\ \sum_{u \in C} \pi(s,u) = \sum_{v \in C} \pi(t,v) \end{cases}$$

$s \sim t$ iff $s\,R\,t$ for some probabilistic bisimulation R.



- $\sim$ is polynomial-time decidable.
- $\sim$ is a congruence.

Christel Baier: Polynomial Time Algorithms for Testing Probabilistic Bisimulation and Simulation. CAV 1996

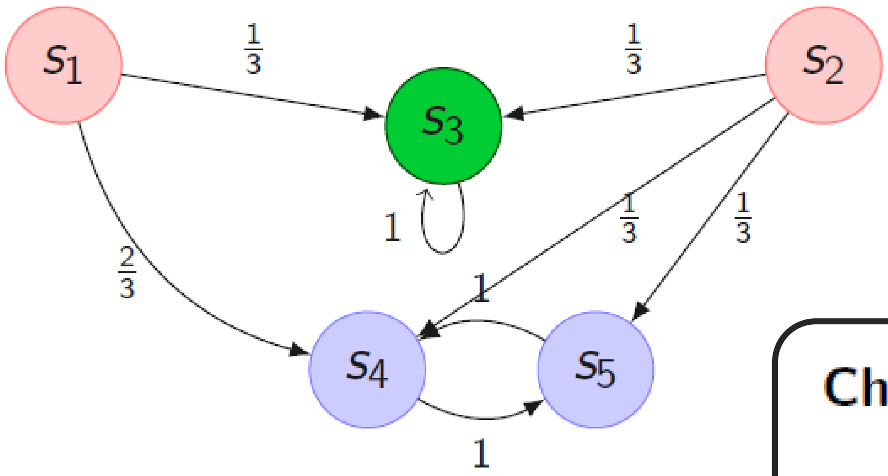
Markov Chains & Probabilistic Bisimulation

Probabilistic Modal Logic

$$F ::= p \mid \neg F \mid F_1 \wedge F_2 \mid \Diamond_{\geq\mu} F$$

where $p \in AP$ and $\mu \in [0,1]$.

$$s \models \Diamond_{\geq\mu} F \text{ iff } \sum_{t \models F} \pi(s,t) \geq \mu$$



$s_1, s_2 \models \Diamond_{\geq 0.5} \bullet$

Characterization

$$s \sim t \text{ iff } (\forall F.\, s \models F \iff t \models F)$$

K. G. Larsen and A. Skou. Bisimulation through probabilistic testing. Information and Computation, 1991

Markov Chains & Probabilistic Bisimulation

Probabilistic Bisimulation [LS]

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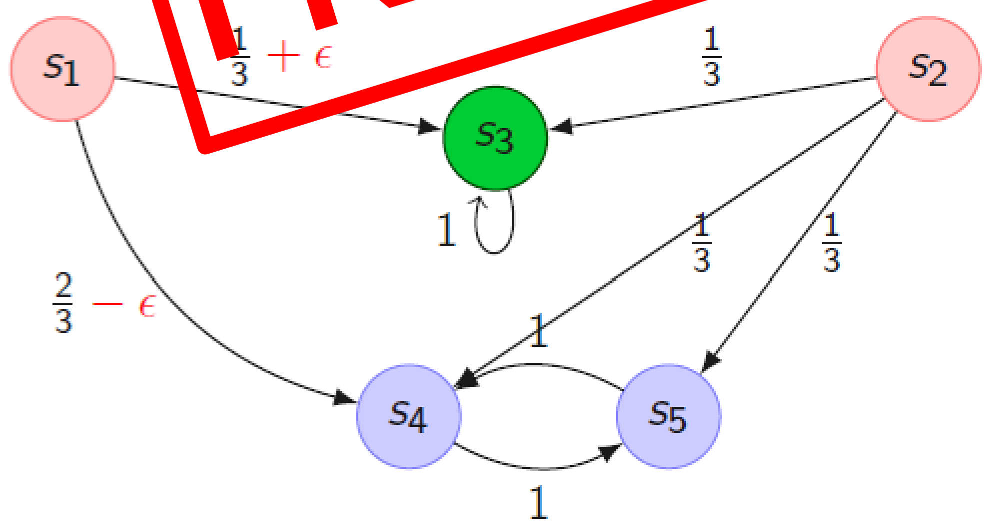
FRAGILE



From Equivalences to Distances

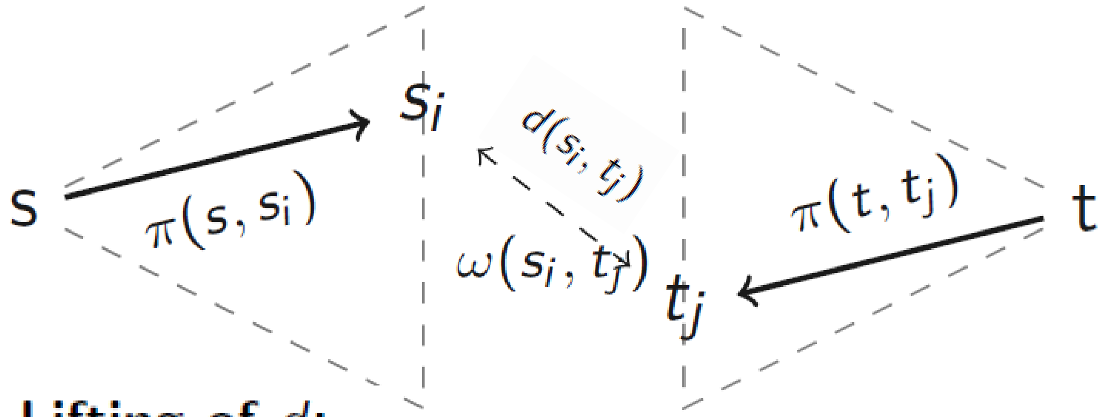
Pseudometrics $d \colon S \times S \to \mathbb{R}_{\geq 0}$ are the quantitative analogue of an equivalence relation

equivalence		pseudometric
$s \equiv s$	$\rightsquigarrow$	$d(s,s) = 0$
$s \equiv t \implies t \equiv s$	$\rightsquigarrow$	$d(s,t) = d(t,s)$
$s \cong u \wedge u \cong t \implies s \cong t$	$\rightsquigarrow$	$d(s,u) + d(u,t) \geq d(s,t)$

Bisimilarity Pseudometric: $d(s,t) = 0 \iff s \sim t$

A. Giacalone, C.-C. Jou, and S. A. Smolka. Algebraic Reasoning for IFIP 1990 In IFIP WG 2.2/2.3, pages 443{458, 1990 .

Fixed Point Characterization



Kantorovich Lifting of d:

$$K(d)(s,t) = \min \left\{ \sum_{u,v \in S} d(u,v) \cdot \omega(u,v) \;\middle|\; \begin{array}{l} \forall u \in S \ \sum_{v \in S} \omega(u,v) = \pi(s,u) \\ \forall v \in S \ \sum_{u \in S} \omega(u,v) = \pi(t,v) \end{array} \right\}$$

matching

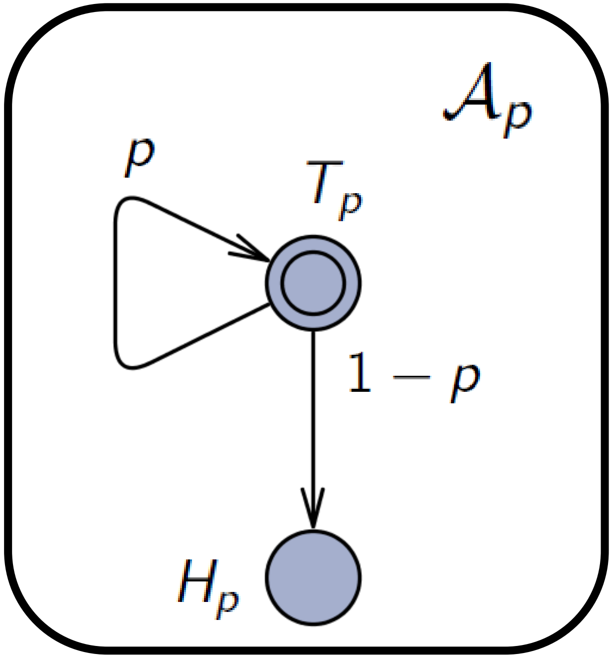
$\omega \in \pi(s,\cdot) \otimes \pi(t,\cdot)$

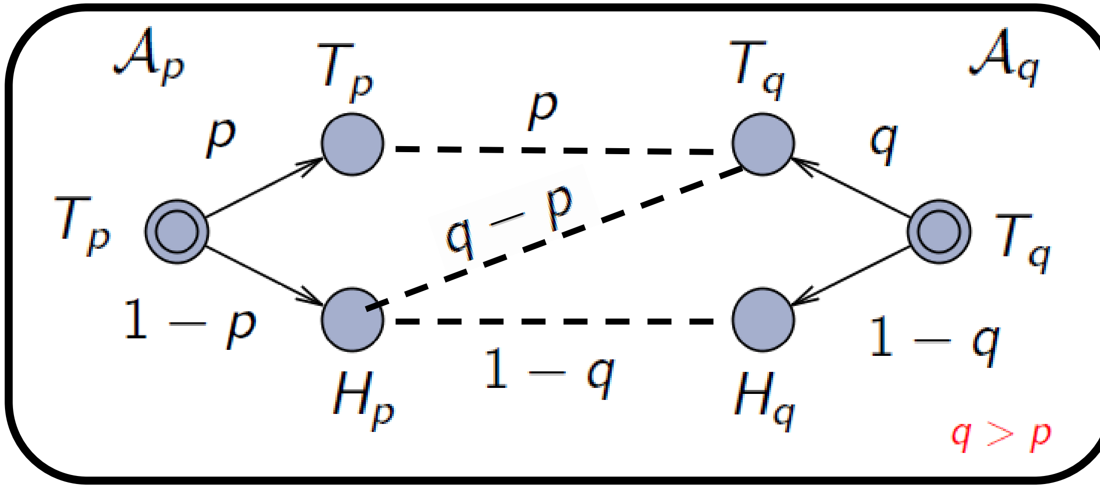
$d_{\mathcal{K}}(s,s')$ is the least fixed point to the following operator on pseudometrics:

$$\mathcal{K}(d)(s,s') = \max \left\{ |\ell s - \ell s'| \, , \, K(d)(s,s') \right\}$$

F. van Breugel and J. Worrell. Towards quantitative verication of probabilistic transition systems. ICALP2001.

Coin Example





$$d_{\mathcal{K}}(T_p, T_q) \ = \ p \cdot d_{\mathcal{K}}(T_p, T_q) + (q-p) \cdot 1$$

Thus $d_{\mathcal{K}}(T_p, T_q) = \frac{q-p}{1-p} \to 0$ when $q \to p$.

Results

Bisimulation Pseudometric: $d_{\mathcal{K}}(s, s') = 0 \iff s \sim s'$.

Complexity: $d_{\mathcal{K}}$ can be computed in polynomial time.

Upper Bound:

$$\sup_{\phi \in \mathrm{LTL}} |\mathbb{P}_s(\phi) - \mathbb{P}_{s'}(\phi)| \leq d_{\mathcal{K}}(s, s')$$

Tools: Efficient on-the-fly algorithm for computing $d_{\mathcal{K}}$ exactly has been developed.

Compositionality: Process constructs are non-expanding, contracting or continuous with respect to $d_{\mathcal{K}}$.

Characterization: $d_{\mathcal{F}} = d_{\mathcal{K}}$

Josee Desharnais, Vineet Gupta, Radha Jagadeesan, Prakash Panangaden: Metrics for Labeled Markov Systems. CONCUR 1999:

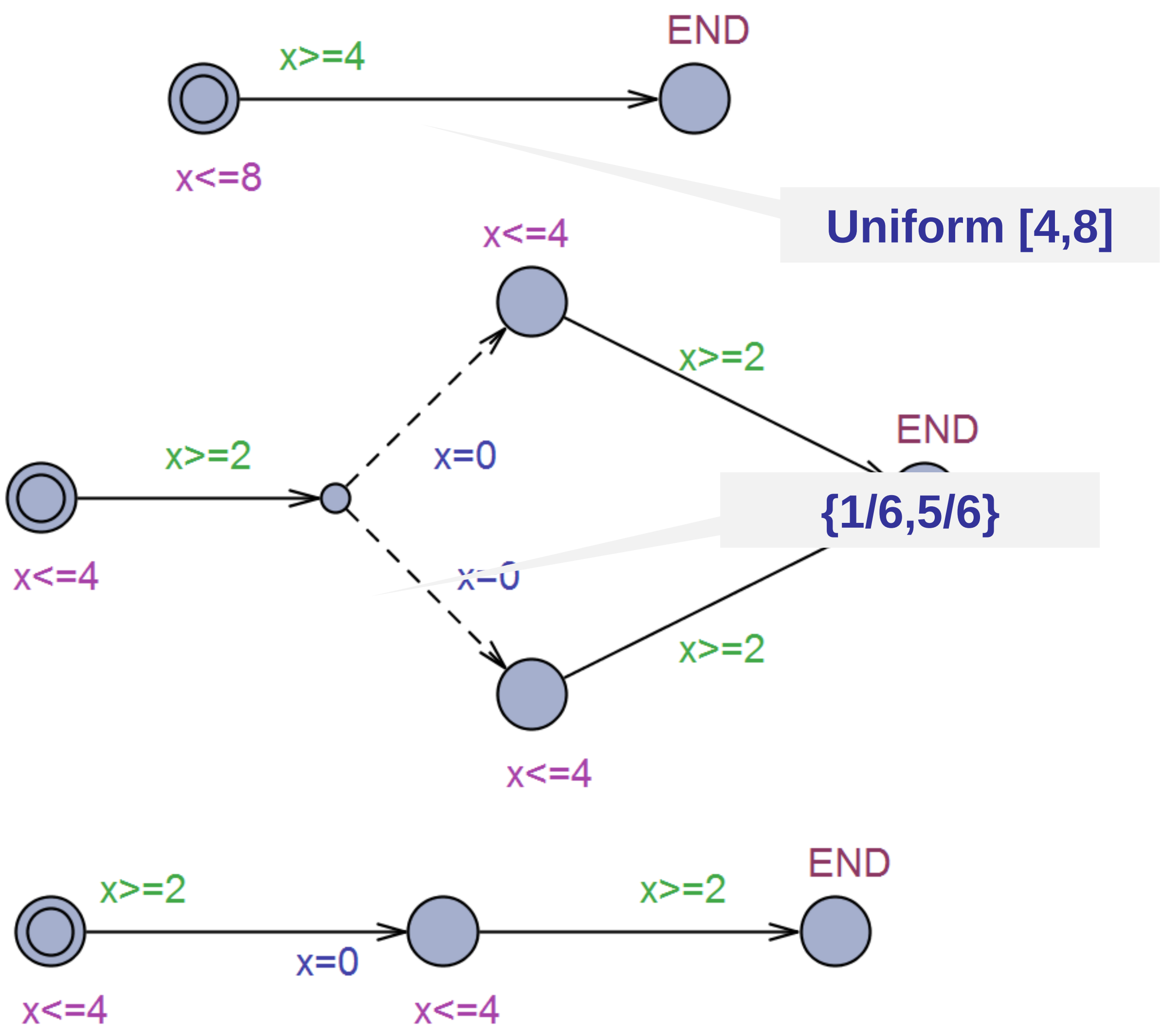
D. Chen, F. van Breugel, and J. Worrell. On the Complexity of Computing Probabilistic Bisimilarity. FOSSACS2012

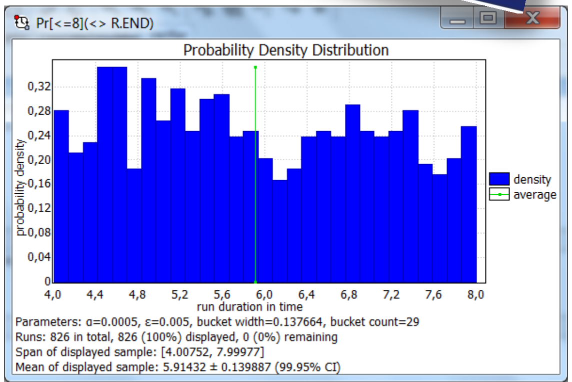
G. Bacci, G. Bacci, K. G. Larsen, and R. Mardare. On-the-Fly Exact Computation of Bisimilarity Distances. TACAS2013

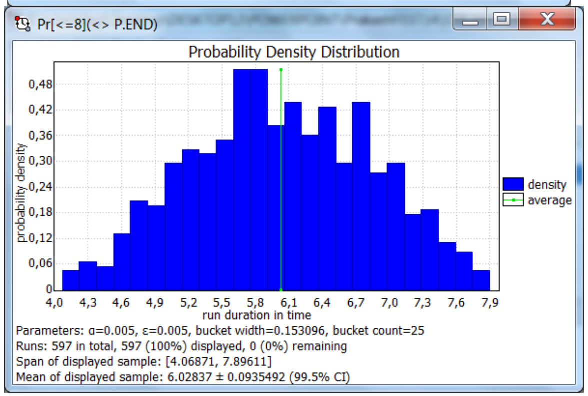
G. Bacci, G. Bacci, K. G. Larsen, and R. Mardare. Topologies of stochastic markov models: Computational aspects.

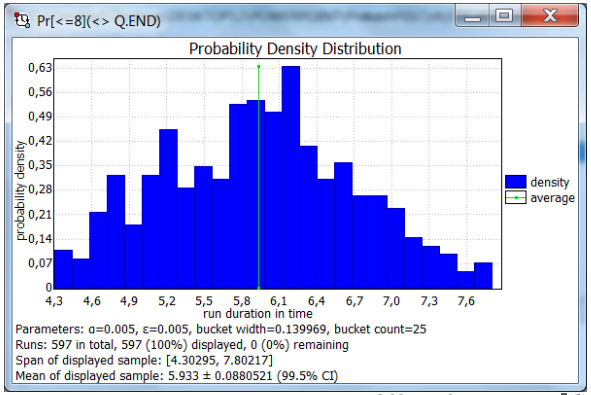
R. D'Argenio, Gebler, David Lee: Axiomatizing Bisimulation Equivalences and Metrics from Probabilistic SOS Rules. FoSSaCS 2014

Stochastic Timed Automata





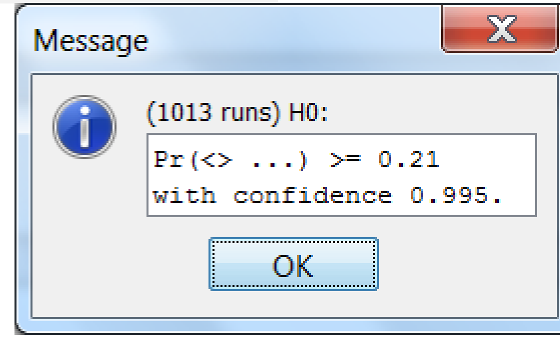


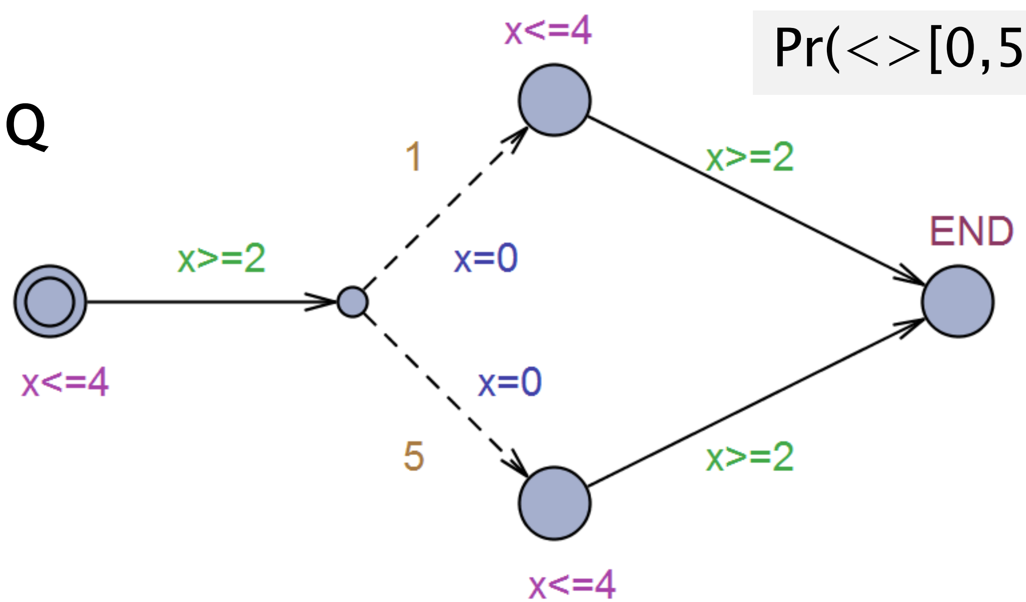


Kim Larsen [11]

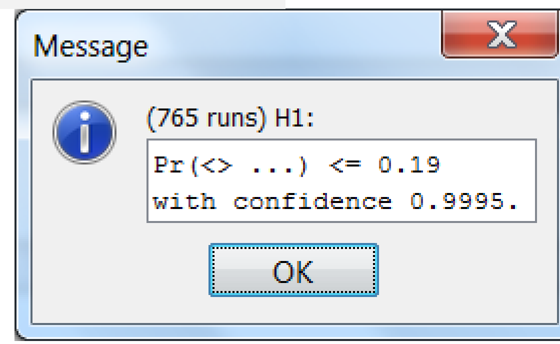
Statistical Model Checking

Pr(<>[0,5] R.END) >= 0.2

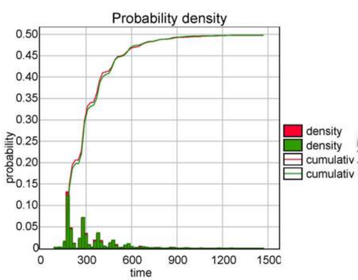


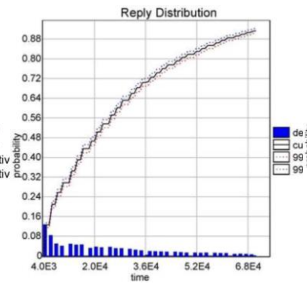


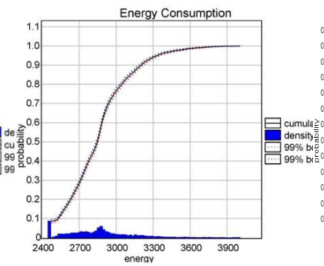
Pr(<>[0,5] Q.END) >= 0.2

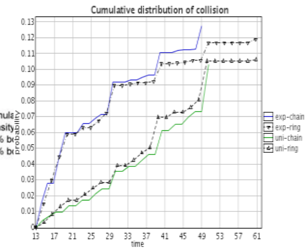


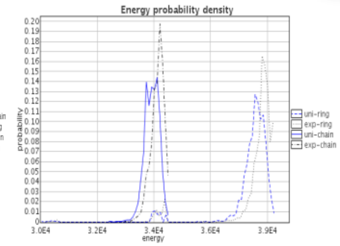
Statistical Model Checking

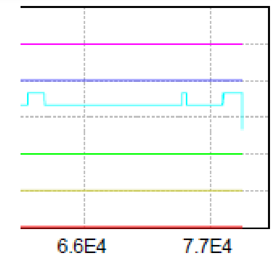










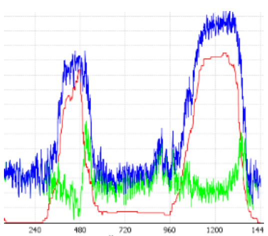


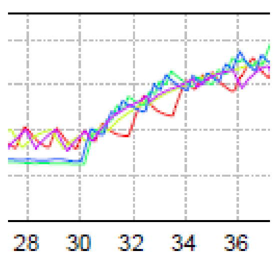
FIREWIRE

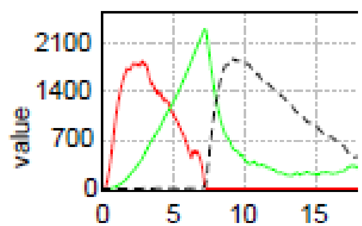
BLUETOOTH

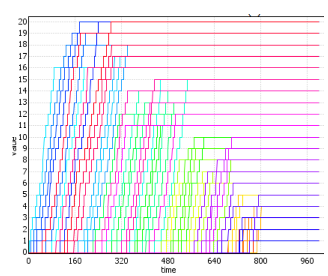
10 node LMAC

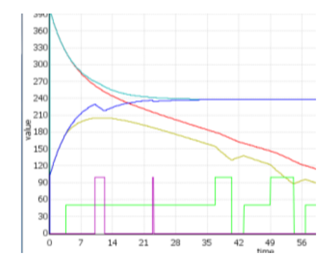
Schedulability Analysis for Mix Cr Sys











Smart Grid Demand / Response

Energy Aware Buildings

Genetic Oscilator (HBS)

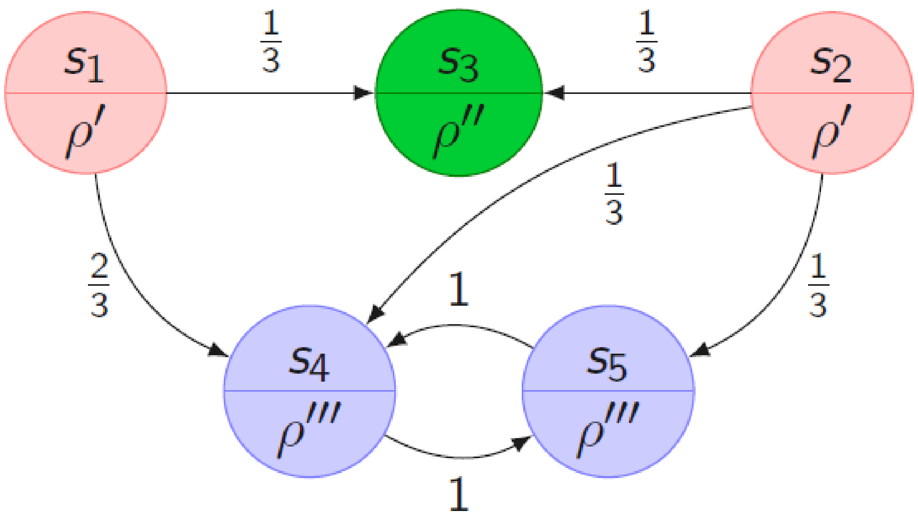
Passenger Seating in Aircraft

Battery Scheduling

Semi–Markov Processes

- S finite set of states
- $\tau\colon S \to \mathcal{D}(S)$ transition probability function
- $\rho\colon S \to \mathcal{D}(\mathbb{R}_+)$ residence-time distribution function
- $\ell\colon S \to 2^{AP}$ labeling function (AP atomic propositions)

$$\mathcal{M} = (S, \tau, \rho, \ell)$$



Probability Measure on Runs

timed path

cylinder set

$$s_0 \xrightarrow{t_0 \in R_0} \cdots \xrightarrow{t_n \in R_n} s_{n+1} \xrightarrow{t_{n+1}} \cdots \quad \in \quad \mathfrak{C}(s_0, R_0, \ldots, R_n, s_{n+1})$$

$$\mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0)) = \begin{cases} 1 & \text{if } s_0 = s \\ 0 & \text{otherwise} \end{cases}$$

delay probability

transition probability

$$\mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0, R_0, \ldots, R_n, s_{n+1})) =$$
$$\mathbb{P}_s^{\mathcal{M}}(\mathfrak{C}(s_0, R_0, \ldots, R_{n-1}, s_n)) \cdot \rho(s_n)(R_n) \cdot \tau(s_n)(s_{n+1})$$

MTL

MTL $\varphi ::= p \mid \bot \mid \varphi \to \varphi \mid X^{[t,t']}\varphi \mid \varphi\, U^{[t,t']}\, \varphi\,,$

timed next

timed until

timed path

$$\mathcal{M}, \pi \models \varphi$$

$\mathcal{M}, \pi \models p$	if $p \in \ell(\pi[0])$,
$\mathcal{M}, \pi \models \bot$	never ,
$\mathcal{M}, \pi \models \varphi \to \psi$	if $\mathcal{M}, \pi \models \psi$ whenever $\mathcal{M}, \pi \models \varphi$,
$\mathcal{M}, \pi \models X^{[t,t']}\varphi$	if $\pi\langle 0\rangle \in [t, t']$, and $\mathcal{M}, \pi\vert_1 \models \varphi$,
$\mathcal{M}, \pi \models \varphi\, U^{[t,t']}\, \psi$	if $\exists i > 0$ s.t. $\sum_{k=0}^{i-1} \pi\langle k\rangle \in [t, t']$, $\mathcal{M}, \pi\vert_i \models \psi$, and $\mathcal{M}, \pi\vert_j \models \varphi$ whenever $0 \le j < i$.

accumulated delay

Variational Pseudo Metrics

Lemma

$\varphi \in MTL \implies [\![\varphi]\!]_{\mathcal{M}} = \{\pi \mid \mathcal{M}, \pi \models \varphi\}$ *is measurable*

$$\delta^{\mathcal{M}}_{\mathrm{MTL}}(s, s') = \sup_{\varphi \in \mathrm{MTL}} |\mathbb{P}^{\mathcal{M}}_{s}([\![\varphi]\!]_{\mathcal{M}}) - \mathbb{P}^{\mathcal{M}}_{s'}([\![\varphi]\!]_{\mathcal{M}})|$$

Lemma

$\mathcal{A} \in DTA \implies [\![\mathcal{A}]\!]_{\mathcal{M}} = \{\pi \mid \pi \in \mathcal{L}(\mathcal{A})\}$ *is measurable*

$$\delta^{\mathcal{M}}_{\mathrm{DTA}}(s, s') = \sup_{\mathcal{A} \in \mathrm{DTA}} |\mathbb{P}^{\mathcal{M}}_{s}([\![\mathcal{A}]\!]_{\mathcal{M}}) - \mathbb{P}^{\mathcal{M}}_{s'}([\![\mathcal{A}]\!]_{\mathcal{M}})|$$

NP-hard !!
Decidable ??

Theorem (MTL and DTA variational distances coincide)

$$\delta^{\mathcal{M}}_{MTL} = \delta^{\mathcal{M}}_{DTA}$$

Taolue Chen and Stefan Kiefer. On the Total Variation Distance of Labelled Markov Chains . LICS2014

G. Bacci, G. Bacci, K. G. Larsen, and R. Mardare. Topologies of stochastic markov models: Computational aspects.

1–RDTA & Denseness

1-RDTA — single-clock resetting DTAs

Lemma (1-RDTAs are $\mathbb{P}^{\mathcal{M}}$-dense in DTAs)

For any $\mathcal{A} \in$ DTA and any $\varepsilon > 0$, there exists $\mathcal{B} \in$ 1-RDTA s.t.

$$|\mathbb{P}_s^{\mathcal{M}}([\![\mathcal{A}]\!]_{\mathcal{M}}) - \mathbb{P}_s^{\mathcal{M}}([\![\mathcal{B}]\!]_{\mathcal{M}})| < \varepsilon$$

Let $\delta^{\mathcal{M}}_{\text{1-RDTA}}(s, s') = \sup_{\mathcal{B} \in \text{1-RDTA}} |\mathbb{P}_s^{\mathcal{M}}([\![\mathcal{B}]\!]_{\mathcal{M}}) - \mathbb{P}_{s'}^{\mathcal{M}}([\![\mathcal{B}]\!]_{\mathcal{M}})|$, then

Theorem

$$\delta^{\mathcal{M}}_{DTA} = \delta^{\mathcal{M}}_{1\text{-}RDTA}$$

Enable approximate MC of MTL for CTMC

Chen, Han, Katoen, Mereacre: Model Checking of Continuous-Time Markov Chains Against Timed Automata Specifications. LMCS 2011

Computable Upper Bound

$$F^{\mathcal{M}}(d)(s,s') = \begin{cases} 1 & \text{if } \ell(s) \neq \ell(s') \\ \alpha + (1-\alpha)\cdot K(d)(\tau(s),\tau(s')) & \text{otherwise} \end{cases}$$

prob. same delay time

prob. different delay time $\|\rho(s) - \rho(s')\|_{TV}$

Kantorovich distance w.r.t. d

Bisimilarity Distance

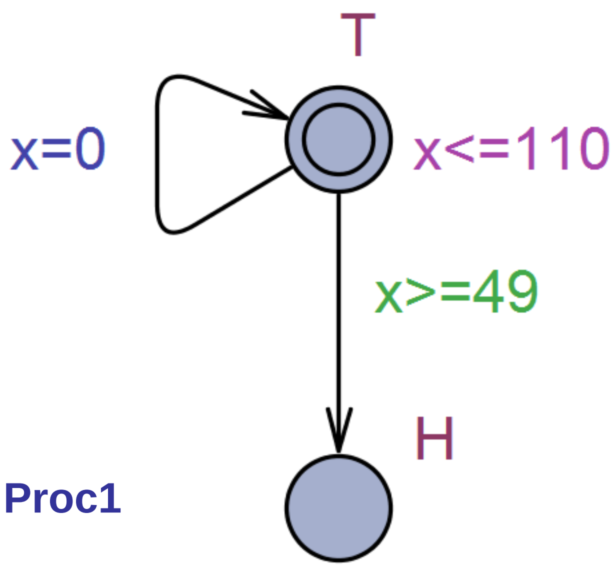
$d^{\mathcal{M}}_{bisim} \colon S \times S \to [0,1]$ is the least fixed point of $F^{\mathcal{M}}$

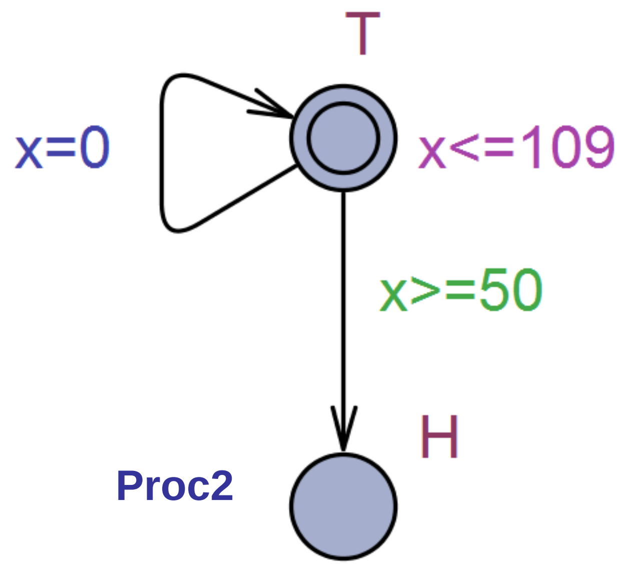
Theorem (Upper bound)

$$\delta^{\mathcal{M}}(s,s') \leq d^{\mathcal{M}}_{bisim}(s,s')$$

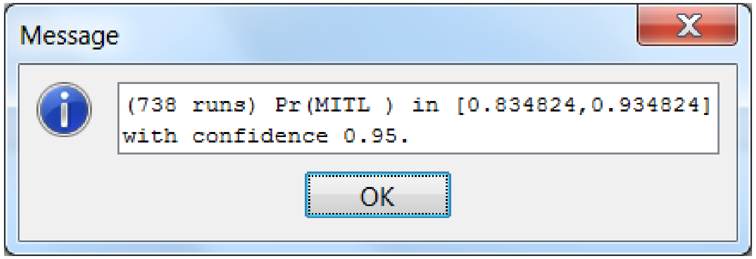
Polynomial

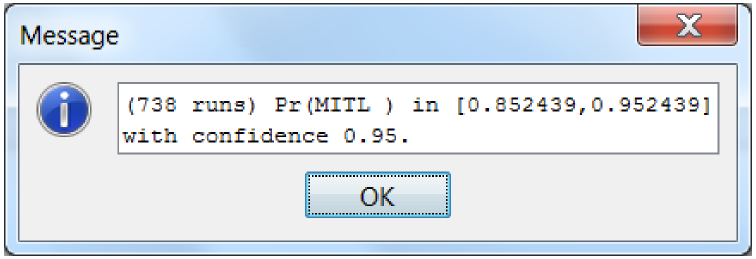
Timed Coins – STA



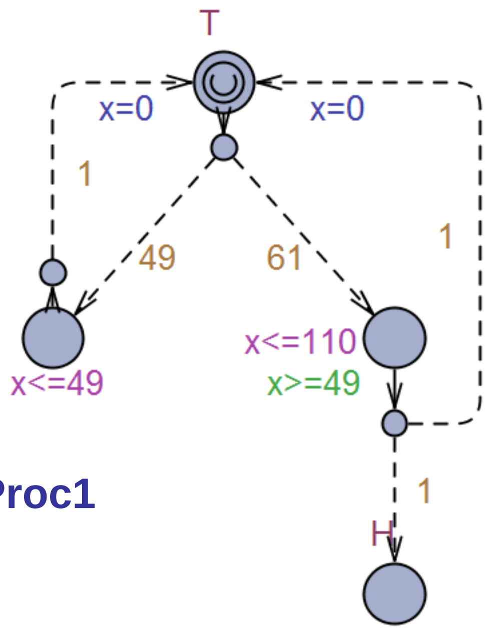


Pr(Proci.T U[5;400] Proci.H)

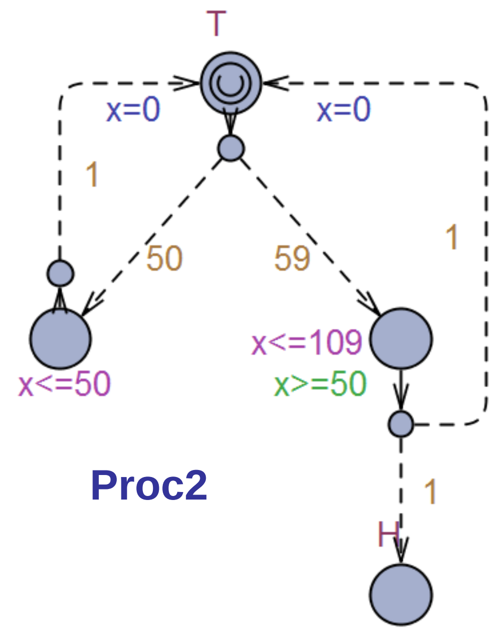




Timed Coins – SMP

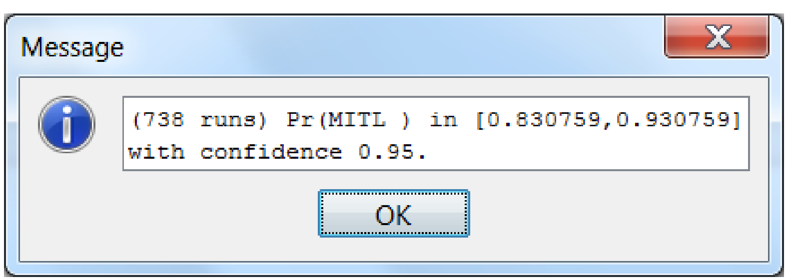


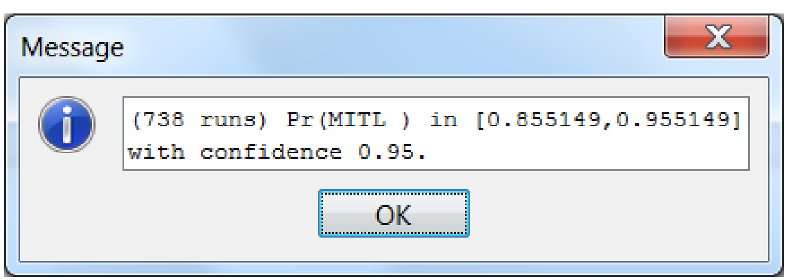
Proc1



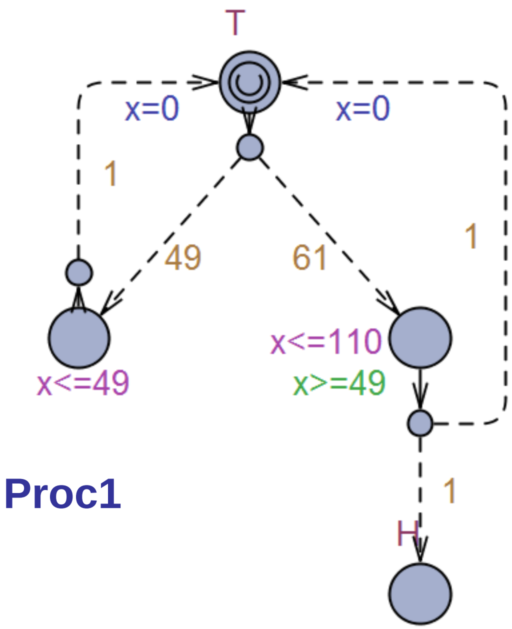
Proc2

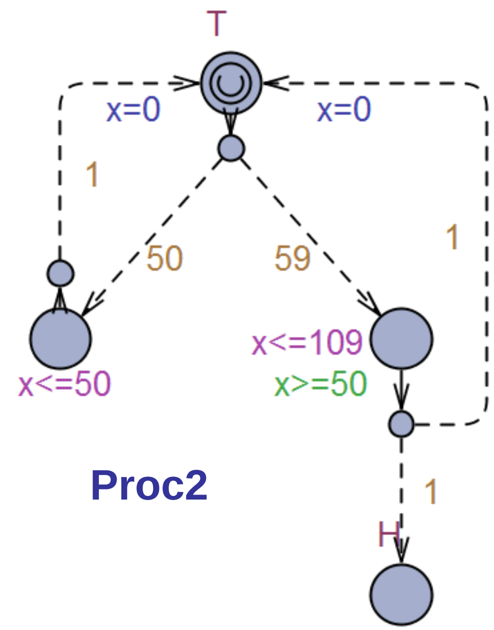
Pr(Proci.T U[5;400] Proci.H)





Timed Coins – SMP





$$||U(49,110)(E) - U(50,109)(E)||_E = \frac{2}{61}$$

$$\boldsymbol{d(T_1, T_2) = 0.132}$$

BisimDist

BisimDist is a Mathematica® library that provides two packages:

MCDist MDPDist CTMCDist

- Data structures (model definition)
- Data structure manipulators & visualizers
- Procedure for computing bisimilarity distances (on-the-fly!)
  - approximated methods (from known upper-bounds)
  - future-discount
- bisimilarity classes / quotient by bisimilarity

Library + Tutorials

http://people.cs.aau.dk/giovbacci/tools.html

Open Problems

- Decidability of **Total Variational MTL distance** ?
- How big is the gap between TV distance and bisimulation distance ?
- **Denseness of CTMC** (1-RDTA) in the class of Semi Markov Process with respect to bisimulation distance ?
- Extension to **Generalised Semi Markov Processes**
- **Weak** bisimulation distances
- **Parameter continuity**, i.e. $d(s(p_1), s(p_2)) \to 0$ when $p_1 \to p_2$?