Indistinguishability Theory

Ueli Maurer

ETH Zurich

FOSAD 2009, Bertinoro, Sept. 2009.

Distinguishing two objects:

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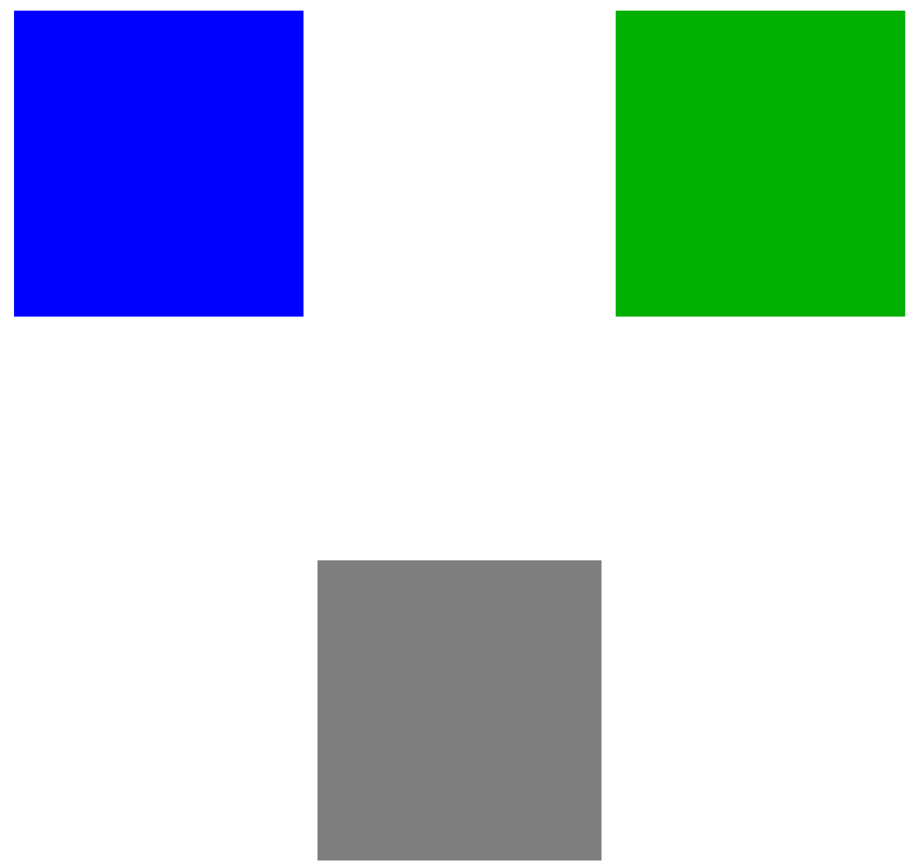

left or right?

Distinguishing two types of numbers

Set A:

2048-bit integers with exactly **2** prime factors, each with at least 512 bits.

Set B:

2048-bit integers with exactly **3** prime factors, each with at least 512 bits.

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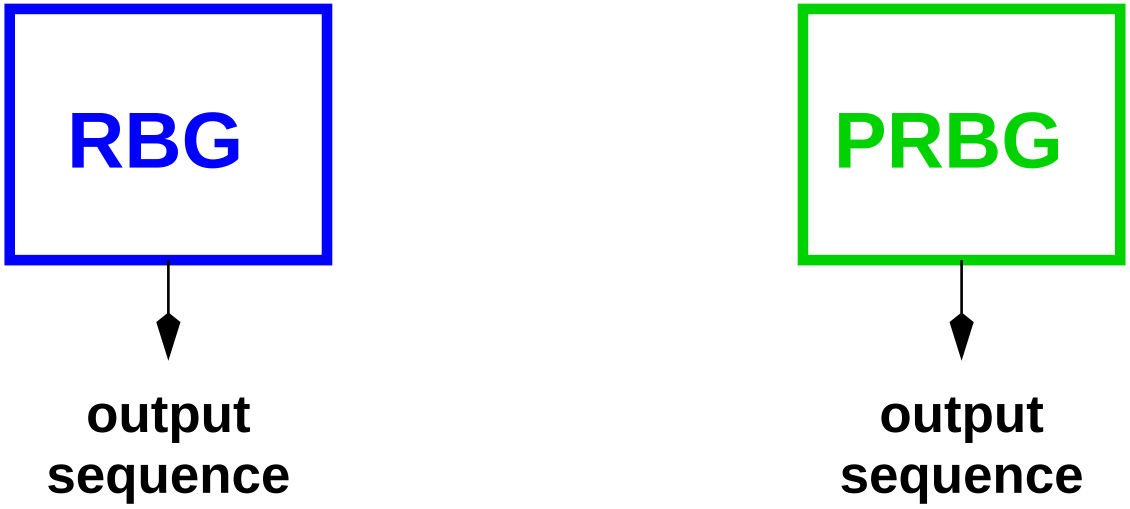
3740957629745118733980567439817539577832546738459678253645098713652955848823336449857660918528256405016387598795387626354856782430914257652536485263740991252317647489855766009633273939475861234987505334958620549877465243510897583932183674432789687645343127364987564354675092736565475849823142537584950243685261

left or right?

Random vs. pseudo-random bit generator



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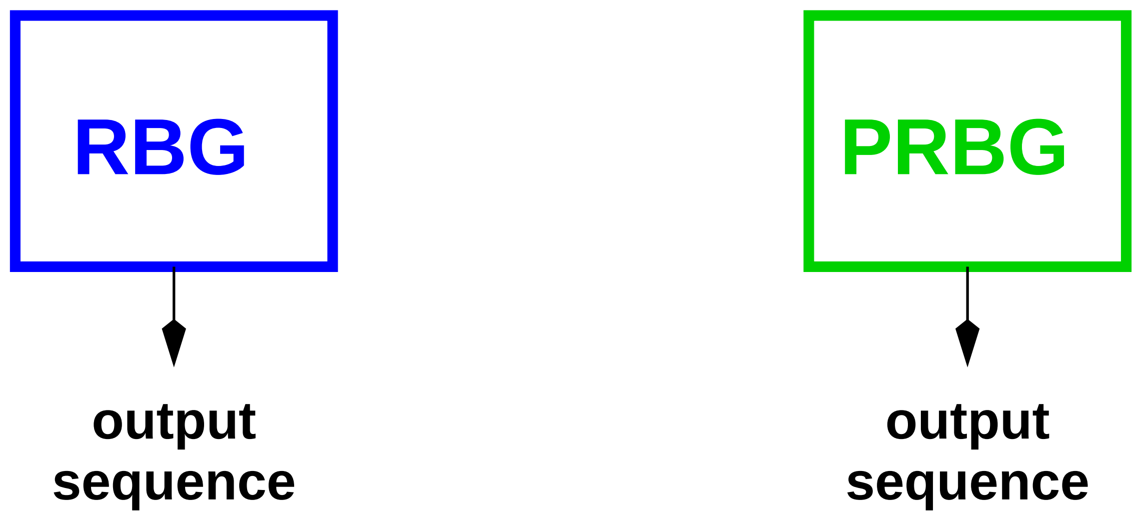

101100011101111001001110100010000011101100101110010111010001101
0000110110101111010100011010110101001001010111101010000001101101
111000111011000101111010010101101001010110000101011010101101001
110011001001100010110100011100101010001011010100001111000101010

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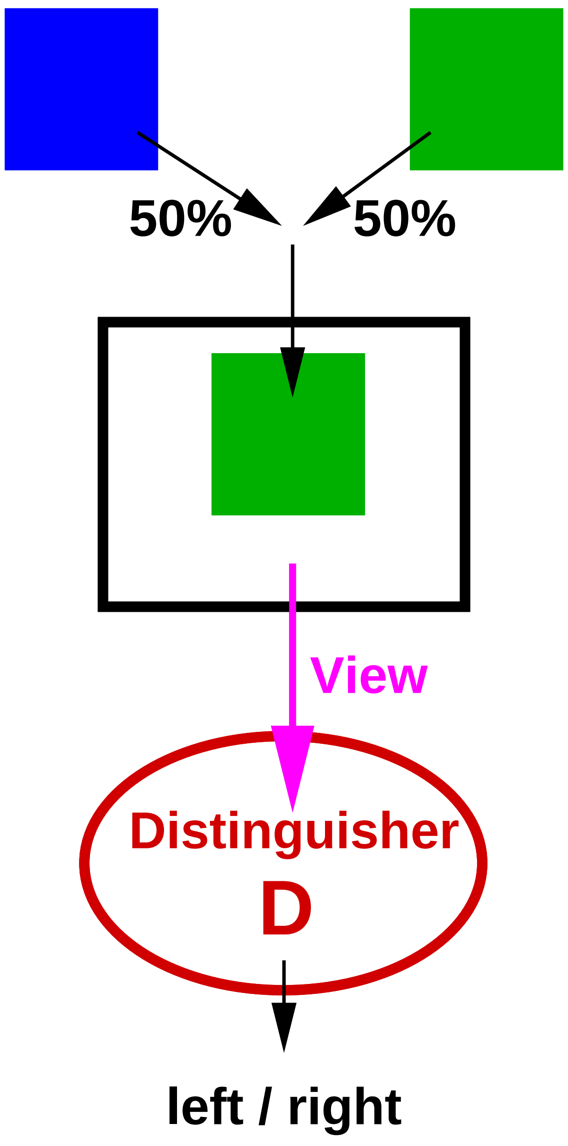
Distinguisher's advantage



D's task: Guess left/right

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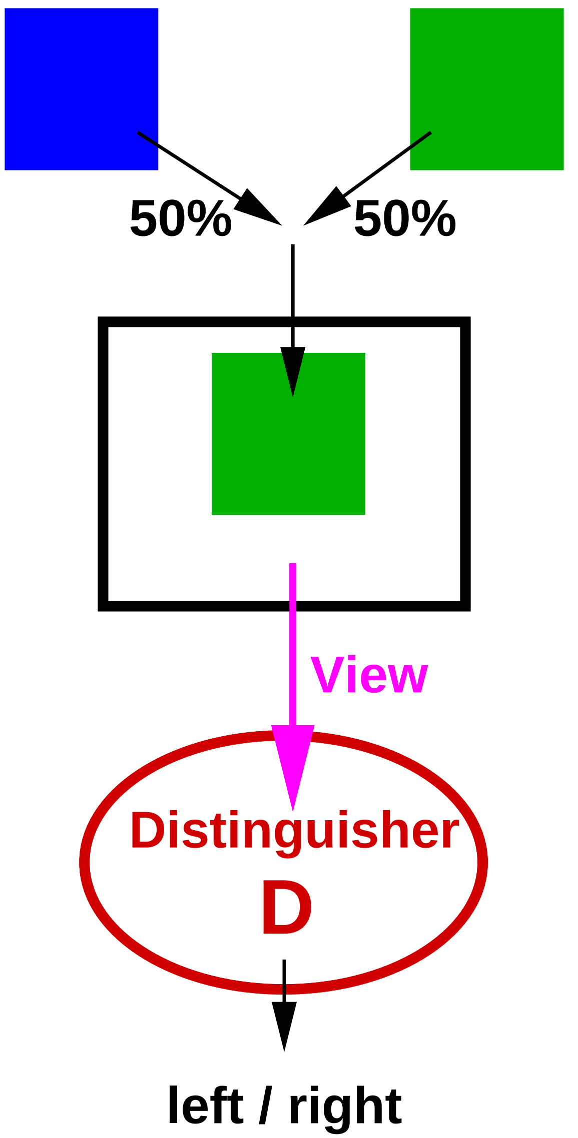

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Prob(correct guess) = 0.5 + α**/2**

$\alpha = \Delta^{D}(\blacksquare, \blacksquare)$ **(D's advantage)**

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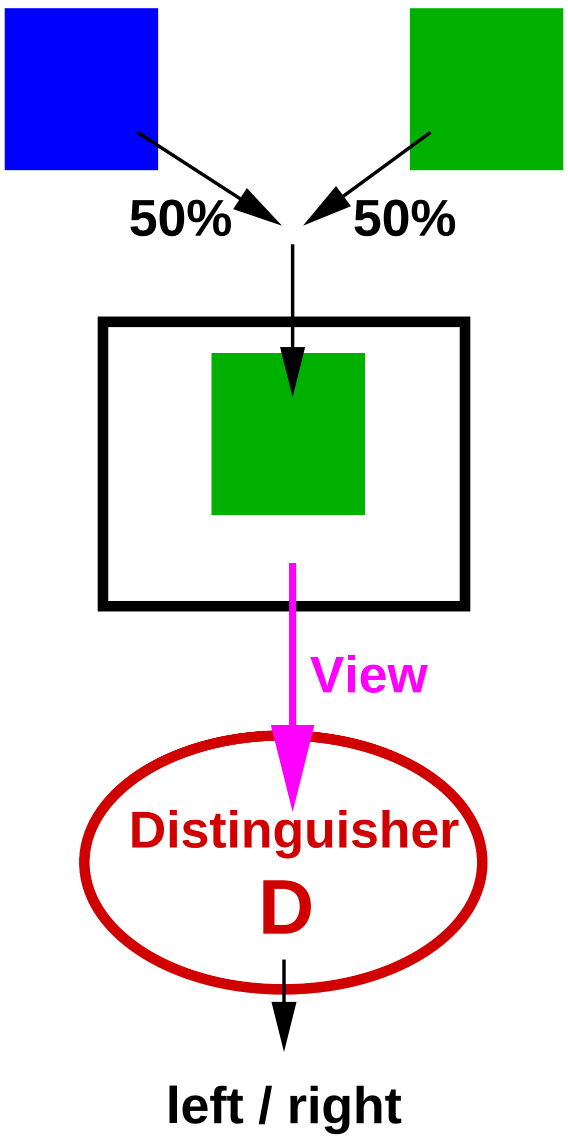

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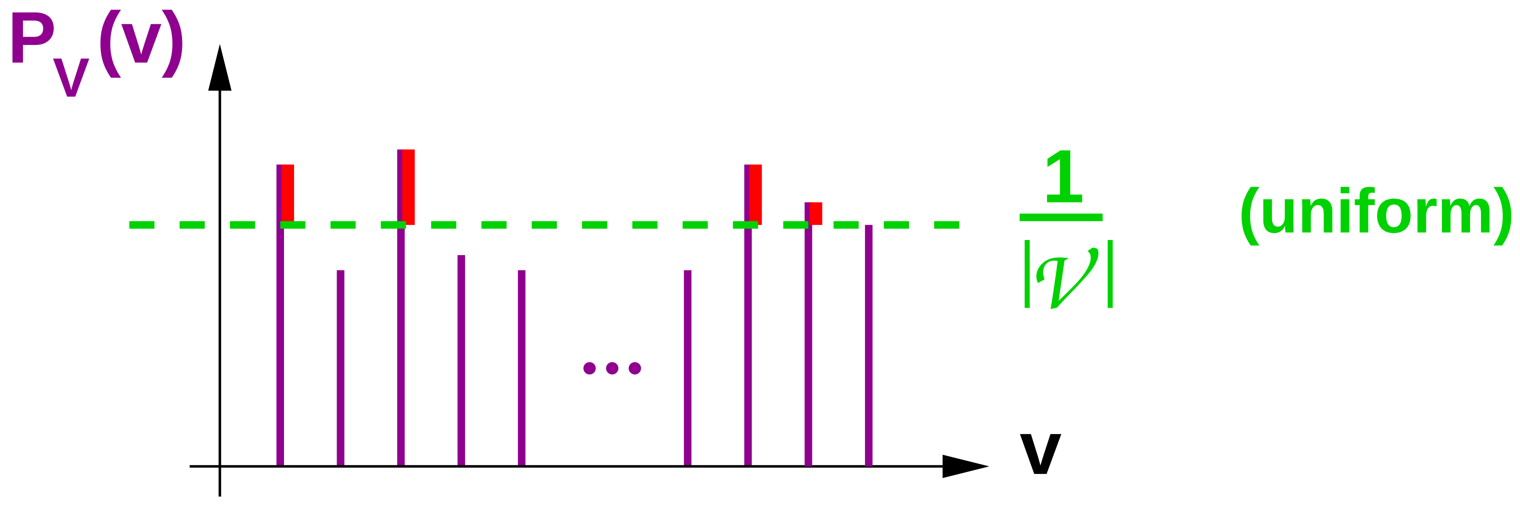
best D: $\Delta(■, ■)$

Distinguishing a RV V from a uniform RV U



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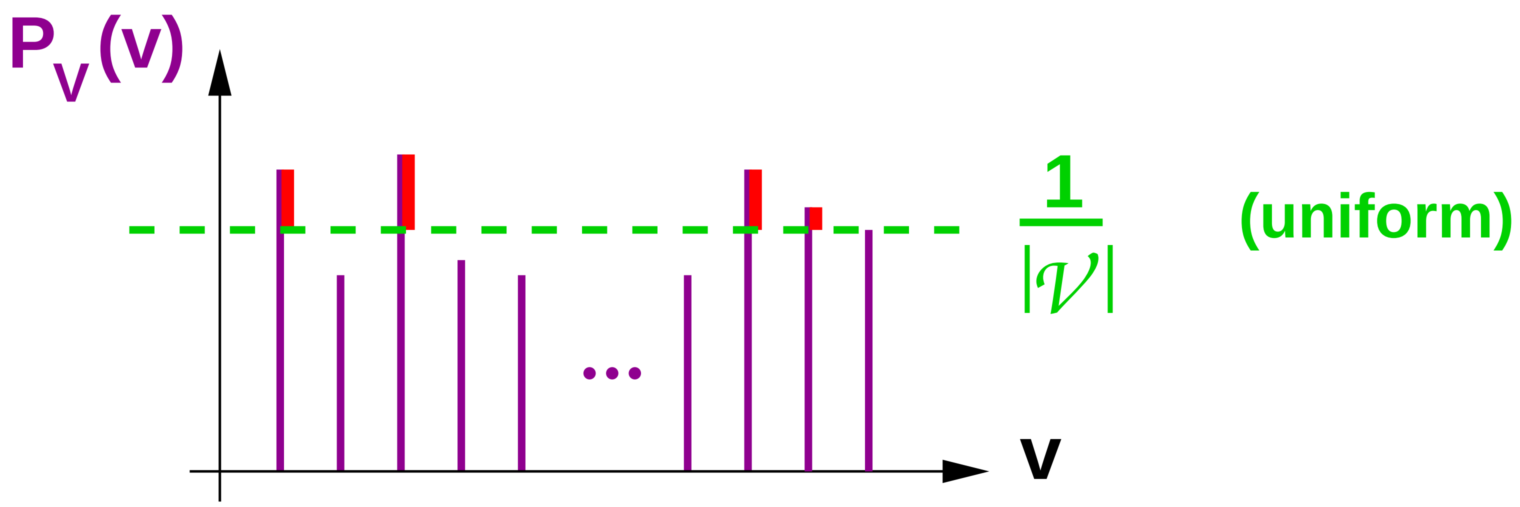

Statistical distance:

$$\mathbf{d}(\mathbf{V},\mathbf{U}) := \frac{1}{2}\sum_{v\in\mathcal{V}}\left|\mathbf{P_V}(v) - \frac{1}{|\mathcal{V}|}\right|$$ **(sum of red quantities)**

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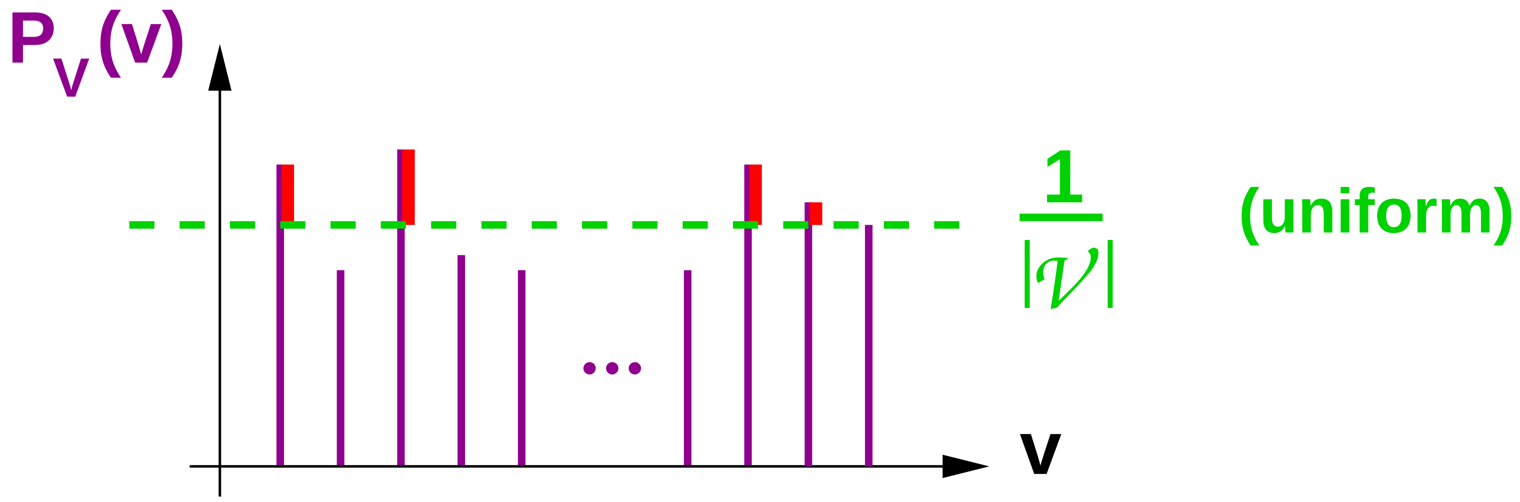

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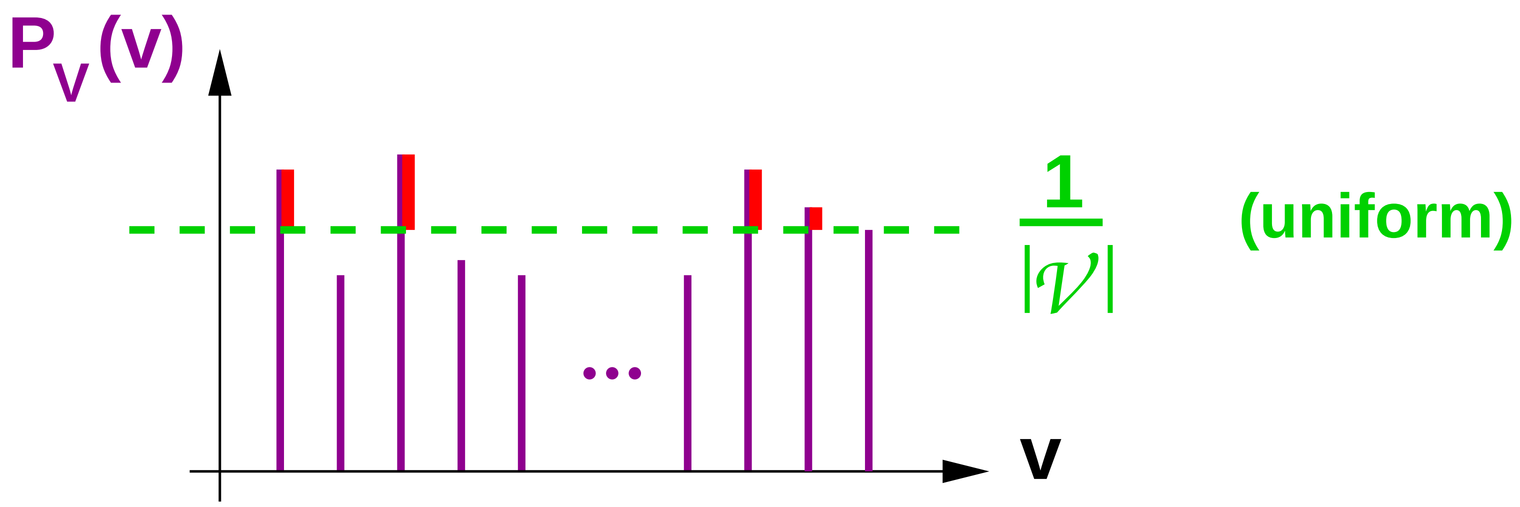

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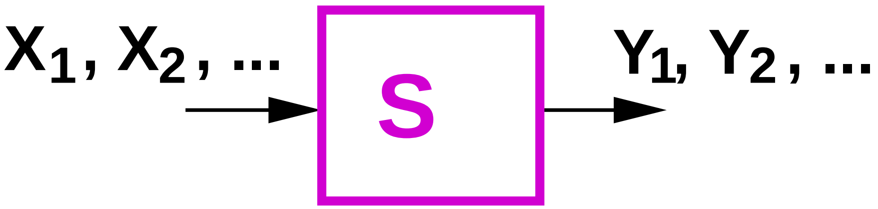
Possible interpretation: $P(\mathbf{V}=\mathbf{U}) \;=\; 1 - \mathbf{d}(\mathbf{V},\mathbf{U})$

Discrete systems



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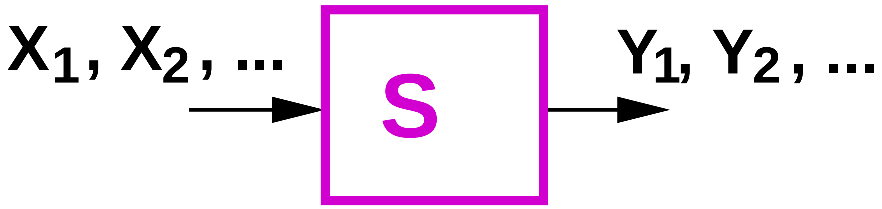
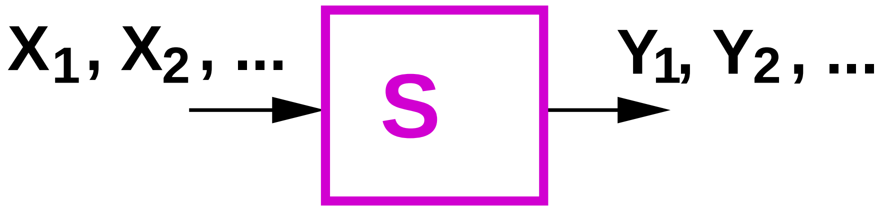


Description of S: pseudo-code, figures, text, ...

Discrete systems



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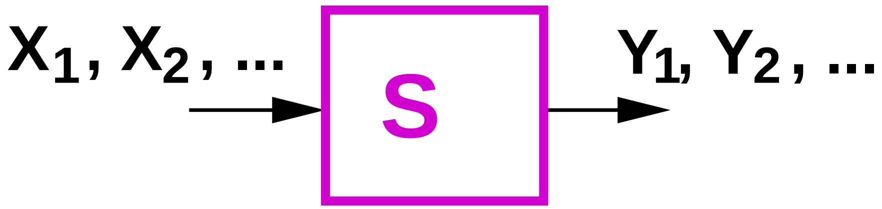

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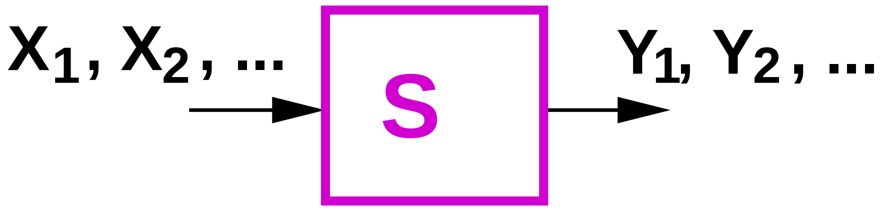

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- Characterized by: $p^{S}_{Y_i|X_1 \cdots X_i Y_1 \cdots Y_{i-1}}$ for $i = 1, 2, \ldots$

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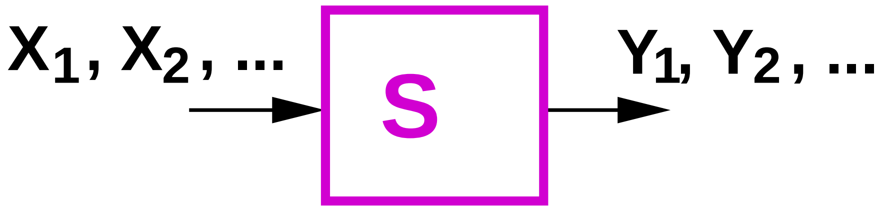

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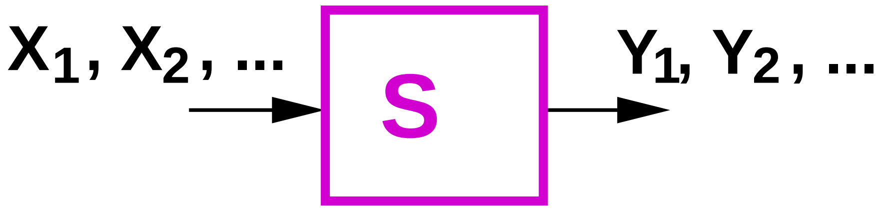

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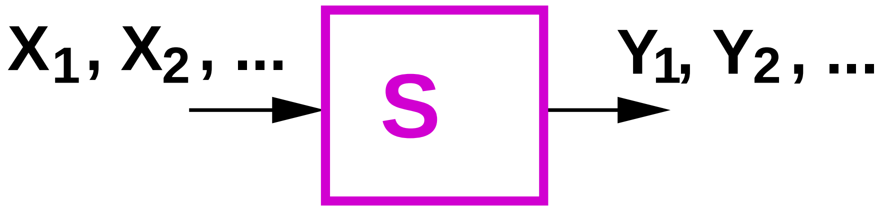

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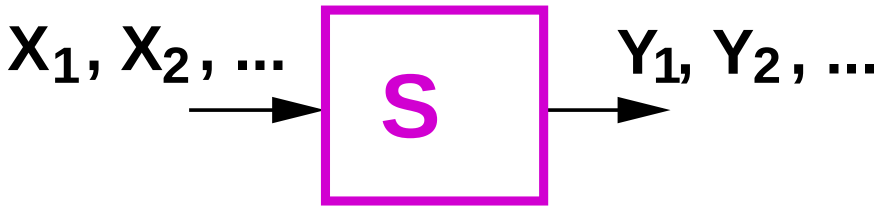
Realization of S from a RV (range $\mathcal{R}$**):** $f_i^{\mathbf{S}} : \mathcal{X}^i \times \mathcal{R} \rightarrow \mathcal{Y}$

Discrete systems



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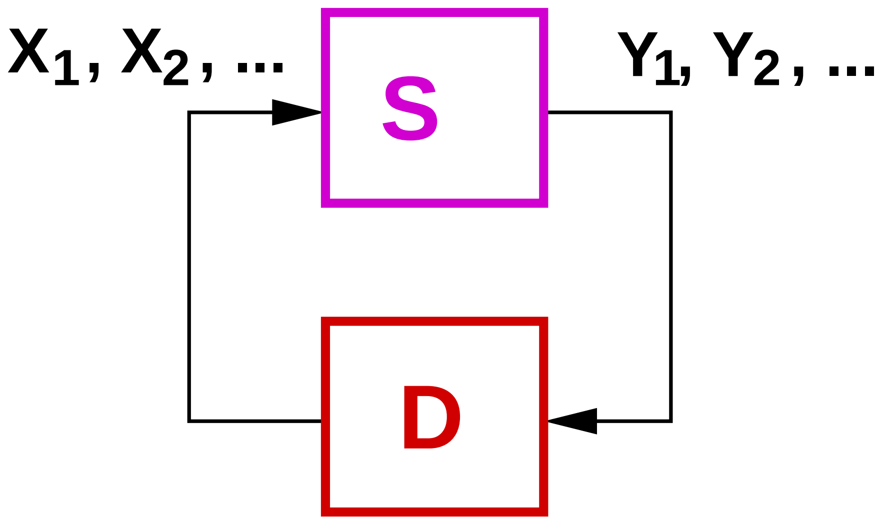
$\rightarrow$ **notion of independence**

Distinguishers



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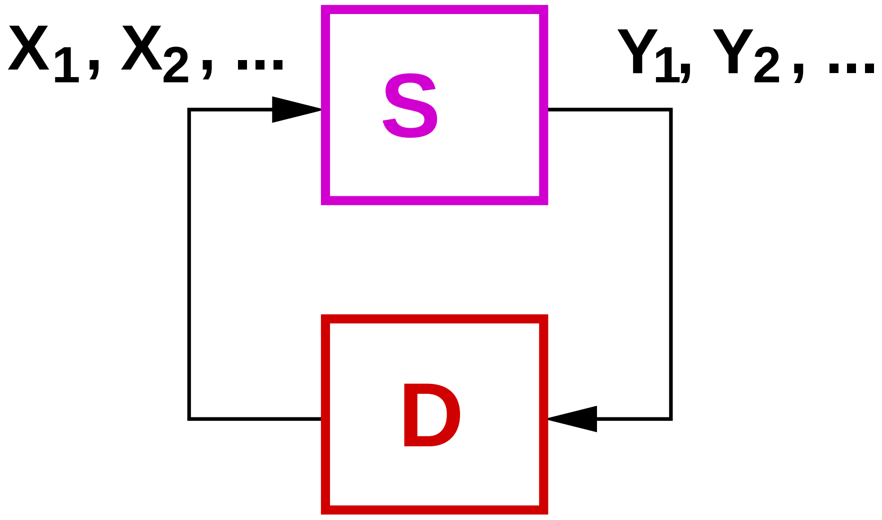

$$P^{DS}_{X^k Y^k} = \prod_{i=1}^{k} p^{S}_{Y_i|X^i Y^{i-1}} \cdot p^{D}_{X_i|X^{i-1} Y^{i-1}}$$

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notation: $X^i = (X_1, \ldots, X_i)$

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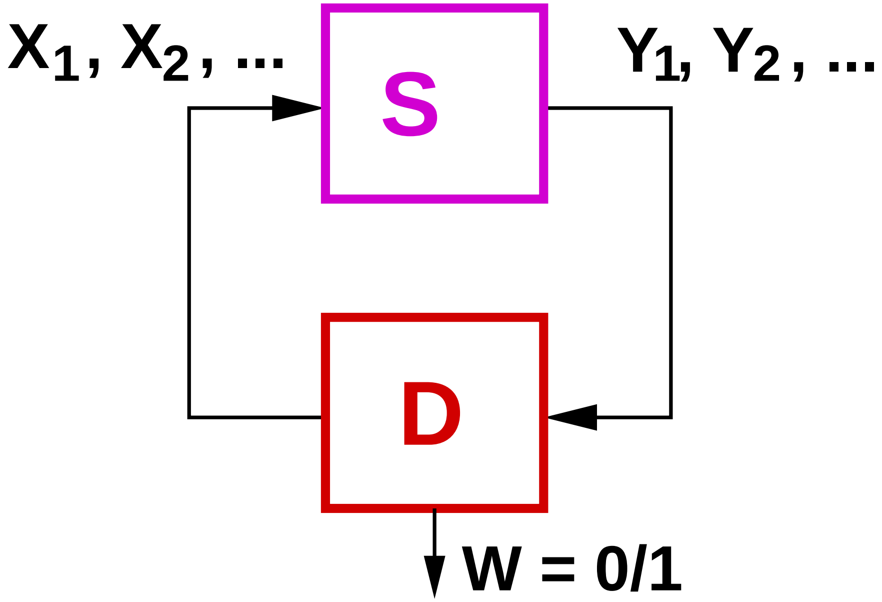

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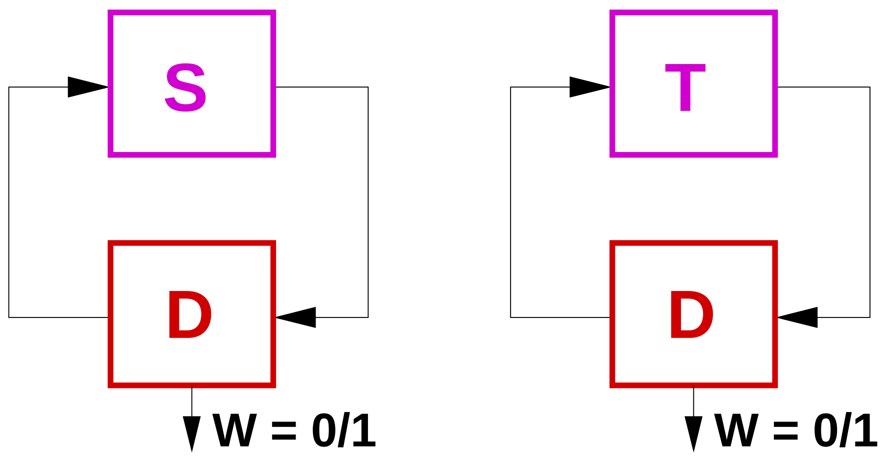
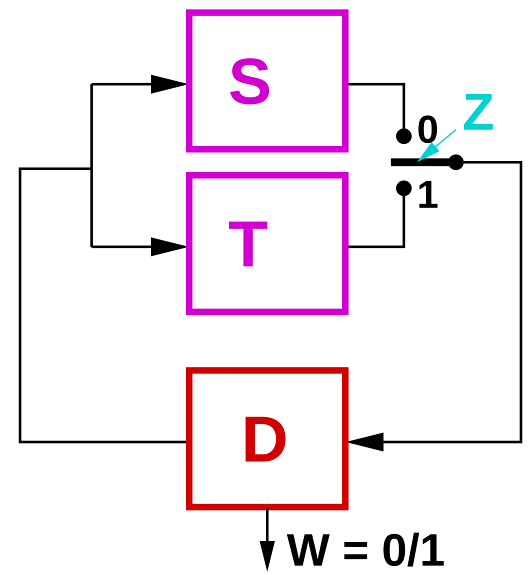
Distinguishing advantage

2 equivalent views:





$$\Delta_k^{\mathbf{D}}(\mathbf{S},\mathbf{T}) \;:=\; \left|\mathbf{P}^{\mathbf{DS}}(\mathbf{W}=1) - \mathbf{P}^{\mathbf{DT}}(\mathbf{W}=1)\right|$$

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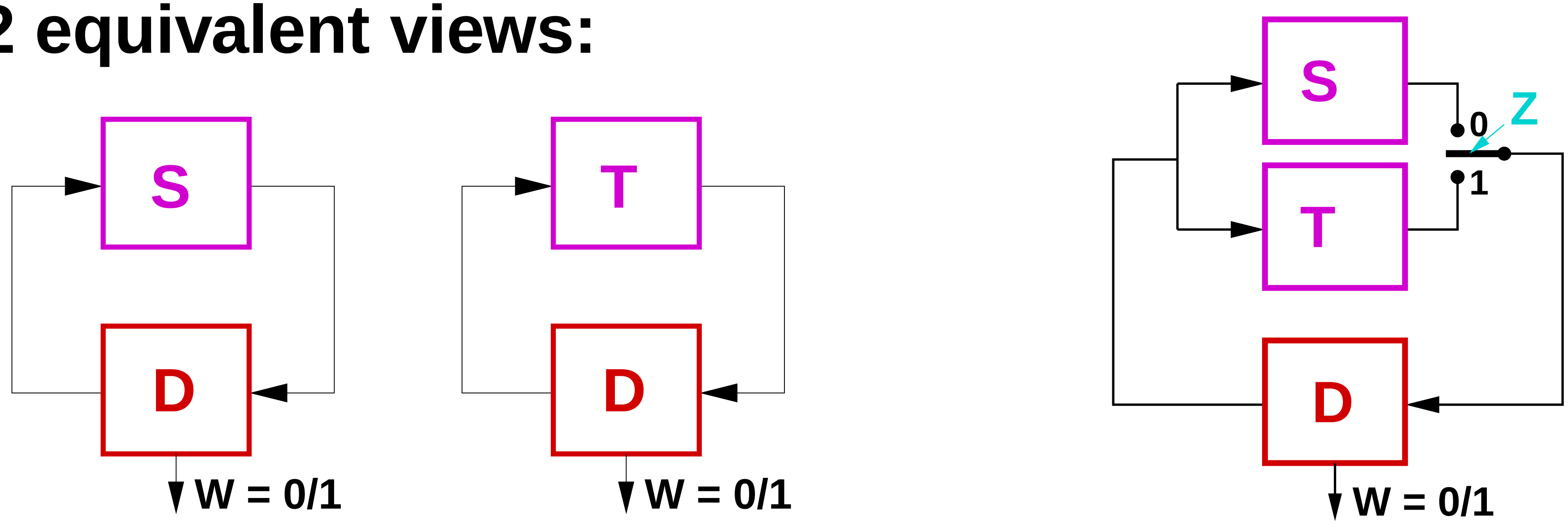

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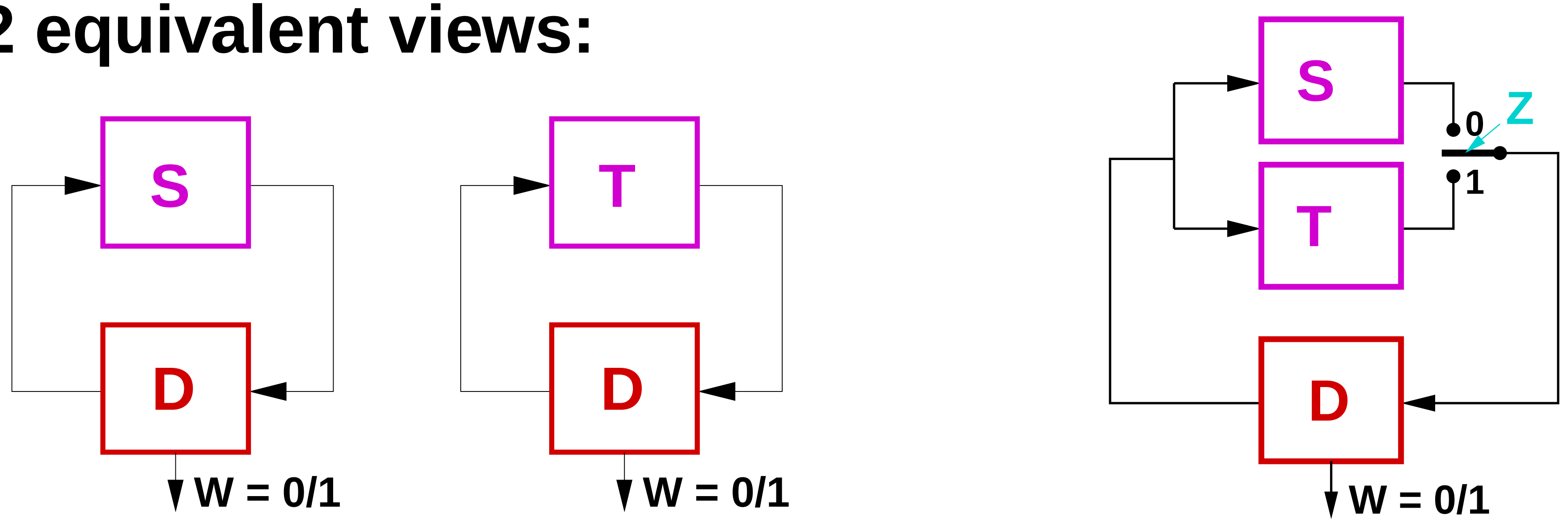

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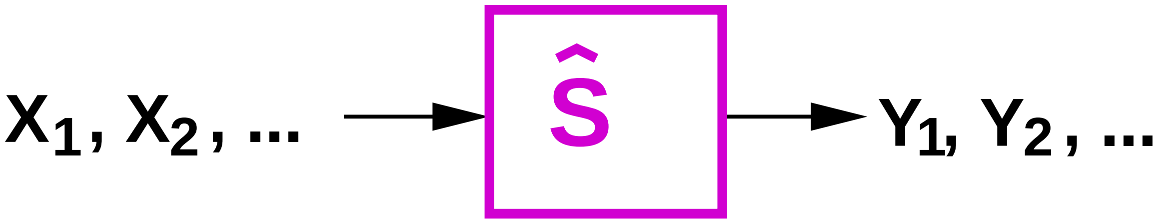
Game-winning



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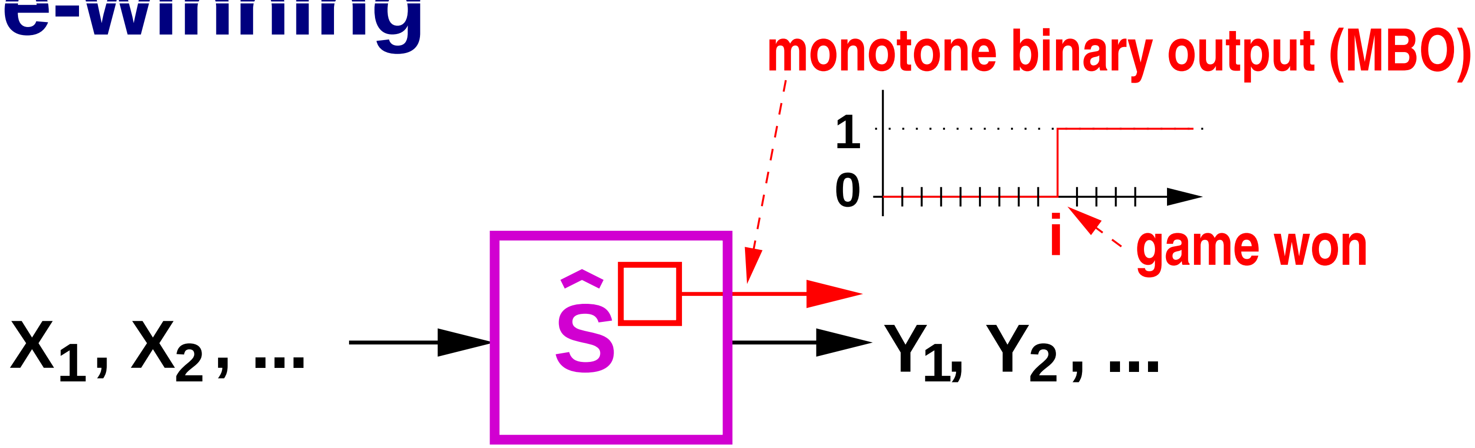

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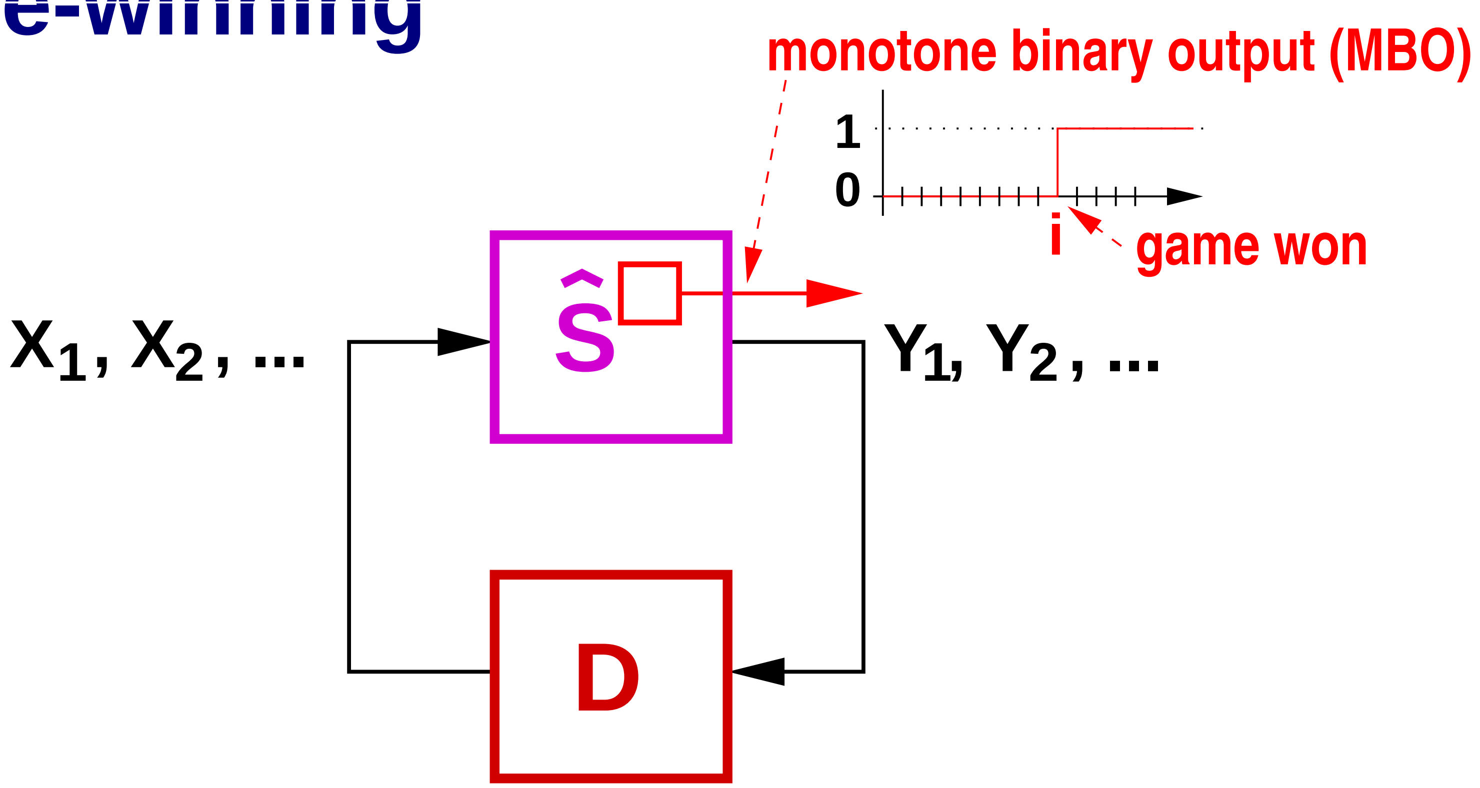

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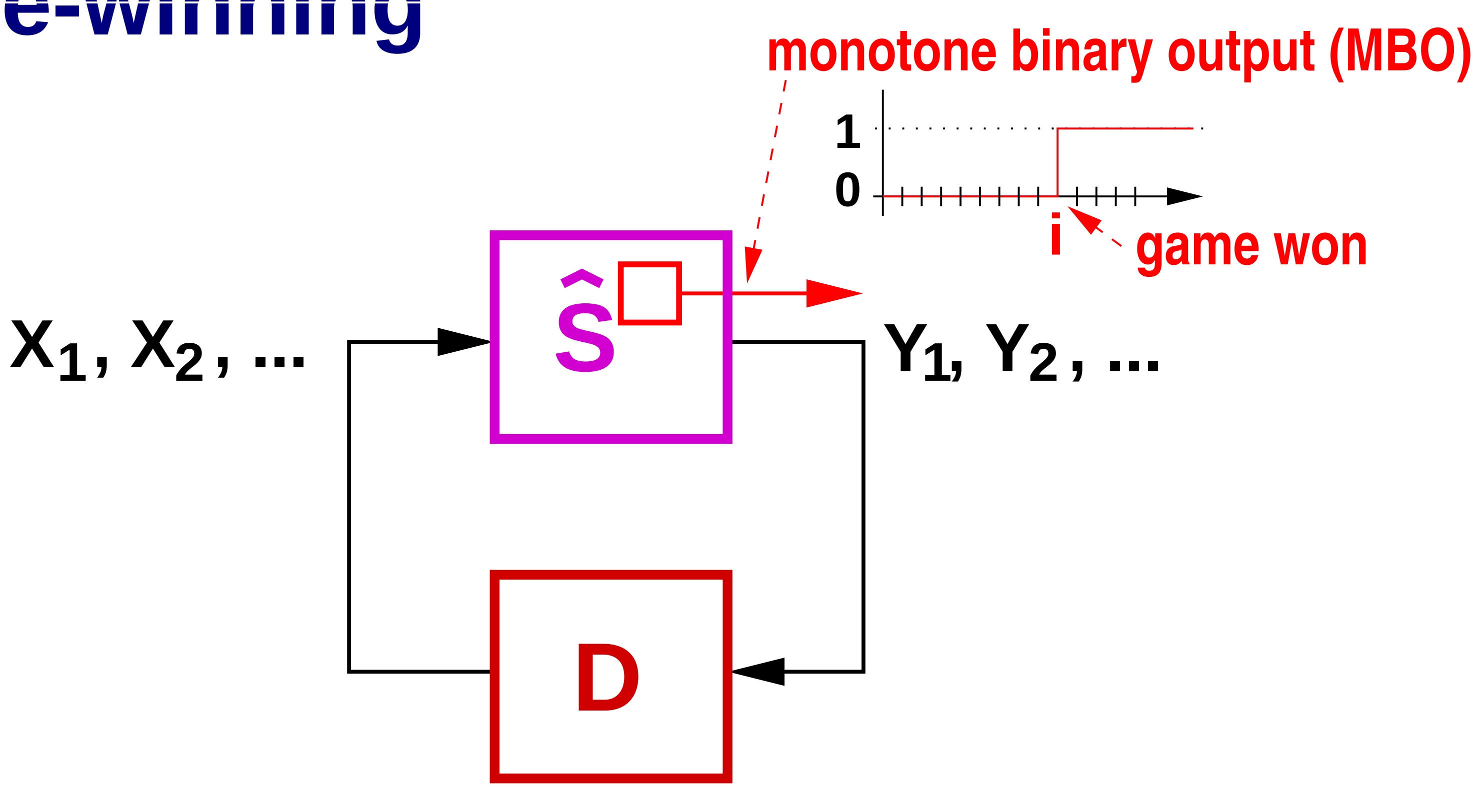

D's prob. of winning with k queries: $\nu_k^{\mathbf{D}}(\hat{\mathbf{S}})$

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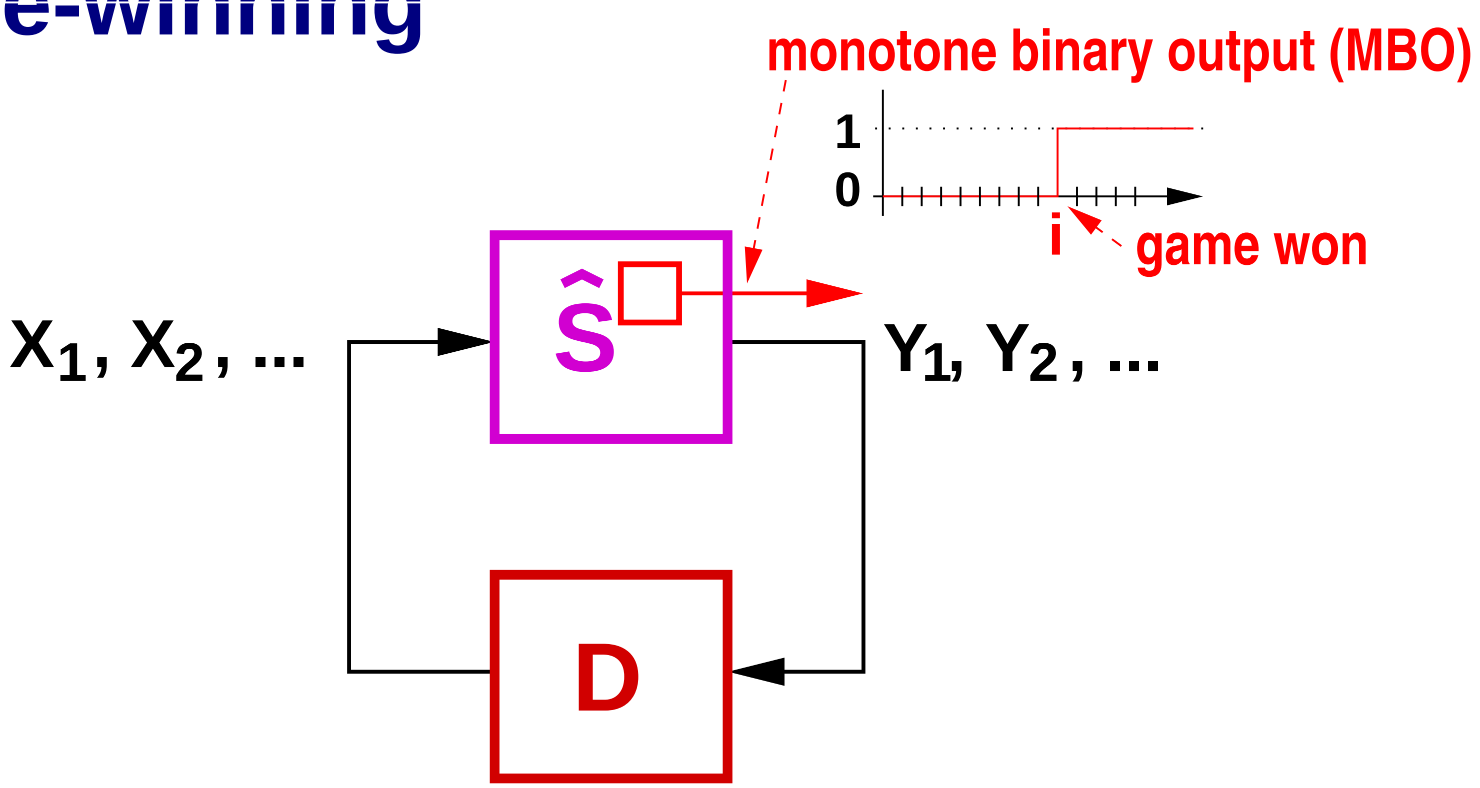

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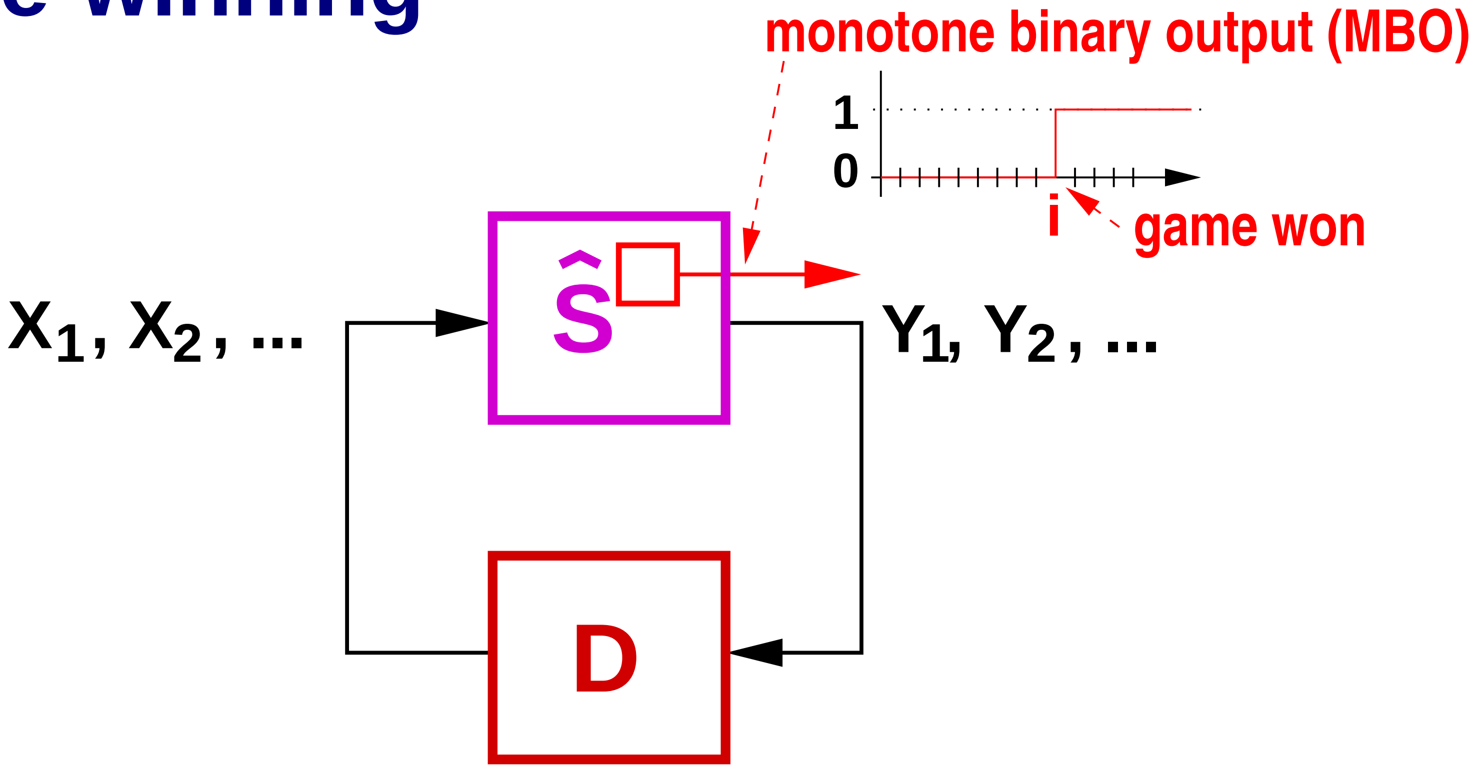

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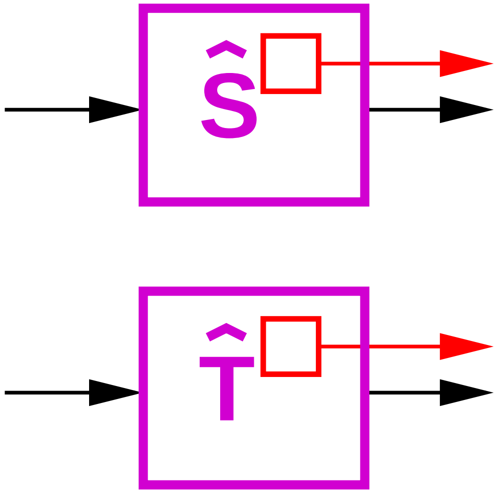
Playing 2 games in parallel



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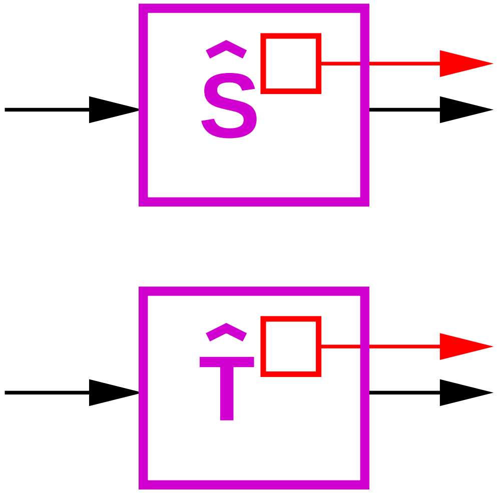

Can a combined strategy be better than optimal individual strategies?

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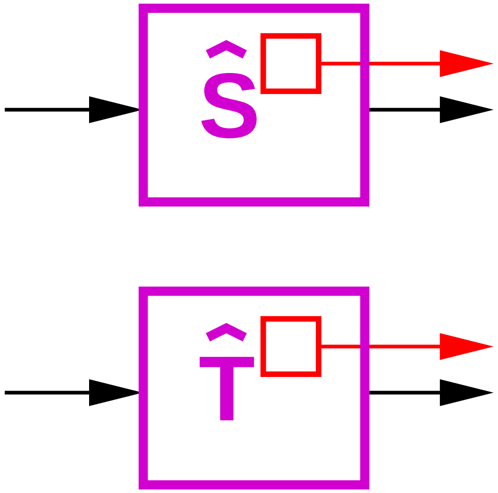

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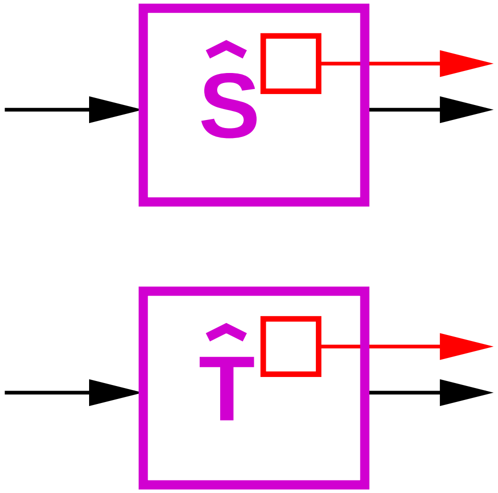

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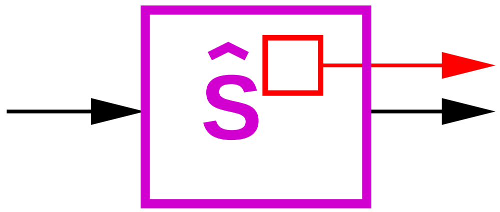
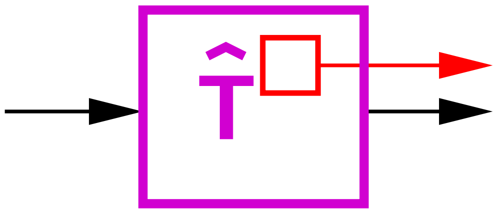
Lemma [MPR07]: For winning both games, playing individual optimal strategies is optimal.

Game-winning $\Longleftrightarrow$ Distinguishing





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Def.: Ŝ and T̂ are restricted equivalent, denoted Ŝ $\stackrel{r}{\equiv}$ T̂, if the I/O behavior is identical as long as MBO =0.

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Def.: $\hat{S}$ and $\hat{T}$ are **restricted equivalent**, denoted $\hat{S} \stackrel{r}{\equiv} \hat{T}$, if the I/O behavior is identical as long as MBO $=0$.

Lemma ($\Rightarrow$) [Mau02]: If $\hat{S} \stackrel{r}{\equiv} \hat{T}$, then, for every D,

$$\Delta_k^{\mathbf{D}}(\mathbf{S},\mathbf{T}) \leq \nu_k^{\mathbf{D}}(\hat{\mathbf{S}}) \qquad \left(= \nu_k^{\mathbf{D}}(\hat{\mathbf{T}}) \right).$$

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In particular, $\Delta_k(\mathbf{S},\mathbf{T}) \leq \nu_k(\hat{\mathbf{S}})$

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Note: This lemma talks about a system as a mathematical object and is independent of the description language used for systems!

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Lemma ($\Leftarrow$) [MPR07]: Any S and T can be enhanced by MBOs to systems $\hat{S}$ and $\hat{T}$ such that $\hat{S} \stackrel{r}{\equiv} \hat{T}$ and, for every D,

$$\nu_k^{D}(\hat{S}) = \Delta_k^{D}(S,T)$$

Security amplification paradigm

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Idea: Combine several mildly secure systems to obtain a highly secure system.

Example: XOR of mildly uniform independent keys yields a highly uniform key!

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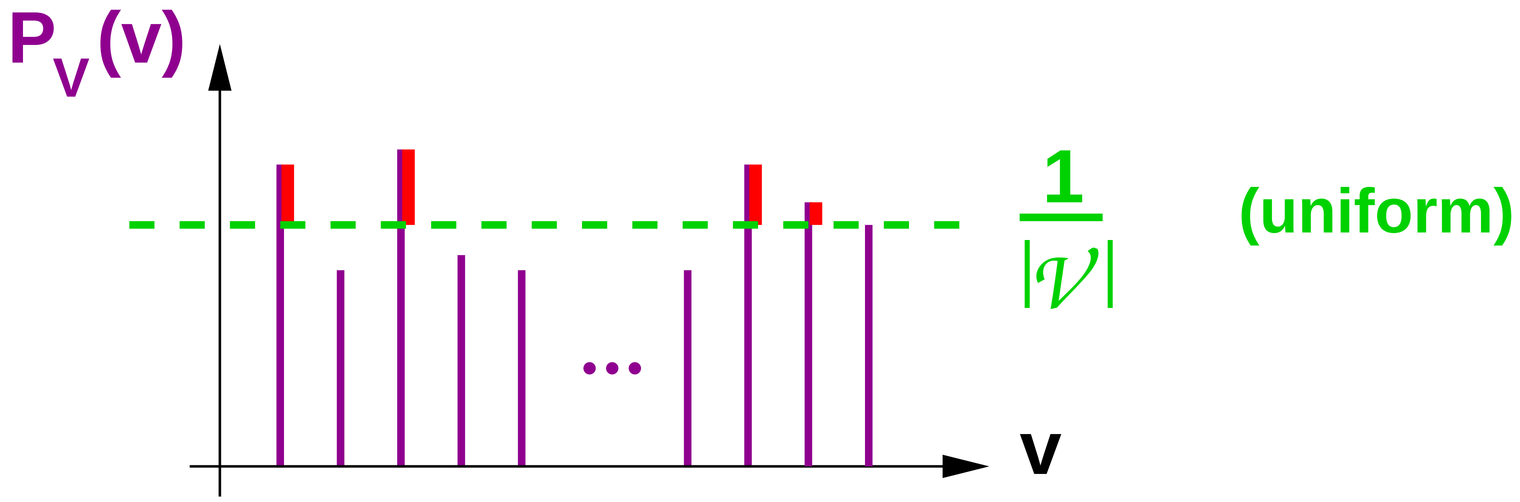
Idea: Combine several mildly secure systems to obtain a highly secure system.

Example: Cascade of mildly secure ciphers yields a highly secure cipher!

Distinguishing a RV V from a uniform RV U



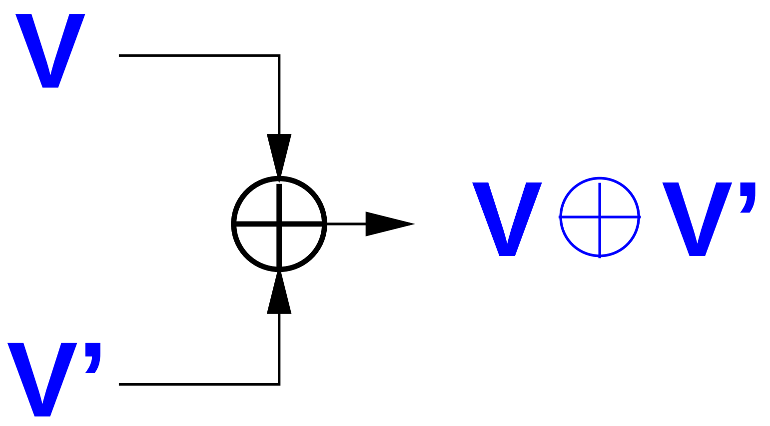
Statistical distance:

$$\mathbf{d}(\mathbf{V},\mathbf{U}) \;:=\; \frac{1}{2}\sum_{v\in\mathcal{V}}\left|P_{\mathbf{V}}(v) - \frac{1}{|\mathcal{V}|}\right| \quad \text{(sum of red quantities)}$$

$$= \;\Delta(\mathbf{V},\mathbf{U})$$

Possible interpretation: $\mathbf{P}(\mathbf{V}=\mathbf{U}) \;=\; 1 - \mathbf{d}(\mathbf{V},\mathbf{U})$

Product theorem for random variables



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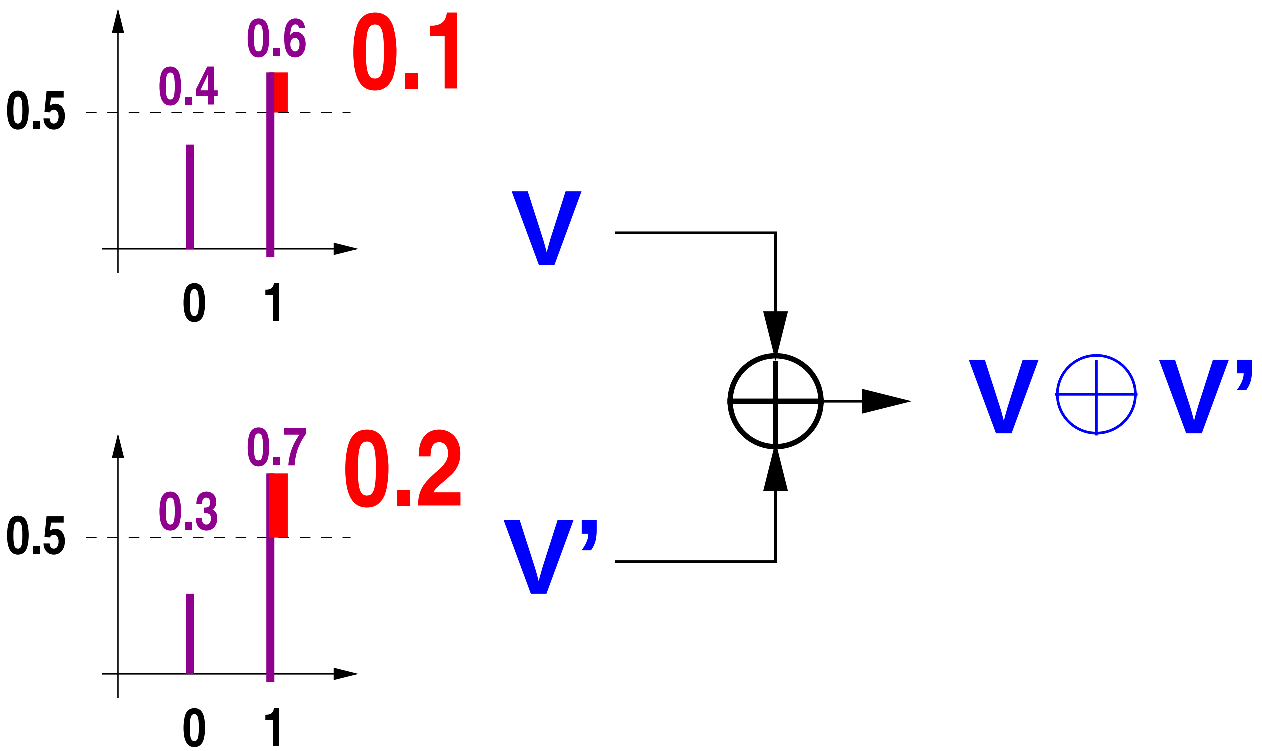

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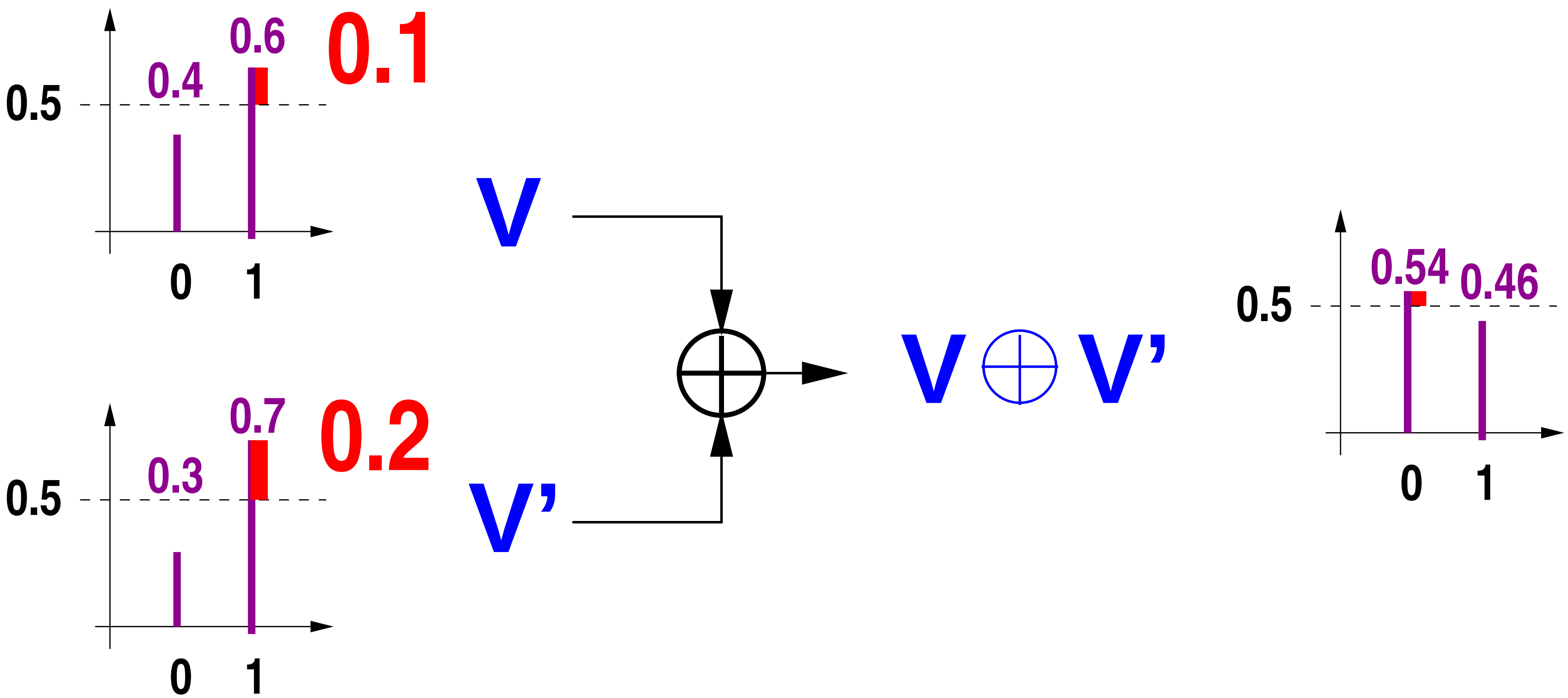

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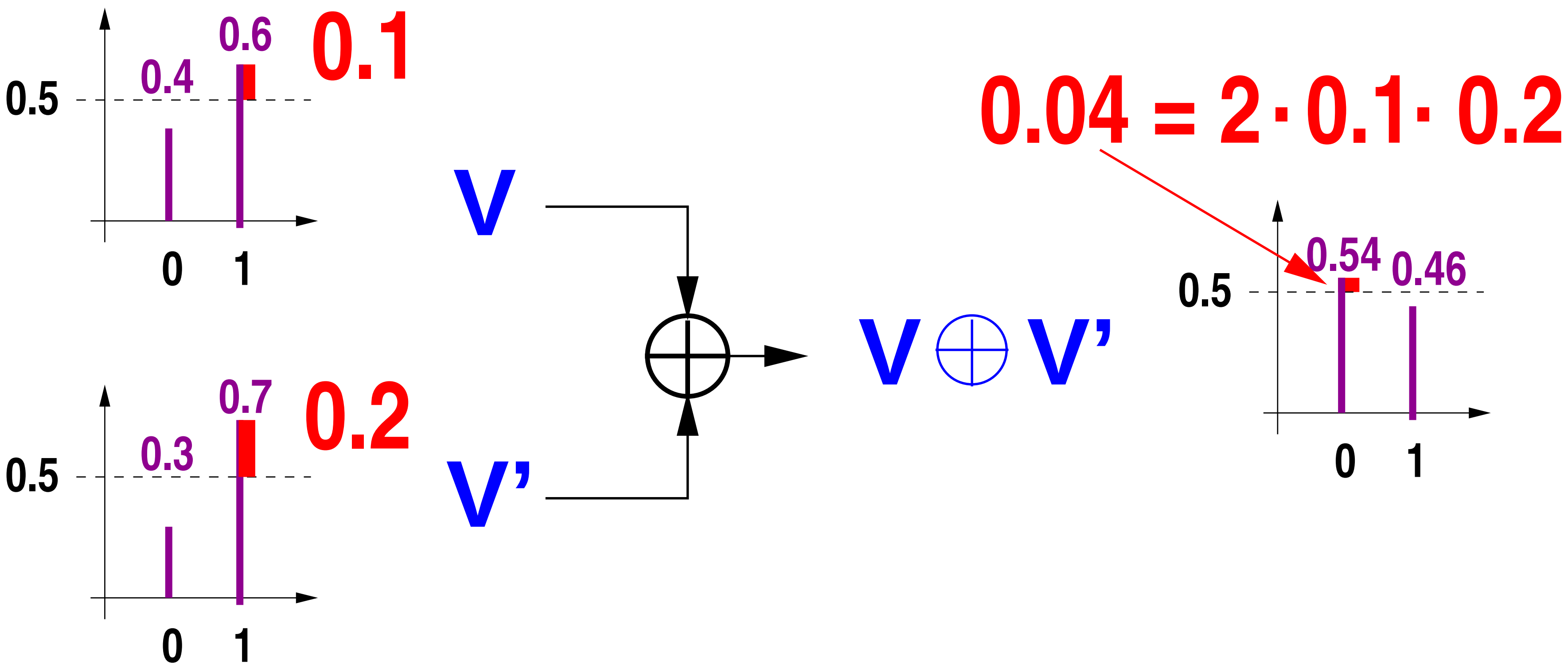
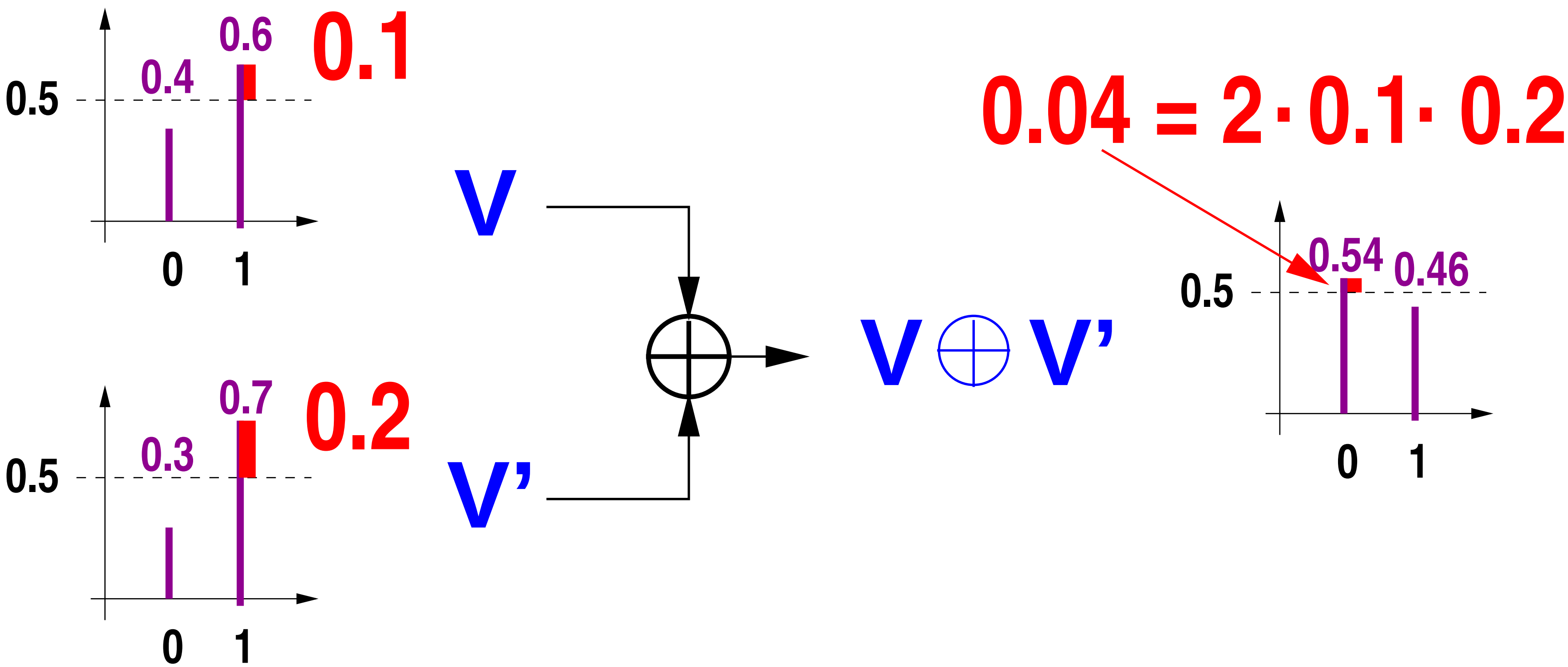
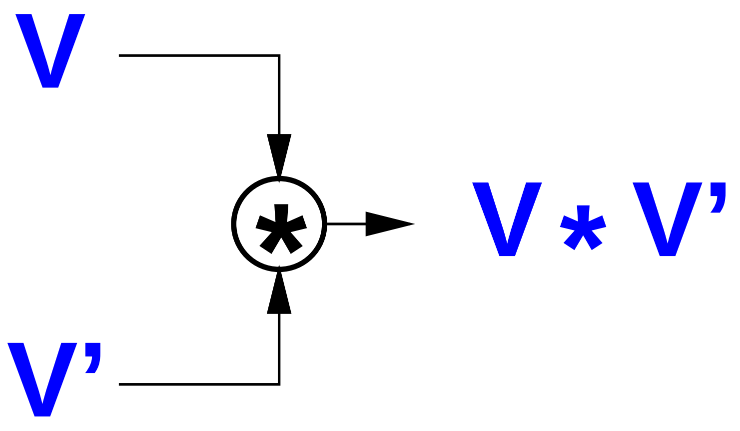


Product theorem for random variables



Theorem: $d(V \oplus V', U) \leq 2 \cdot d(V, U) \cdot d(V', U)$

Product theorem for random variables

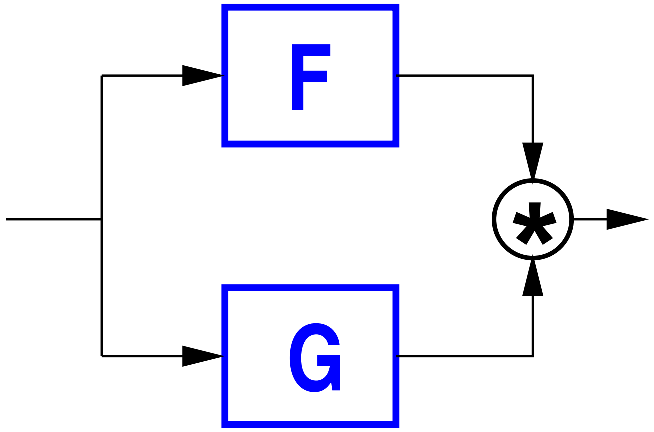


Theorem: $d(V \star V', U) \leq 2 \cdot d(V, U) \cdot d(V', U)$

for any quasi-group operation $\star$

Product theorems for systems ?

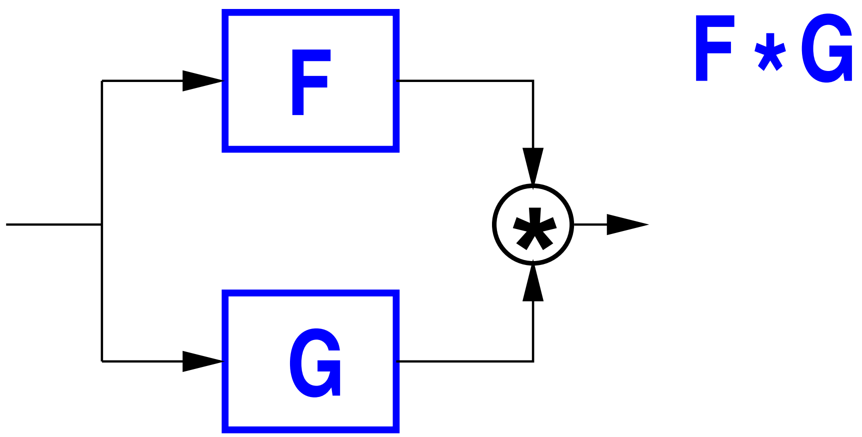
Let F and G be (possibly stateful) functions.



F*G

Product theorems for systems ? [MPR07]

Let F and G be (possibly stateful) functions.



Theorem: $\Delta_k(\mathbf{F} \star \mathbf{G}, \mathbf{R}) \;\leq\; \mathbf{2} \cdot \Delta_k(\mathbf{F}, \mathbf{R}) \cdot \Delta_k(\mathbf{G}, \mathbf{R})$

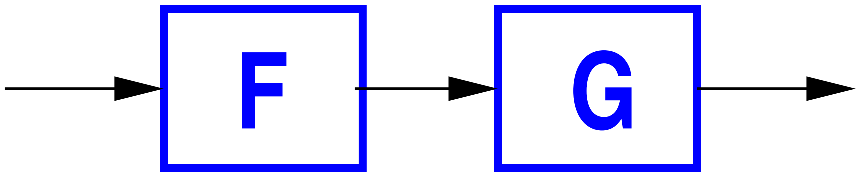
for any quasi-group operation $\star$.

($\mathbf{R}$ = uniform random function)

Product theorems for systems [MPR07]

Let F and G be (possibly stateful) permutations.

$F \triangleright G$



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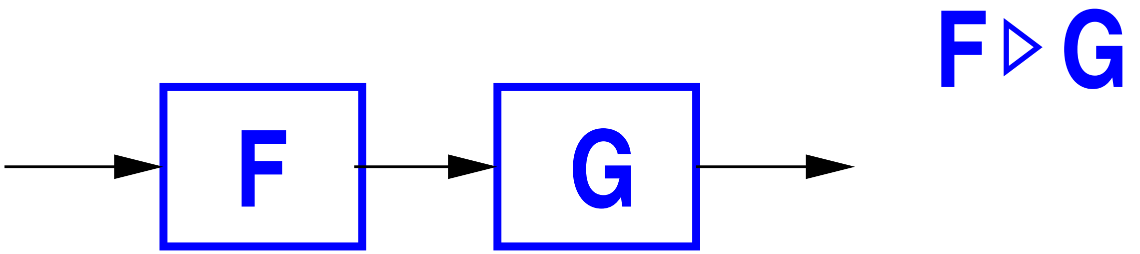

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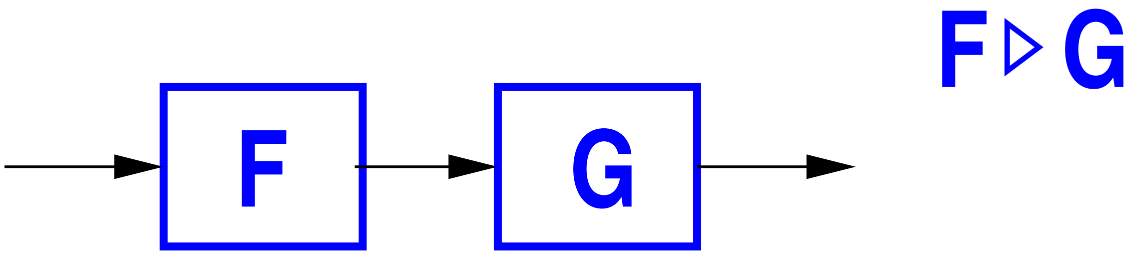

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Special case: Vaudenay's decorrelation theorem

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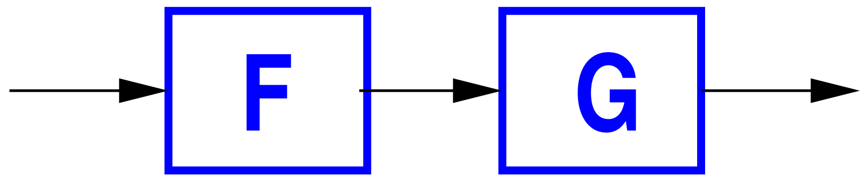

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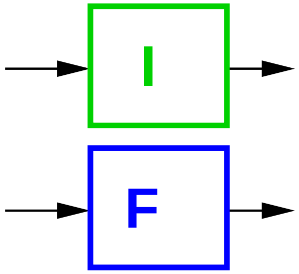
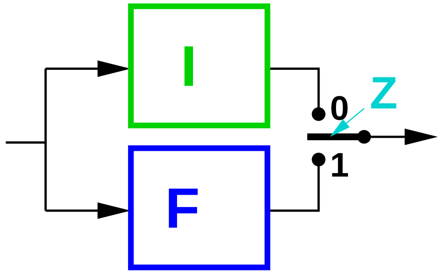
What is the general principle?

Neutralizing constructions [MPR07]



$$\Delta_k(\mathbf{F}, \mathbf{I})$$

Neutralizing constructions [MPR07]



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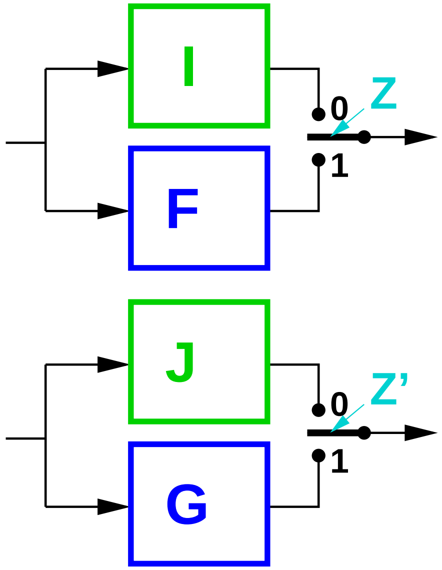

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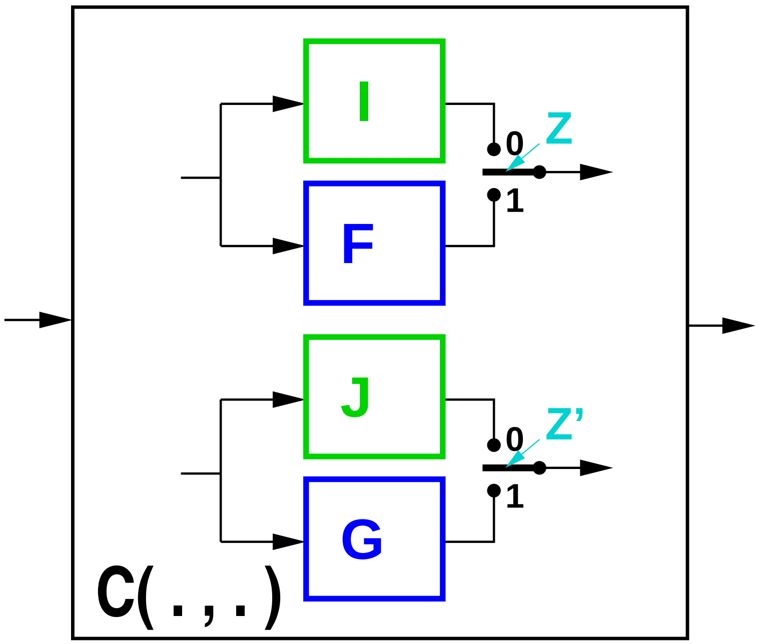

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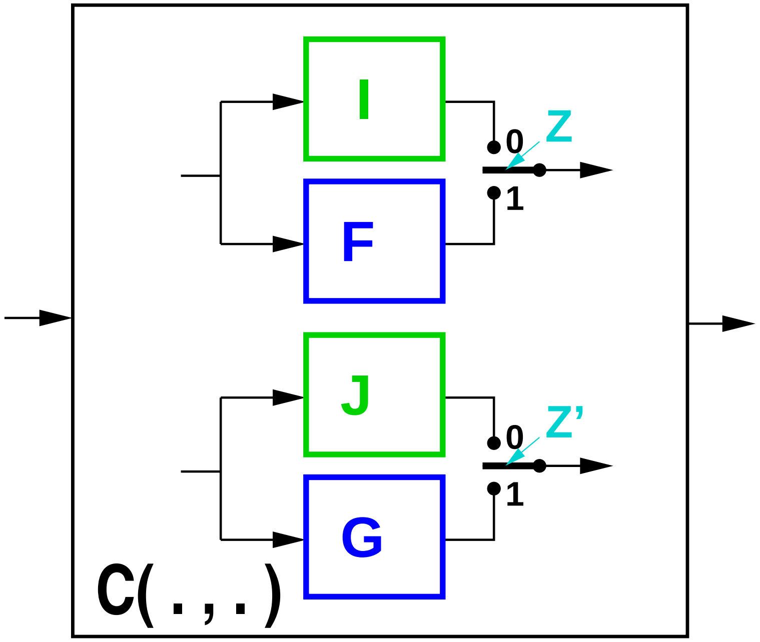
Neutralizing constructions [MPR07]



$\Delta_k($**F**, **I**$)$

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Def.: **C**$(.,.)$ **is neutralizing if C**$($**I**, **G**$) \equiv$ **C**$($**F**, **J**$) \equiv$ **C**$($**I**, **J**$) \equiv$ **Q**

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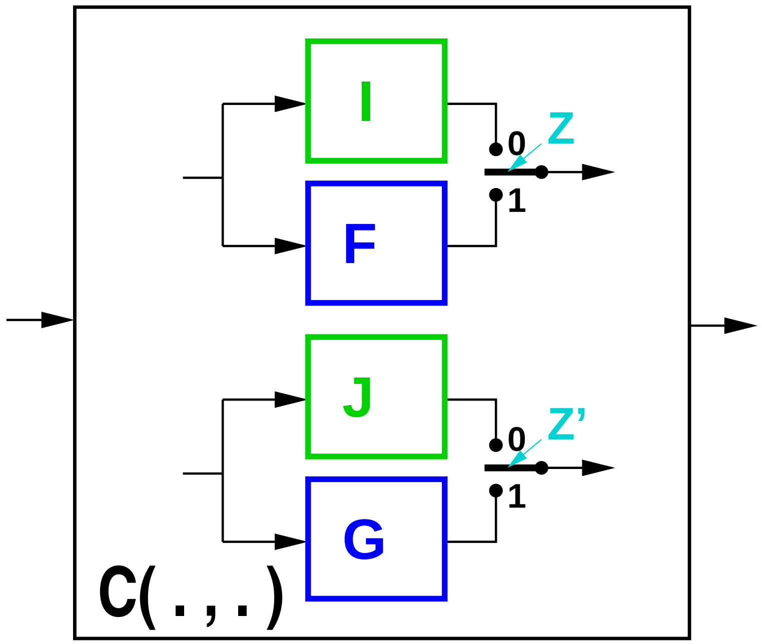

$\Delta_k($**F**, **I**$)$

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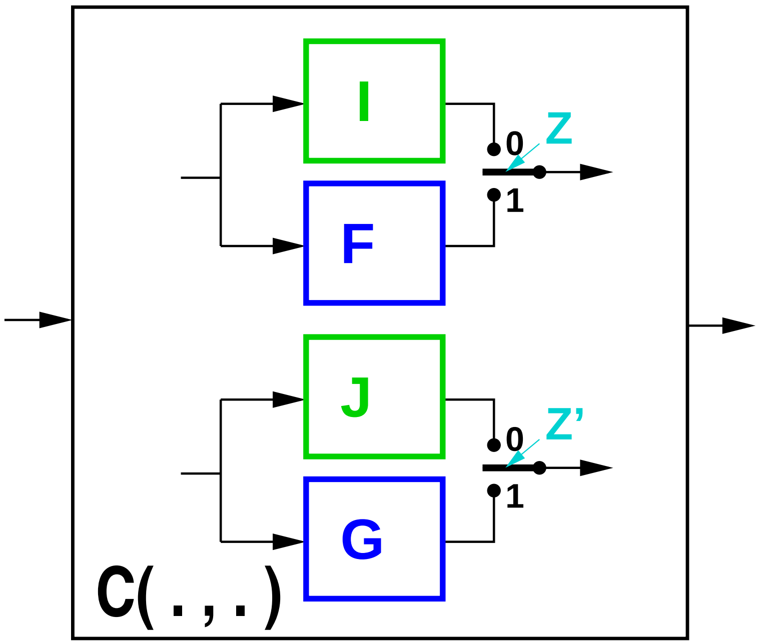
Examples: C$($F, G$) =$ F $\star$ G, I $=$ J $=$ Q $=$ R

Neutralizing constructions [MPR07]



$\Delta_k(\mathbf{F}, \mathbf{I})$

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Def.: $\mathbf{C}(., .)$ is neutralizing if $\mathbf{C}(\mathbf{I}, \mathbf{G}) \equiv \mathbf{C}(\mathbf{F}, \mathbf{J}) \equiv \mathbf{C}(\mathbf{I}, \mathbf{J}) \equiv \mathbf{Q}$

Examples: $\mathbf{C}(\mathbf{F}, \mathbf{G}) = \mathbf{F} \star \mathbf{G}$, $\quad \mathbf{I} = \mathbf{J} = \mathbf{Q} = \mathbf{R}$

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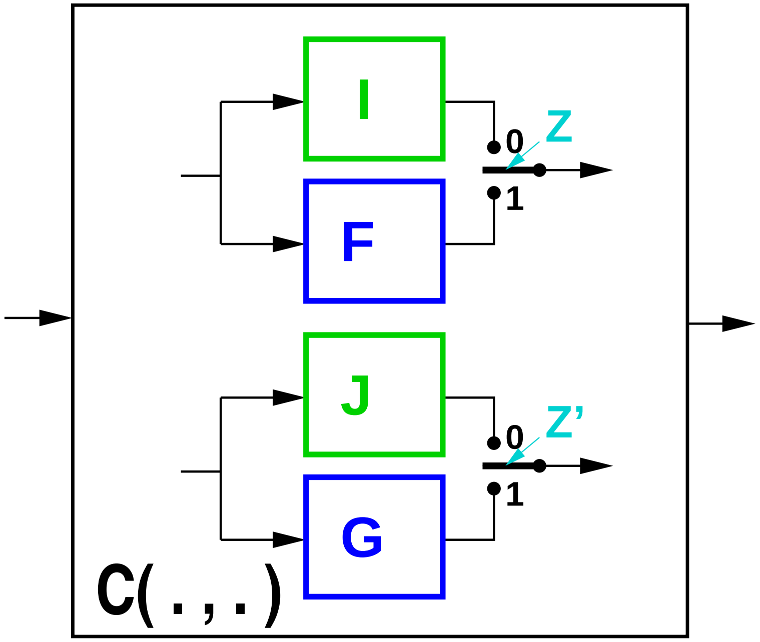

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$\mathbf{C}(\mathbf{F}, \mathbf{G}) = \mathbf{F} \triangleright \mathbf{G}$, $\mathbf{I} = \mathbf{J} = \mathbf{Q} = \mathbf{P}$

Theorem: $\Delta_k(\mathbf{C}(\mathbf{F}, \mathbf{G}), \mathbf{Q}) \leq 2 \cdot \Delta_k(\mathbf{F}, \mathbf{I}) \cdot \Delta_k(\mathbf{G}, \mathbf{J})$

Proof of the product theorem (1)

Theorem: $\Delta_k(\mathbf{C}(\mathbf{F}, \mathbf{G}), \mathbf{Q}) \leq \mathbf{2} \cdot \Delta_k(\mathbf{F}, \mathbf{I}) \cdot \Delta_k(\mathbf{G}, \mathbf{J})$

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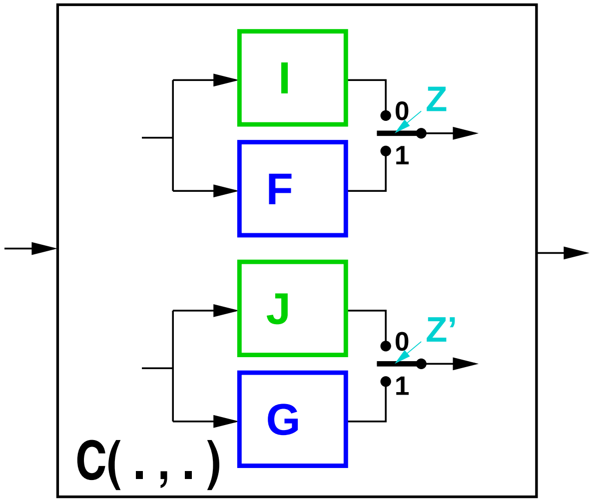

$\mathbf{C}(\mathbf{I}/\mathbf{F},\mathbf{J}/\mathbf{G})$

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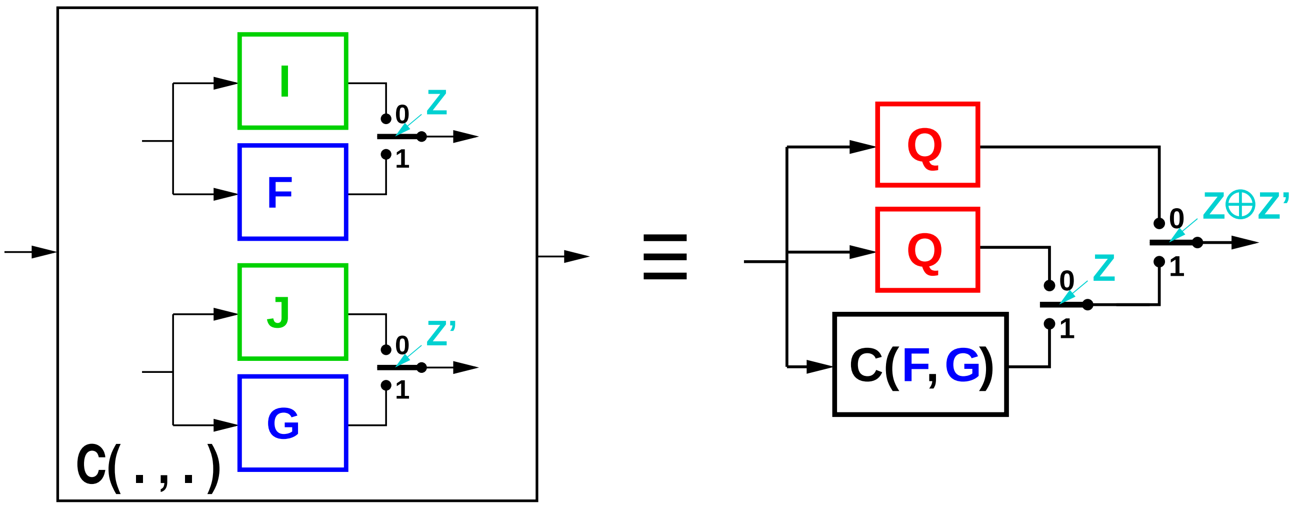

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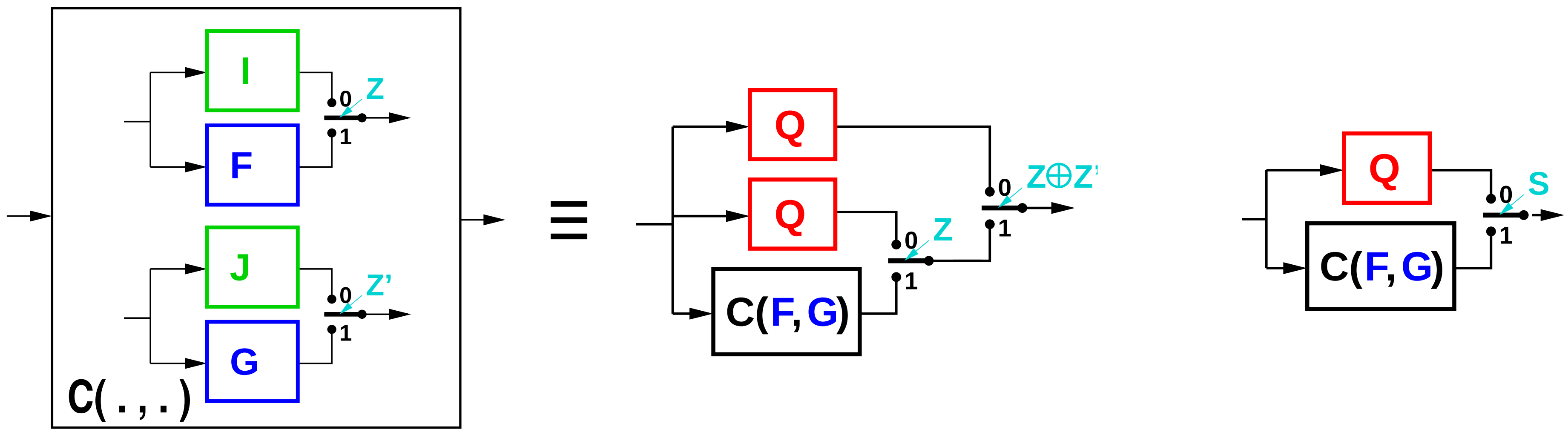

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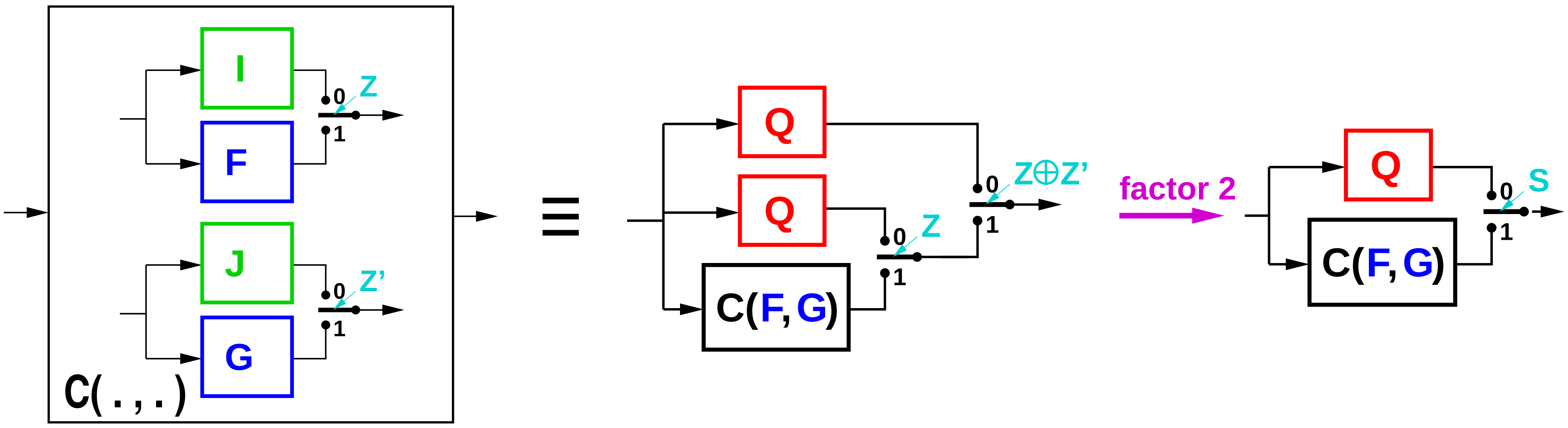

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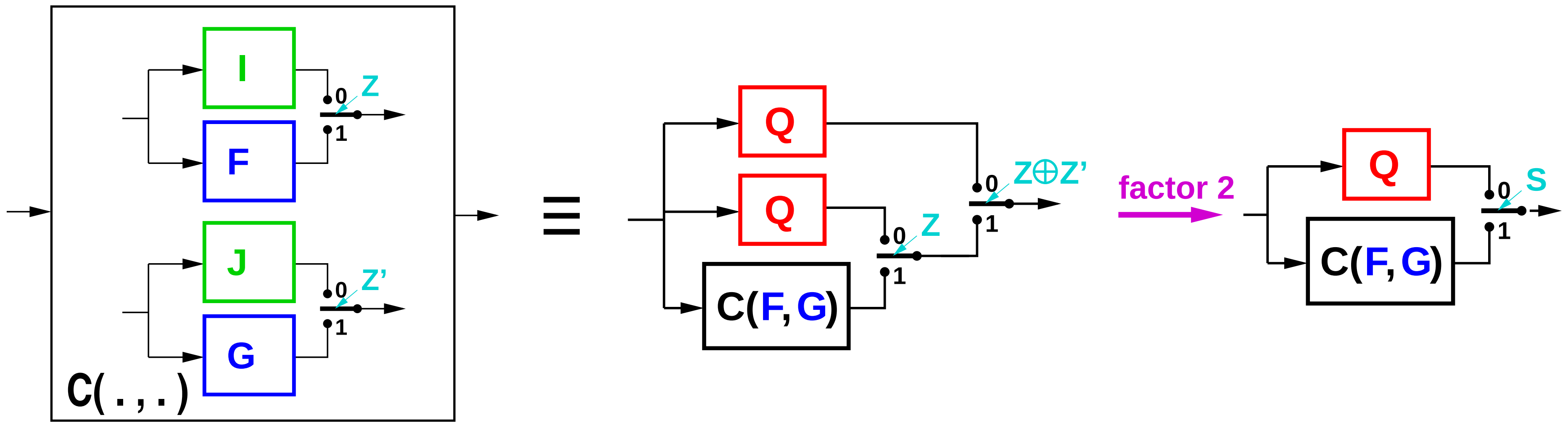


$\mathbf{C}(\mathbf{I}/\mathbf{F},\mathbf{J}/\mathbf{G})$

$\Delta_k(\mathbf{C}(\mathbf{F},\mathbf{G}),\mathbf{Q}) = \mathbf{2}\cdot$ **adv. in guessing** $\mathbf{Z}\oplus\mathbf{Z}'$ **in** $\mathbf{C}(\mathbf{I}/\mathbf{F},\mathbf{J}/\mathbf{G})$

Game-winning $\Longleftrightarrow$ Indistinguishability

Def.: $\hat{\mathsf{S}}$ and $\hat{\mathsf{T}}$ are restricted equivalent, denoted $\hat{\mathsf{S}} \stackrel{r}{\equiv} \hat{\mathsf{T}}$, if the I/O behavior is identical as long as MBO $=$0.

Lemma ($\Rightarrow$) [Mau02]: If $\hat{\mathsf{S}} \stackrel{r}{\equiv} \hat{\mathsf{T}}$, then, for every D,

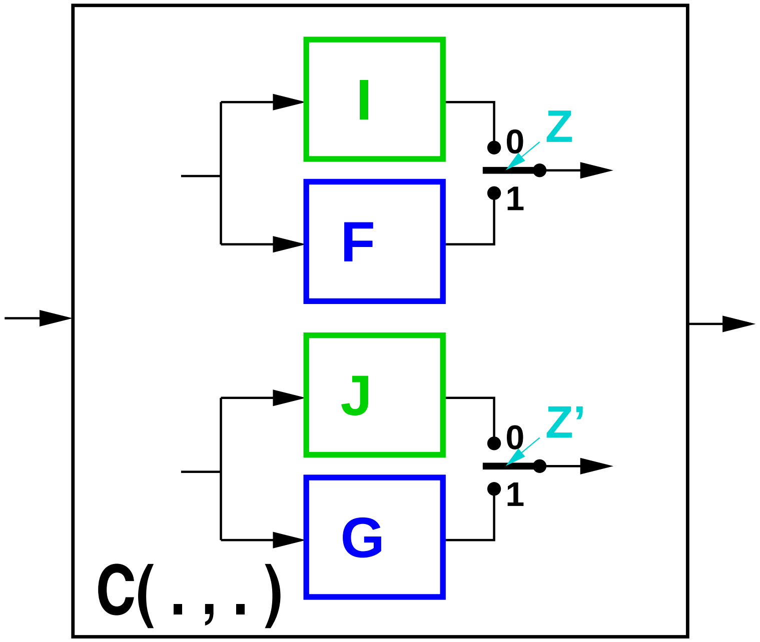
$$\Delta_k^{\mathsf{D}}(\mathsf{S},\mathsf{T}) \leq \nu_k^{\mathsf{D}}(\hat{\mathsf{S}}) \qquad (= \nu_k^{\mathsf{D}}(\hat{\mathsf{T}})\ ).$$

In particular, $\Delta_k(\mathsf{S},\mathsf{T}) \leq \nu_k(\hat{\mathsf{S}})$

Lemma ($\Leftarrow$) [MPR07]: Any S and T can be enhanced by MBOs to systems $\hat{\mathsf{S}}$ and $\hat{\mathsf{T}}$ such that $\hat{\mathsf{S}} \stackrel{r}{\equiv} \hat{\mathsf{T}}$ and, for every D,

$$\nu_k^{\mathsf{D}}(\hat{\mathsf{S}}) = \Delta_k^{\mathsf{D}}(\mathsf{S},\mathsf{T})$$

Proof of the product theorem (2)



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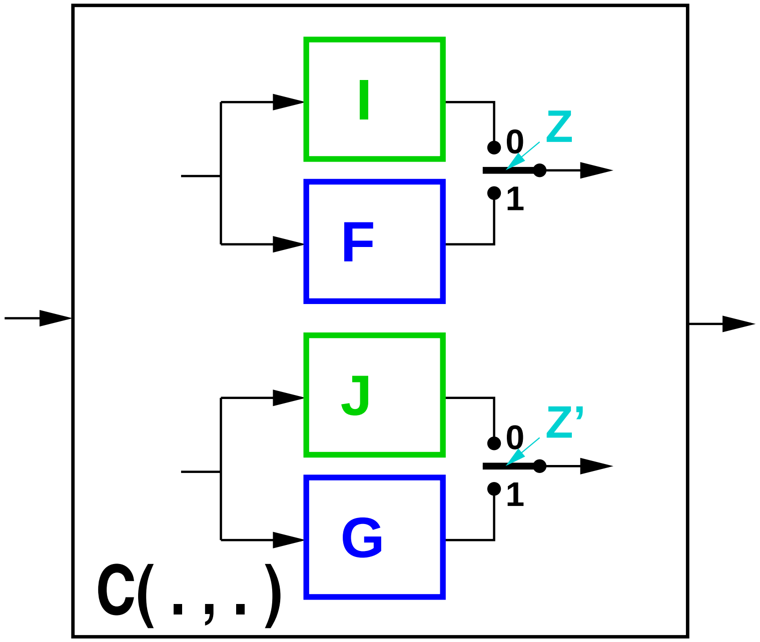
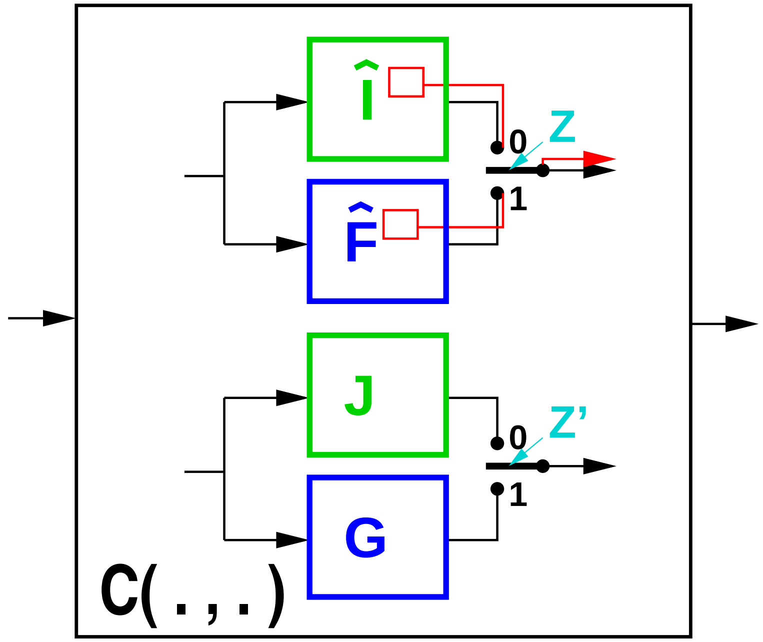


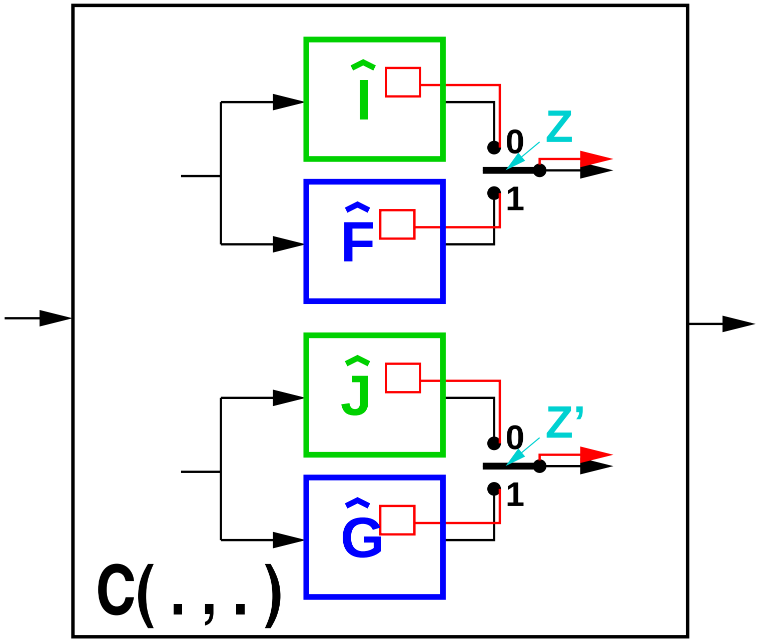
- **Task: Guess $Z \oplus Z'$**

Proof of the product theorem (2)



- **Task: Guess $Z \oplus Z'$**
- **Define MBOs and give the guesser access to them.**

Proof of the product theorem (2)



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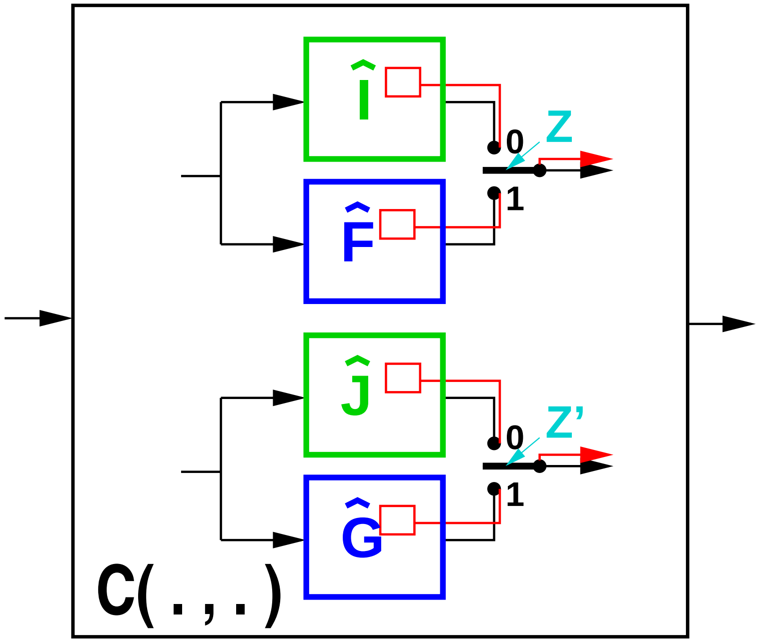

- **Task: Guess Z $\oplus$ Z$'$**
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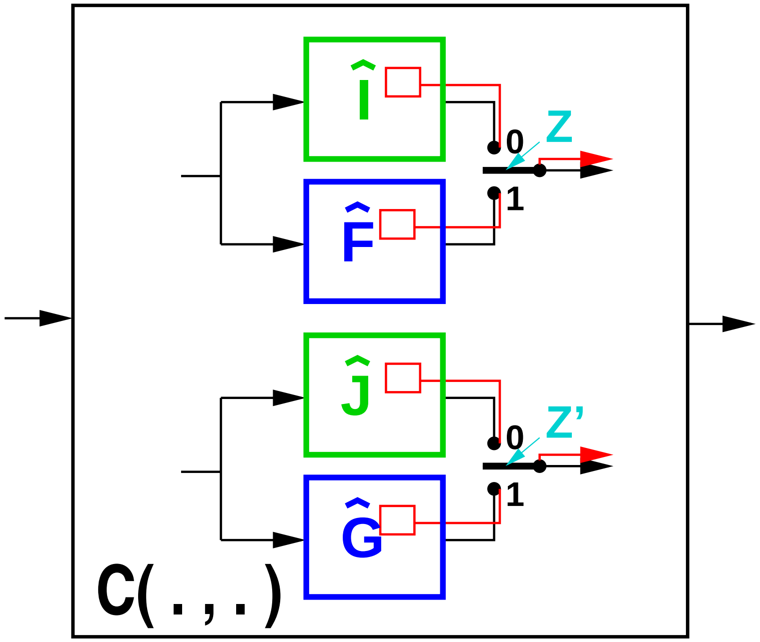
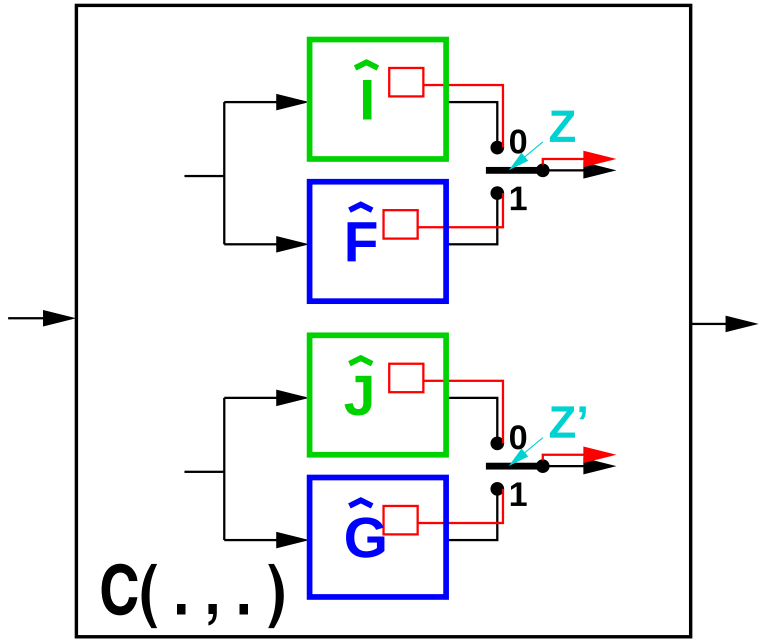
- **Game 1 not won $\Rightarrow$ advantage 0 in guessing Z**

Proof of the product theorem (2)



- **Task: Guess $Z \oplus Z'$**
- **Define MBOs and give the guesser access to them.**
- **Game 2 not won $\Rightarrow$ advantage 0 in guessing Z'**

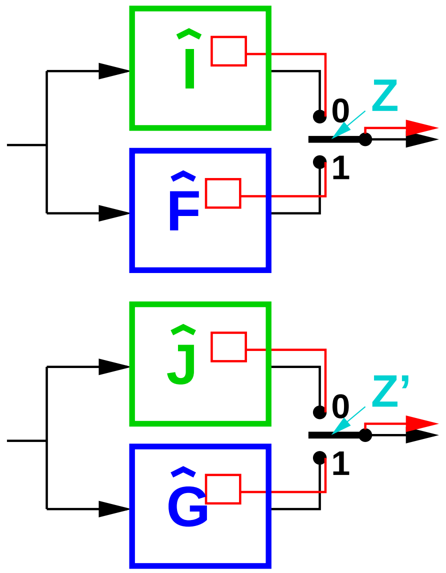
Proof of the product theorem (2)



- **Task: Guess $Z \oplus Z'$**
- **Define MBOs and give the guesser access to them.**
- **Game 2 not won $\Rightarrow$ advantage 0 in guessing Z'**
- **Game 1 or game 2 not won $\Rightarrow$ adv. 0 in guessing $Z \oplus Z'$.**

  $\Rightarrow$ advantage $\leq$ probability that both games won

Proof of the product theorem (2)



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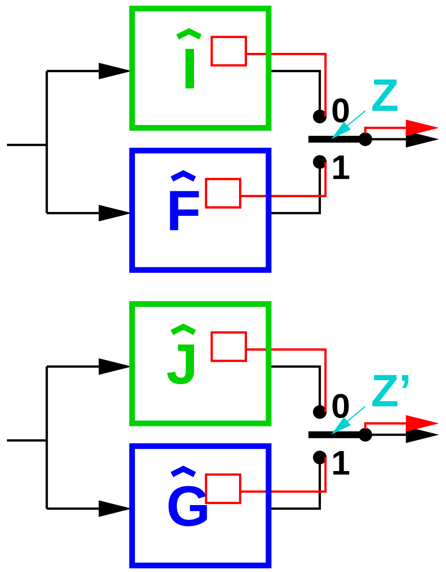

- Task: Guess $\mathsf{Z} \oplus \mathsf{Z}'$
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- We give the guesser direct access to the 2 games.
- Prob. of winning $=$ product of winning games 1 and 2.

  $= \Delta_k(\mathsf{F}, \mathsf{I}) \cdot \Delta_k(\mathsf{G}, \mathsf{J})$ q.e.d.

Computational indisting. amplification

Theorem [M-Tessaro09]: The previous statements hold also for computational indistinguishability.

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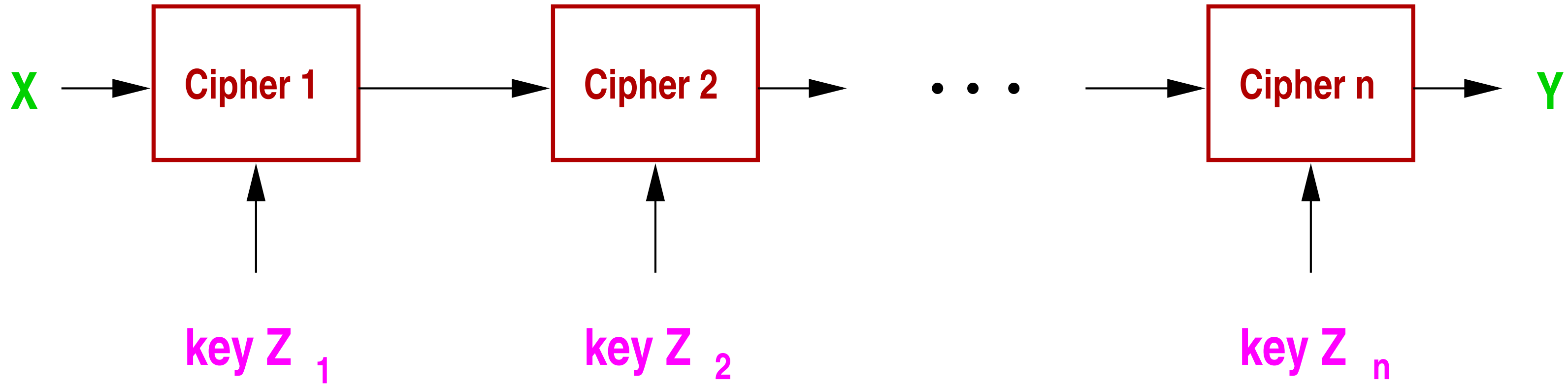
Example:



$$\Delta^{\mathcal{E}}(\mathbf{C}_i, \mathbf{P}) \leq \epsilon \;\Rightarrow\; \Delta^{\mathcal{E}}(\mathbf{C}_1 \cdots \mathbf{C}_n, \mathbf{P}) \approx 2^{n-1}\epsilon^n + \gamma$$

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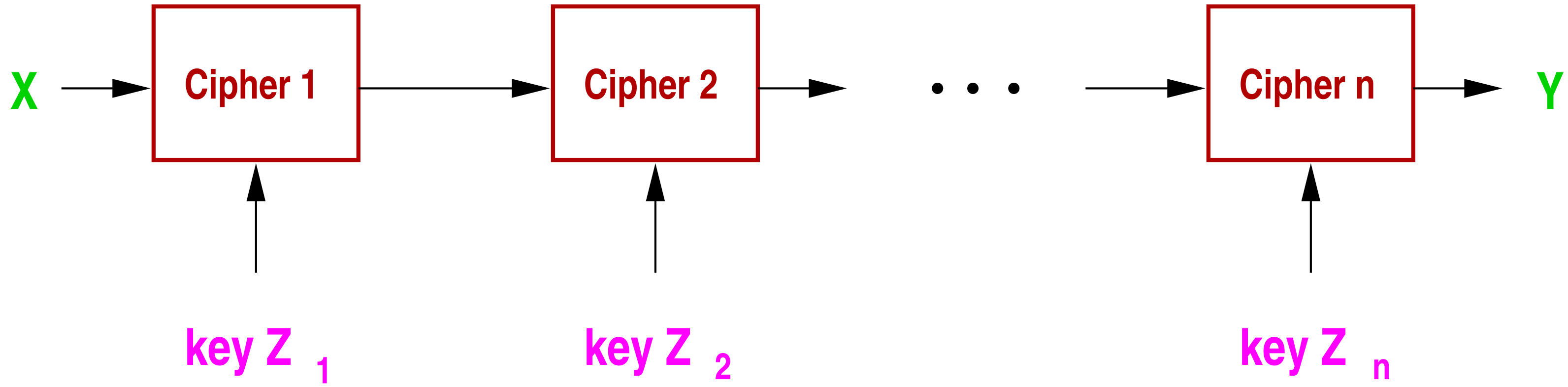

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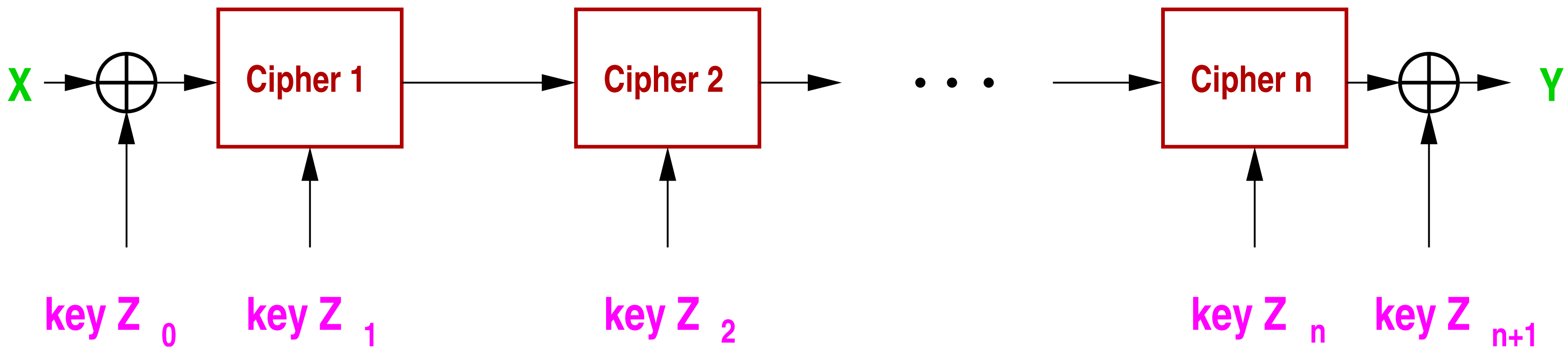
Problem: Amplification only if $\epsilon < 0.5$.

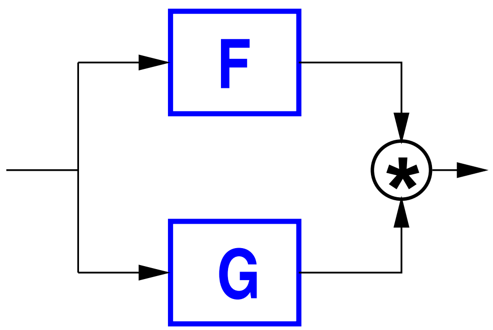
Strong security amplification



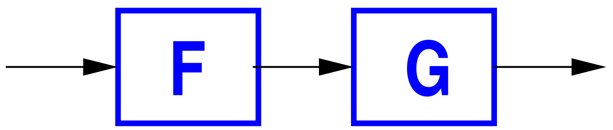
Theorem [MT09]:

$$\Delta^{\mathcal{E}}(\mathbf{C}_i, \mathbf{P}) \leq \epsilon \;\Rightarrow\; \Delta^{\mathcal{E}}(\oplus \mathbf{C}_1 \cdots \mathbf{C}_n \oplus, \mathbf{P}) \approx \epsilon^n + \gamma$$

Indistinguishability amplification: Type 2



F ⋆ G



F ▷ G

Theorem: $\Delta_k(\mathbf{F} \star \mathbf{G}, \mathbf{R}) \;\leq\; \Delta_k^{\mathrm{NA}}(\mathbf{F}, \mathbf{R}) + \Delta_k^{\mathrm{NA}}(\mathbf{G}, \mathbf{R}).$

Theorem: $\Delta_k(\mathbf{F} \triangleright \mathbf{G}, \mathbf{P}) \;\leq\; \Delta_k^{\mathrm{NA}}(\mathbf{F}, \mathbf{P}) + \Delta_k^{\mathrm{NA}}(\mathbf{G}, \mathbf{P}).$