ULTRAS *at Work: Compositionality and Equational Metaresults for Bisimulation and Trace Metaequivalences*

Marco Bernardo

University of Urbino – Italy

Objectives of a Behavioral Metamodel

- Unifying theory: offering a uniform view of existing behavioral models for a deeper understanding of their similarities and differences.
- Reuse facilities: providing general methodologies and tools for the development of new models, calculi, languages, ...
- A behavioral metamodel should *reduce* the effort needed for:
  - Defining syntax, semantics, and behavioral relations.
  - Investigating compositionality properties.
  - Studying alternative characterizations (equational, logical, ...).
  - Designing verification algorithms.
- Which existing models does it capture?
  Which new models may be generated from it?
  Which results are valid for all the specific models embodied in it?

Towards Behavioral Metamodels

- Some frameworks may be viewed as metamodels:
  - SOS rule formats.
  - Probabilistic automata.
  - Weighted automata.
- But they were *not* developed with the *explicit* purpose of paving the way to unifying theories and reuse facilities.
- Their focus is on ensuring certain *properties in a general setting* or achieving a *higher level of expressivity*.
- Categorical representations based on coalgebras and bialgebras can be considered metamodels.
- Need for something that is *less abstract* and *easier to use*, hopefully closer to automata and languages.

Some Recent Proposals

- The WLTS metamodel by Klin (no internal nondeterminism):
  - *Commutative monoids* to express and combine weights attached to transition labels under a *weight determinacy condition*.
  - Equipped with a notion of *weighted bisimilarity* and a *rule format* guaranteeing the compositionality of bisimulation semantics.
- The FuTS metamodel by De Nicola, Massink, Latella & Loreti:
  - *Commutative semirings* for a compositional and compact definition of the *operational semantics*, useful for a precise understanding of similarities and differences among process calculi of the same class.
  - Bisimilarity addressed from a *coalgebraic* viewpoint with De Vink.
- The ULTraS metamodel by Bernardo, De Nicola & Loreti:
  - *Preordered sets equipped with minimum* to describe reachability.
  - Emphasis on bisimulation and trace-based *metaequivalences*.
  - *Rule format* and *coalgebraic* characterization by Miculan & Peressotti.

Definition of the ULTRAS Metamodel

- $(D, \sqsubseteq_D, \bot_D)$ is a preordered set equipped with minimum $\bot_D$:
  - $d \in D$ represents a *degree of one-step reachability*.
  - $\bot_D$ denotes *unreachability*.
- $\Delta \in (S \to D)_{\mathrm{nefs}}$ is a *reachability distribution* such that $0 < |supp(\Delta)| < \omega$ where $supp(\Delta) = \{s \in S \mid \Delta(s) \neq \bot_D\}$.
- A uniform labeled transition system on $(D, \sqsubseteq_D, \bot_D)$, or D-ULTRAS, is a triple $(S, A, \longrightarrow)$ where:
  - $S \neq \emptyset$ is an at most countable set of states.
  - $A \neq \emptyset$ is a countable set of transition-labeling actions.
  - $\longrightarrow \subseteq S \times A \times (S \to D)_{\mathrm{nefs}}$ is a transition relation.
- Given a transition $s \xrightarrow{a} \Delta$:
  - $\Delta(s')$ quantifies the reachability degree of any $s' \in S$.
  - The set of reachable states is $supp(\Delta)$, which is nonempty and finite.

Generality of the ULTRAS Metamodel

- ULTRAS is much more *parsimonious* than WLTS and FUTS, preordered sets with minimum are enough to represent reachability.
- Algebraic structures are really necessary only when defining *behavioral relations* or process language *semantics*.
- The ULTRAS metamodel is general enough to encompass:
  - Nondeterministic models (LTS).
  - Probabilistic models (ADTMC, MDP, PA).
  - Stochastically timed models (ACTMC, CTMDP, MA).
  - Deterministically timed models (TA, PTA).
- Preordered sets to be used:
  - $(\mathbb{B}, \sqsubseteq_{\mathbb{B}}, \bot)$, where $\bot \sqsubseteq_{\mathbb{B}} \top$, for capturing LTS and TA.
  - $(\mathbb{R}_{[0,1]}, \leq, 0)$ for capturing ADTMC, MDP, PA, PTA, MA.
  - $(\mathbb{R}_{\geq 0}, \leq, 0)$ for capturing ACTMC and CTMDP.

Ingredients for Behavioral Metaequivalences

- Importing resolutions of nondeterminism with a formalization inspired by testing theories for nondeterministic and probabilistic processes.
- Adding a reachability-consistent semiring structure for:
  - Calculating multistep reachability values.
  - The overall reachability of a set of states.
- Defining measure schemata, based on the semiring operations, that consist of a reachability measure function for each resolution.
- Playing with the order of certain universal quantifiers in the definition of the metaequivalences thus obtaining pre-/post-metaequivalences.

Importing Resolutions in the ULTRAS Metamodel

- Behavioral metaequivalences on ULTRAS requires calculations that may be hampered by the presence of nondeterminism.
- A resolution of a state s belonging to an ULTRAS $\mathcal{U} = (S, A, \longrightarrow)$ is the result of a possible way of resolving choices starting from s.
- As if a *deterministic scheduler* were applied that, at the current state, selects one of its outgoing transitions or no transitions at all.
- Formalized as an acyclic *deterministic* ULTRAS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ obtained by unfolding the graph structure of $\mathcal{U}$ (special case of WLTS).
- Defined through a correspondence function from Z to S inspired by testing theories for probabilistic and nondeterministic processes.
- $Res(s)$ is the set of resolutions of s (for trace semantics).
- $k\text{-}Res(s)$ is the set of k-resolutions of s (for bisimulation semantics).

Formal Definition of Resolution of Nondeterminism

- Given an ULTRAS $\mathcal{U} = (S, A, \longrightarrow)$, a resolution of $s \in S$ is an ULTRAS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$, with no cycles and Z disjoint from S, for which there exists a correspondence function $corr_{\mathcal{Z}} : Z \to S$ such that $s = corr_{\mathcal{Z}}(z_s)$, for some $z_s \in Z$, and for all $z \in Z$:
  - If $z \xrightarrow{a}_{\mathcal{Z}} \Delta$ then $corr_{\mathcal{Z}}(z) \xrightarrow{a} \Delta'$, with $corr_{\mathcal{Z}}$ being bijective between $supp(\Delta)$ and $supp(\Delta')$ and $\Delta(z') = \Delta'(corr_{\mathcal{Z}}(z'))$ for all $z' \in supp(\Delta)$.
  - At most one transition departs from z.
- In the case of a k-resolution for $k \in \mathbb{N}_{\geq 1}$, if z is reachable from z_s with a sequence of less than k transitions then:
  - $z \notin S$;
  - z cannot be part of a cycle;
  - z has at most one outgoing transition;

  otherwise z is equal to $corr_{\mathcal{Z}}(z) \in S$ and has the same outgoing transitions that it has in $\mathcal{U}$.

Adding a Reachability-Consistent Semiring Structure

- The calculations required by ULTRAS behavioral metaequivalences refer to degrees of *multistep reachability* taken from $(D, \sqsubseteq_D, \perp_D)$.
- Need for a *commutative semiring* $(D, \oplus, \otimes, 0_D, 1_D)$ where:
  - $\otimes$ enables the calculation of multistep reachability from values of consecutive single-step reachability along the same trajectory.
  - $\oplus$ is useful for aggregating values of multistep reachability along different trajectories starting from the same state.
- The semiring must be consistent with the notion of reachability:
  - $0_D = \perp_D$ (both represent unreachability);
  - $d_1 \otimes d_2 \neq 0_D$ whenever $d_1 \neq 0_D \neq d_2$ (so consecutive steps cannot yield unreachability);
  - the sum via $\oplus$ of finitely many values 1_D is $\neq 0_D$ – *characteristic zero*

    (it ensures that two nonzero values sum up to zero only if they are one the inverse of the other w.r.t. $\oplus$, thus avoiding inappropriate zero results when aggregating values of trajectories from the same state; no $\mathbb{Z}_n$).

Measuring Multistep Reachability

- A measure schema $\mathcal{M}$ for an ULTRAS $\mathcal{U} = (S, A, \longrightarrow)$ on a reachability-consistent semiring $(D, \oplus, \otimes, 0_D, 1_D)$ is a set of *measure functions* $\mathcal{M}_{\mathcal{Z}} : Z \times A^* \times 2^Z \to D$, one for each $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ and $s \in S$:

$$\mathcal{M}_{\mathcal{Z}}(z, \alpha, Z') = \begin{cases} f_{\mathcal{Z}}\Big(\bigoplus_{z' \in supp(\Delta)} (\Delta(z') \otimes \mathcal{M}_{\mathcal{Z}}(z', \alpha', Z')), z, a, \Delta\Big) & \\ & \text{if } \alpha = a\,\alpha' \text{ and } z \xrightarrow{a}_{\mathcal{Z}} \Delta \\ 1_D & \text{if } \alpha = \varepsilon \text{ and } z \in Z' \\ 0_D & \text{otherwise} \end{cases}$$

- $f_{\mathcal{Z}} : D \times Z \times A \times (Z \to D)_{\text{nefs}} \to D$ provides some flexibility.
- The definition applies to $\mathcal{Z} \in k\text{-}Res(s)$ by restricting to traces $\alpha \in A^*$ such that $|\alpha| \leq k$.
- $\mathcal{M}_{\text{nd}}$ denotes the measure schema for $(\mathbb{B}, \vee, \wedge, \bot, \top)$.
- $\mathcal{M}_{\text{pb}}$ denotes the measure schema for $(\mathbb{R}_{\geq 0}, +, \times, 0, 1)$.
- $\mathcal{M}_{\text{ete}}$ and $\mathcal{M}_{\text{sbs}}$, developed for the stochastic case, also exploit $f_{\mathcal{Z}}$.

Measure Schemata for the Stochastic Case

- The *end-to-end* option originates a measure schema $\mathcal{M}_{\mathrm{ete}}(t)$ that expresses the probability of performing within $t \in \mathbb{R}_{\geq 0}$ time units a computation from state z labeled with trace α to a state in Z'

(convolution of two probability distributions when $\alpha = a\,\alpha'$ and $t > 0$ built by taking $x \in \mathbb{R}_{[0,t]}$):

$$\mathcal{M}_{\mathrm{ete}}(z, \alpha, Z')(t) \;=\; f_{\mathrm{ete}}(\sum_{z' \in supp(\Delta)} (\Delta(z') \times \mathcal{M}_{\mathrm{ete}}(z', \alpha', Z')(t - x)), z, a, \Delta)(t)$$

$$f_{\mathrm{ete}}(d, z, a, \Delta)(t) \;=\; \int_0^t \mathrm{e}^{-\mathrm{E}(z) \times x} \times d \,\mathrm{d}x$$

- The *step-by-step* option originates a measure schema $\mathcal{M}_{\mathrm{sbs}}(\theta)$ that expresses the prob. of perf. within a sequence of time units $\theta \in (\mathbb{R}_{\geq 0})^*$ a computation from state z labeled with trace α to a state in Z'

(product of two probability distributions when $\alpha = a\,\alpha'$ and $\theta = t\,\theta'$ with $t > 0$):

$$\mathcal{M}_{\mathrm{sbs}}(z, \alpha, Z')(\theta) \;=\; f_{\mathrm{sbs}}(\sum_{z' \in supp(\Delta)} (\Delta(z') \times \mathcal{M}_{\mathrm{sbs}}(z', \alpha', Z')(\theta')), z, a, \Delta)(t)$$

$$f_{\mathrm{sbs}}(d, z, a, \Delta)(t) \;=\; \frac{1 - \mathrm{e}^{-\mathrm{E}(z) \times t}}{\mathrm{E}(z)} \times d$$

Bisimulation Pre-/Post-Metaequivalences

- $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ and $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ are defined in the style of Larsen & Skou and differ for the *position* of the univ. quantif. over *sets* of equivalence classes.
- *Partially* matching transitions, i.e., with respect to *one* destination.
- An equivalence relation $\mathcal{B}$ over S is an $\mathcal{M}$-pre-bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all $a \in A$ and for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s_2)$ such that:

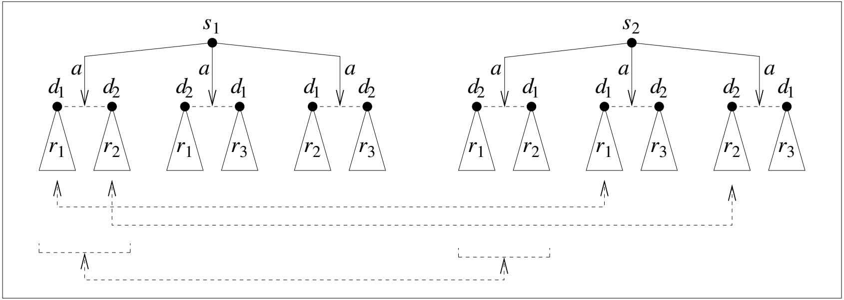
$$\mathcal{M}(z_{s_1}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s_2}, a, \bigcup \mathcal{G})$$

- *Fully* matching transitions, i.e., with respect to *all* destinations.
- An equivalence relation $\mathcal{B}$ over S is an $\mathcal{M}$-post-bisimulation iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s_2)$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$:

$$\mathcal{M}(z_{s_1}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s_2}, a, \bigcup \mathcal{G})$$

Pre-Metaequivalences vs. Post-Metaequivalences

- D-ULTRAS models identified by $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ but distinguished by $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ for $d_1 \neq d_2$ and inequivalent continuations:



- Internal nondeterminism due to three initial a-transitions.
- Continuations and their D-values are the same in both models.
- Continuations and their D-values are *shuffled within* each model.
- Only D-values are *shuffled across* the two models too.

Generality of Bisimulation Metaequivalences

- Specific bisimulation equivalences captured by both metaequivalences:
  - Park/Milner bisimilarity for LTS.
  - Giacalone, Jou & Smolka bisimilarity for ADTMC.
  - Larsen & Skou bisimilarity for MDP.
  - Hillston bisimilarity for ACTMC.
  - Neuhäußer & Katoen bisimilarity for CTMDP.
- Differences emerge in the case of specific models in which there are internal nondeterminism & probabilities/stochasticity.
- Only $\sim^{\rm post}_{\rm B, \mathcal{M}_{pb}}$ coincides with the strong bisimulation equivalence of Segala & Lynch for PA.
- $\sim^{\rm pre}_{\rm B, \mathcal{M}_{pb}}$ coincides with a new bisimulation equivalence for PA, which is logically characterized by Larsen & Skou PML (like in the case of fully prob. processes, reactive prob. processes, alternating PA).

Trace Pre-/Post-Metaequivalences

- $\sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}}$ and $\sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}}$ differ for the *position* of the universal quantifiers over traces w.r.t. the computations of the challenger and the defender.
- *Partially* matching resolutions, i.e., with respect to *one* trace.
- $s_1 \sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}} s_2$ iff for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 \in Res^{\mathrm{c}}(s_1)$ there exists $\mathcal{Z}_2 \in Res^{\mathrm{c}}(s_2)$ such that:

  $$\mathcal{M}(z_{s_1}, \alpha, Z_1) \; = \; \mathcal{M}(z_{s_2}, \alpha, Z_2)$$

  and symmetrically ... for each $\mathcal{Z}_2 \in Res^{\mathrm{c}}(s_2)$ there exists $\mathcal{Z}_1 \in Res^{\mathrm{c}}(s_1)$ ...
- *Fully* matching resolutions, i.e., with respect to *all* traces.
- $s_1 \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}} s_2$ iff for each $\mathcal{Z}_1 \in Res^{\mathrm{c}}(s_1)$ there exists $\mathcal{Z}_2 \in Res^{\mathrm{c}}(s_2)$ such that for all $\alpha \in A^*$:

  $$\mathcal{M}(z_{s_1}, \alpha, Z_1) \; = \; \mathcal{M}(z_{s_2}, \alpha, Z_2)$$

  and symmetrically ... for each $\mathcal{Z}_2 \in Res^{\mathrm{c}}(s_2)$ there exists $\mathcal{Z}_1 \in Res^{\mathrm{c}}(s_1)$ ...

Coherent Resolutions for Trace Semantics

- ULTRAS submodels rooted in the support of the target distribution of a transition:
  - are not necessarily distinct;
  - can have several outgoing transitions.
- The scheduler thus has the freedom of making *different* decisions in different occurrences of the *same* submodel.
- Overdiscriminating trace metaequivalences (violation of desirable properties).
- Coherent resolutions are resolutions in which the *same* decisions are made in different occurrences of the *same* submodel.
- If two states in the target distribution of a transition of $\mathcal{U}$ possess the same traces of a certain length, then so do the two states to which they correspond in $\mathcal{Z}$.
- $Res^{\rm c}(s)$ is the set of coherent resolutions of s.

Generality of Trace Metaequivalences

- Specific trace equivalences captured by both metaequivalences:
  - Brookes, Hoare & Roscoe trace equivalence for LTS.
  - Jou & Smolka trace equivalence for ADTMC.
  - Seidel trace equivalence for MDP.
  - Wolf, Baier & Majster-Cederbaum trace equiv. for ACTMC (ete option).
  - Bernardo trace equivalence for ACTMC (sbs option).
- Differences again emerge in the case of specific models in which there are internal nondeterminism & probabilities/stochasticity.
- Only $\sim^{\rm post}_{\rm T, \mathcal{M}_{pb}}$ coincides with the strong trace-distribution equivalence of Segala for PA.
- $\sim^{\rm pre}_{\rm T, \mathcal{M}_{pb}}$ coincides with a new trace equivalence for PA, which is a congruence with respect to parallel composition (this is not the case with any other probabilistic trace equivalence).

Discriminating Power of the Metaequivalences

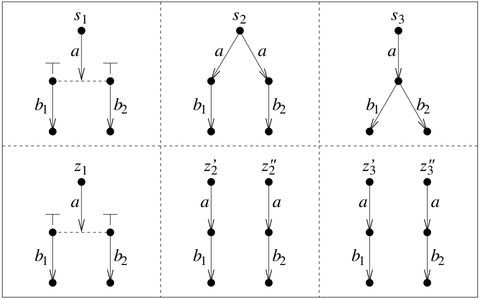
- $\sim^{\rm post}_{\rm B,\mathcal{M}}$ is finer than $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ (obvious from their definitions).
- $\sim^{\rm post}_{\rm T,\mathcal{M}}$ is finer than $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ (obvious from their definitions).
- $\sim^{\rm post}_{\rm B,\mathcal{M}}$ is finer than $\sim^{\rm post}_{\rm T,\mathcal{M}}$ (requires coherent resolutions).
- $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ and $\sim^{\rm post}_{\rm T,\mathcal{M}}$ / $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ are incomparable if there is internal nondet.

- $\sim^{\rm post}_{\rm B,\mathcal{M}}$ and $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ coincide on ULTRAS without internal nondet.
- $\sim^{\rm post}_{\rm T,\mathcal{M}}$ and $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ may not coincide even if there is no internal nondet.

- Weighted bisimilarity for WLTS coincides with both $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ and $\sim^{\rm post}_{\rm B,\mathcal{M}}$ when the same commutative monoid is considered.
- Bisimilarity for FUTS coincides with both $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ and $\sim^{\rm post}_{\rm B,\mathcal{M}}$ when the same commut. semiring is considered (deterministic state spaces).

Strictness of Inclusions and Internal Nondeterminism

- Three $\mathbb{B}$-ULTRAS models and their maximal resolutions ($b_1 \neq b_2$):



- $s_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} s_2$ but $s_1 \not\sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{nd}}} s_2$ | $s_1 \not\sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} s_3$ hence $s_1 \not\sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{nd}}} s_3$
  (s_2 and s_3 have different maximal 1-resolutions).
- $s_1 \sim^{\text{pre}}_{\text{T},\mathcal{M}_{\text{nd}}} s_2$ but $s_1 \not\sim^{\text{post}}_{\text{T},\mathcal{M}_{\text{nd}}} s_2$ | $s_1 \sim^{\text{pre}}_{\text{T},\mathcal{M}_{\text{nd}}} s_3$ but $s_1 \not\sim^{\text{post}}_{\text{T},\mathcal{M}_{\text{nd}}} s_3$
  (s_1 and s_3 have no internal nondeterminism, but s_1 is *not* the canonical representation of any LTS).
- $s_2 \sim^{\text{post}}_{\text{T},\mathcal{M}_{\text{nd}}} s_3$ but $s_2 \not\sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{nd}}} s_3$.

A Process Algebraic View of ULTRAS

- Search for metaresults for behavioral metaequivalences.
- UPROC – *uniform process calculus* over $(D, \oplus, \otimes, 0_D, 1_D)$.
- Syntax of the set $\mathbb{P}$ of *process terms*:

$$\boxed{P \ ::= \ \underline{0} \mid a \,.\, \mathcal{D} \mid P + P \mid P \parallel_L P}$$

  where $a \in A$ and $L \subseteq A$.
- Syntax of the set $\mathbb{D}$ of *distribution terms*:

$$\boxed{\mathcal{D} \ ::= \ d \triangleright P \mid \mathcal{D} \oplus \mathcal{D}}$$

  where $d \in D \setminus \{0_D\}$.
- Operator $+$ describes a nondeterministic choice.
- A probabilistic choice like in $P' \,{}_p\!+ P''$, where $p \in \mathbb{R}_{]0,1[}$, is rendered as $\tau \,.\, (p \triangleright P' \oplus (1 - p) \triangleright P'')$ with τ invisible action.

Operational Semantics of Dynamic Process Operators

- The operational semantic rules generate a D-ULTRAS $(\mathbb{P}, A, \longrightarrow)$.
- Action prefix:

$$\frac{\mathcal{D} \longmapsto \Delta}{a \,.\, \mathcal{D} \xrightarrow{a} \Delta}$$

- Alternative composition:

$$\frac{P_1 \xrightarrow{a} \Delta}{P_1 + P_2 \xrightarrow{a} \Delta} \qquad \frac{P_2 \xrightarrow{a} \Delta}{P_1 + P_2 \xrightarrow{a} \Delta}$$

Operational Semantics of Distribution Operators

- Singleton support distribution:

$$\boxed{d \triangleright P \longmapsto \{(P, d)\}}$$

- $\{(P, d)\}$ is a shorthand for the reachability distribution identically equal to 0_D except in P where its value is d.
- Distribution composition:

$$\boxed{\frac{\mathcal{D}_1 \longmapsto \Delta_1 \qquad \mathcal{D}_2 \longmapsto \Delta_2}{\mathcal{D}_1 \oplus \mathcal{D}_2 \longmapsto \Delta_1 \oplus \Delta_2}}$$

- $(\Delta_1 \oplus \Delta_2)(P) \;=\; \Delta_1(P) \oplus \Delta_2(P)$.
- Whenever $\mathcal{D} \longmapsto \Delta$, we let $supp(\mathcal{D}) = supp(\Delta)$.

Operational Semantics of Static Process Operators

- Parallel composition:

$$\frac{P_1 \xrightarrow{a} \Delta_1 \qquad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a} \Delta_1 \otimes \delta_{P_2}} \qquad \frac{P_2 \xrightarrow{a} \Delta_2 \qquad a \notin L}{P_1 \parallel_L P_2 \xrightarrow{a} \delta_{P_1} \otimes \Delta_2}$$

$$\frac{P_1 \xrightarrow{a} \Delta_1 \qquad P_2 \xrightarrow{a} \Delta_2 \qquad a \in L}{P_1 \parallel_L P_2 \xrightarrow{a} \Delta_1 \otimes \Delta_2}$$

- $(\Delta_1 \otimes \Delta_2)(P_1 \parallel_L P_2) \;=\; \Delta_1(P_1) \otimes \Delta_2(P_2)$.
- δ_P is identically equal to 0_D except in P where its value is 1_D.

Compositionality Metaresults

- Investigating whether the behavioral metaequivalences are *compositional* with respect to the various operators of UProC.
- Search for congruence results *independent from specific models*.
- Achieved for distribution operators and dynamic process operators.
- Confirm the existence, between bisimulation and trace semantics, of a foundational difference with respect to parallel composition, which shows up in the presence of internal nondeterminism:
  - Bisimilarity: only the post-metaequivalence is always a congruence.
  - Trace: it is the pre-metaequivalence that is always a congruence.
- *Is there a semantics for which both pre- and post-metaequivalences are always congruences with respect to parallel composition?*

Compositionality of Bisimulation Metaequivalences

- $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ and $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ are both congruences with respect to distribution operators, action prefix, alternative composition.
- $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ is a congruence with respect to parallel composition too, hence so is $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ in the absence of internal nondeterminism.
- $\sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}}$ is a congruence with respect to parallel composition, because in the only reachability-consistent semiring with $|D| = 2$, which is $(\mathbb{B}, \vee, \wedge, \bot, \top)$, parallel composition cannot generate values different from $\bot$ and $\top$.
- $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ is *not* a congruence with respect to parallel composition when $|D| \geq 3$ *and* there is internal nondeterminism.
- $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ is the coarsest congruence contained in $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ w.r.t. parallel composition in the case of an *image-finite* ULTRAS on a reachability-consistent *field* (algebraic and topological properties of vector spaces).

Compositionality of Trace Metaequivalences

- $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$ and $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$ are both congruences with respect to distribution operators, action prefix, alternative composition (action prefix requires coherent resolutions).
- $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$ is a congruence with respect to parallel composition too.
- The proof is based on the alternative characterization of $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$, which associates with each state the *set of traces* it can perform in the various resolutions, each extended with its *degree of executability*.
- $\sim_{\mathrm{T},\mathcal{M}_{\mathrm{nd}}}^{\mathrm{post}}$ is a congruence with respect to parallel composition if we rule out ULTRAS that are not canonical representations of LTS.
- $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$ is *not* a congruence with respect to parallel composition whenever it does *not* coincide with $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$ (due to internal nondeterminism).
- A coarsest congruence result is not possible for trace semantics because $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$ is finer than $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$.

Equational Characterization Metaresults

- Sound and complete axiom systems *independent from specific models*.
- Core axioms valid for all metaequivalences and measure schemata:

$$
\begin{array}{|lrcl|}
\hline
(\mathcal{A}_1) & (P_1 + P_2) + P_3 & = & P_1 + (P_2 + P_3) \\
(\mathcal{A}_2) & P_1 + P_2 & = & P_2 + P_1 \\
(\mathcal{A}_3) & P + \underline{0} & = & P \\
(\mathcal{A}_4) & (\mathcal{D}_1 \oplus \mathcal{D}_2) \oplus \mathcal{D}_3 & = & \mathcal{D}_1 \oplus (\mathcal{D}_2 \oplus \mathcal{D}_3) \\
(\mathcal{A}_5) & \mathcal{D}_1 \oplus \mathcal{D}_2 & = & \mathcal{D}_2 \oplus \mathcal{D}_1 \\
\hline
\end{array}
$$

- $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$ are typical of nondeterministic process calculi.
- $\mathcal{A}_4$, $\mathcal{A}_5$ are typical of probabilistic process calculi:
  - $P' \,_{p}{+}\, P'' \; = \; P'' \,_{1-p}{+}\, P'$.
  - $(P' \,_{p}{+}\, P'') \,_{q}{+}\, P''' \; = \; P' \,_{p \cdot q}{+}\, (P'' \,_{(1-p) \cdot q / (1 - p \cdot q)}{+}\, P''')$.

Idempotency Axioms for $\sim_{\mathrm{B}}^{\mathrm{post}}$

- Additional axioms for $\sim_{\mathrm{B}}^{\mathrm{post}}$:

$(\mathcal{A}_{\mathrm{B},1}^{\mathrm{post}})$	$P + P \;=\; P$
$(\mathcal{A}_{\mathrm{B},2}^{\mathrm{post}})$	$d_1 \triangleright P \oplus d_2 \triangleright P \;=\; (d_1 \oplus d_2) \triangleright P$

- $\mathcal{A}_{\mathrm{B},1}^{\mathrm{post}}$ is typical of bisimilarity over nondeterministic process calculi.
- $\mathcal{A}_{\mathrm{B},2}^{\mathrm{post}}$ encodes bisimilarity axioms such as:
  - $P\,{}_p\!+ P \;=\; P$ typical of probabilistic process calculi.
  - $\lambda_1 \,.\, P + \lambda_2 \,.\, P \;=\; (\lambda_1 + \lambda_2) \,.\, P$ typical of stochastic process calculi.
- *Sum normal form* of a process term $P \in \mathbb{P}$ for studying completeness:
  - either $\underline{0}$,
  - or $\sum_{i \in I} a_i \,.\, (\sum_{j \in J_i} d_{i,j} \triangleright P_{i,j})$ with every $P_{i,j}$ in sum normal form.

Shuffling Axiom for $\sim_{\rm B}^{\rm pre}$

- Additional axiom for $\sim_{\rm B}^{\rm pre}$ (all index sets are nonempty and finite):

$$(\mathcal{A}_{\rm B,1}^{\rm pre}) \quad \sum_{i \in I_1} a \,.\, (\textstyle\sum_{j \in J_{1,i}} d_{1,i,j} \triangleright P_{1,i,j}) \;=\; \sum_{i \in I_2} a \,.\, (\textstyle\sum_{j \in J_{2,i}} d_{2,i,j} \triangleright P_{2,i,j})$$

  - For all $i_1 \in I_1$ and $\emptyset \neq J_1 \subseteq J_{1,i_1}$ containing the indices of *all* the occurrences of any process indicated by an index in J_1 itself, there exist $i_2 \in I_2$ and $\emptyset \neq J_2 \subseteq J_{2,i_2}$ containing the indices of *all* the occurrences of any process indicated by an index in J_2 itself, s.t.:
    1. $\forall j_1 \in J_1 \,.\, (\exists j_2 \in J_2 \,.\, P_{1,i_1,j_1} = P_{2,i_2,j_2} \vee \nexists j_2 \in J_{2,i_2} \,.\, P_{1,i_1,j_1} = P_{2,i_2,j_2})$.
    2. $\{P_{1,i_1,j} \mid j \in J_1\} \;\supseteq\; \{P_{2,i_2,j} \mid j \in J_2\}$.
    3. $\bigoplus_{j \in J_1} d_{1,i_1,j} \;=\; \bigoplus_{j \in J_2} d_{2,i_2,j}$.
  - Symmetric condition obtained by exchanging I_1, J_1 with I_2, J_2.

- $\mathcal{A}_{\rm B,1}^{\rm pre}$ subsumes:
  - Both idempotency axioms $\mathcal{A}_{\rm B,1}^{\rm post}$ and $\mathcal{A}_{\rm B,2}^{\rm post}$.
  - $a \,.\, \mathcal{D}_1 + a \,.\, \mathcal{D}_2 \;=\; a \,.\, (\mathcal{D}_1 \oplus \mathcal{D}_2)$ under the same constraints.

Choice-Deferring Axioms for $\sim_{\rm T}^{\rm post}$

- Additional axioms for $\sim_{\rm T}^{\rm post}$ with respect to $\sim_{\rm B}^{\rm post}$:

$$(\mathcal{A}_{\rm T,1}^{\rm post})\quad \begin{array}{l} a_1 \,.\, (\mathcal{D}_1 \oplus d_1 \triangleright (P_1 + a_2 \,.\, (\mathcal{D}_2 \oplus d_2 \triangleright (\ldots \triangleright (P_{n-1} + a_n \,.\, (\mathcal{D}_n \oplus d_n \triangleright P'))\ldots)))) \\ + \\ a_1 \,.\, (\mathcal{D}_1 \oplus d_1 \triangleright (P_1 + a_2 \,.\, (\mathcal{D}_2 \oplus d_2 \triangleright (\ldots \triangleright (P_{n-1} + a_n \,.\, (\mathcal{D}_n \oplus d_n \triangleright P''))\ldots)))) \\ = \\ a_1 \,.\, (\mathcal{D}_1 \oplus d_1 \triangleright (P_1 + a_2 \,.\, (\mathcal{D}_2 \oplus d_2 \triangleright (\ldots \triangleright (P_{n-1} + a_n \,.\, (\mathcal{D}_n \oplus d_n \triangleright (P' + P'')))\ldots)))) \\ \quad \text{where } P', P'' \notin supp(\mathcal{D}_n) \end{array}$$

$$(\mathcal{A}_{\rm T,2}^{\rm post})\quad \begin{array}{l} a \,.\, (\mathcal{D} \oplus d_1 \triangleright (\sum_{j \in J} b_j \,.\, \mathcal{D}_{1,j}) \oplus d_2 \triangleright (\sum_{j \in J} b_j \,.\, \mathcal{D}_{2,j})) \\ = \\ a \,.\, (\mathcal{D} \oplus d' \triangleright (\sum_{j \in J} b_j \,.\, (\mathcal{D}'_{1,j} \oplus \mathcal{D}'_{2,j}))) \\ \quad \text{if } d' = d_1 \oplus d_2 \text{ and for } 1 \le i \le 2 \text{ there exists } d'_i \in D \text{ such that } d' \otimes d'_i = d_i \\ \quad \text{where } \mathcal{D}'_{i,j} \text{ is obtained from } \mathcal{D}_{i,j} \text{ by multiplying each of its } D\text{-values by } d'_i \end{array}$$

- Simplest instance of $\mathcal{A}_{\rm T,1}^{\rm post}$, typical of nondeterministic process calculi:
  $a \,.\, (d \triangleright P') + a \,.\, (d \triangleright P'') \; = \; a \,.\, (d \triangleright (P' + P''))$.
- Application of $\mathcal{A}_{\rm T,2}^{\rm post}$, typical of probabilistic process calculi:
  $a \,.\, (d_1 \triangleright (b \,.\, (1_D \triangleright P_1)) \oplus d_2 \triangleright (b \,.\, (1_D \triangleright P_2))) \; =$
  $a \,.\, (1_D \triangleright b \,.\, (d_1 \triangleright P_1 \oplus d_2 \triangleright P_2))$ where $d_1 \oplus d_2 = 1_D$.

Expansion Law for Parallel Composition

- The validity of this law (for all the behavioral metaequivalences) stems from the operational semantic rules.
- Let P_1 and P_2 be in sum normal form (with I_1 and I_2 possibly empty):

$$\sum_{i \in I_1} a_{1,i} \,.\, \big(\bigoplus_{j \in J_{1,i}} d_{1,i,j} \triangleright P_{1,i,j} \big)$$

$$\sum_{i \in I_2} a_{2,i} \,.\, \big(\bigoplus_{j \in J_{2,i}} d_{2,i,j} \triangleright P_{2,i,j} \big)$$

- The axiom (where any empty summation yields $\underline{0}$):

$$\begin{aligned} P_1 \parallel_L P_2 \;=\; & \sum_{i \in I_1}^{a_{1,i} \notin L} a_{1,i} \,.\, \big(\bigoplus_{j \in J_{1,i}} d_{1,i,j} \triangleright (P_{1,i,j} \parallel_L P_2) \big) \\ + \; & \sum_{i \in I_2}^{a_{2,i} \notin L} a_{2,i} \,.\, \big(\bigoplus_{j \in J_{2,i}} d_{2,i,j} \triangleright (P_1 \parallel_L P_{2,i,j}) \big) \\ + \; & \sum_{i \in I_1}^{a_{1,i} \in L} \sum_{i' \in I_2}^{a_{2,i'} = a_{1,i}} a_{1,i} \,.\, \big(\bigoplus_{j \in J_{1,i}} \bigoplus_{j' \in J_{2,i'}} (d_{1,i,j} \otimes d_{2,i',j'}) \triangleright (P_{1,i,j} \parallel_L P_{2,i',j'}) \big) \end{aligned}$$

  is sound with respect to all considered metaequivalences.

Future Work

- Keep putting ULTRAS at work on behavioral metaequivalences to further extend the resulting unifying process theory:
  - Logical characterization metaresults.
  - Metaresults for other bisimulation-/trace-based metaequivalences.
  - Metaresults for testing metaequivalences.
  - The spectrum of metaequivalences.
- Defining and studying properties of:
  - Behavioral metapreorders.
  - Weak variants of behavioral metarelations.
  - Approximate variants of behavioral metarelations.
- On the metamodel side, capturing also:
  - Interleaving models with continuous state spaces.
  - Truly concurrent models such as Petri nets and event structures.

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