A Companion of "Relating Strong Behavioral Equivalences for Processes with Nondeterminism and Probabilities"

Marco Bernardo[a], Rocco De Nicola[b], Michele Loreti[c]

[a] *Dipartimento di Scienze di Base e Fondamenti – Università di Urbino – Italy*
[b] *IMT – Institute for Advanced Studies Lucca – Italy*
[c] *Dipartimento di Statistica, Informatica, Applicazioni – Università di Firenze – Italy*

Abstract

In the paper "Relating Strong Behavioral Equivalences for Processes with Nondeterminism and Probabilities" to appear in Theoretical Computer Science, we present a comparison of behavioral equivalences for nondeterministic and probabilistic processes. In particular, we consider strong trace, failure, testing, and bisimulation equivalences. For each of these groups of equivalences, we examine the discriminating power of three variants stemming from three approaches that differ for the way probabilities of events are compared when nondeterministic choices are resolved via deterministic schedulers. The established relationships are summarized in a so-called spectrum. However, the equivalences we consider in that paper are only a small subset of those considered in the original spectrum of equivalences for nondeterministic systems introduced by Rob van Glabbeek. In this companion paper, we enlarge the spectrum by considering variants of trace equivalences (completed-trace equivalences), additional decorated-trace equivalences (failure-trace, readiness, and ready-trace equivalences), and variants of bisimulation equivalences (kernels of simulation, completed-simulation, failure-simulation, and ready-simulation preorders). Moreover, we study how the spectrum changes when randomized schedulers are used instead of deterministic ones.

Keywords: bisimulation equivalence, simulation equivalence, testing equivalence, readiness equivalence, failure equivalence, trace equivalence, nondeterminism, probability

1. Introduction

In [1], a systematic account of the main known probabilistic equivalences for nondeterministic *and* probabilistic systems has been presented by defining them over an extension of the LTS model combining nondeterminism and probability that we call NPLTS, in which every action-labeled transition goes from a source state to a probability distribution over target states rather than to a single target state [12, 14]. Schedulers, which can be viewed as external entities that select the next action to perform, are used to resolve nondeterminism [14]. By "playing" with schedulers, a number of possibilities for defining behavioral equivalences over NPLTS models emerge. We concentrated on three approaches that differ for the way probabilities of events are compared when nondeterministic choices are resolved via schedulers:

1. **Fully Matching Resolutions** Two resolutions are compared with respect to the probability distributions of all considered events.

2. **Partially Matching Resolutions** The probabilities of the set of events of a resolution are required to be individually matched by the probabilities of the same events in possibly different resolutions.

3. **Max-Min-Matching Resolution Sets** Only the extremal probabilities of each event stemming from the different resolutions are compared.

In [1], we have studied the relationships among the probabilistic variants of the main equivalences for nondeterministic systems that stem from the three approaches outlined above.

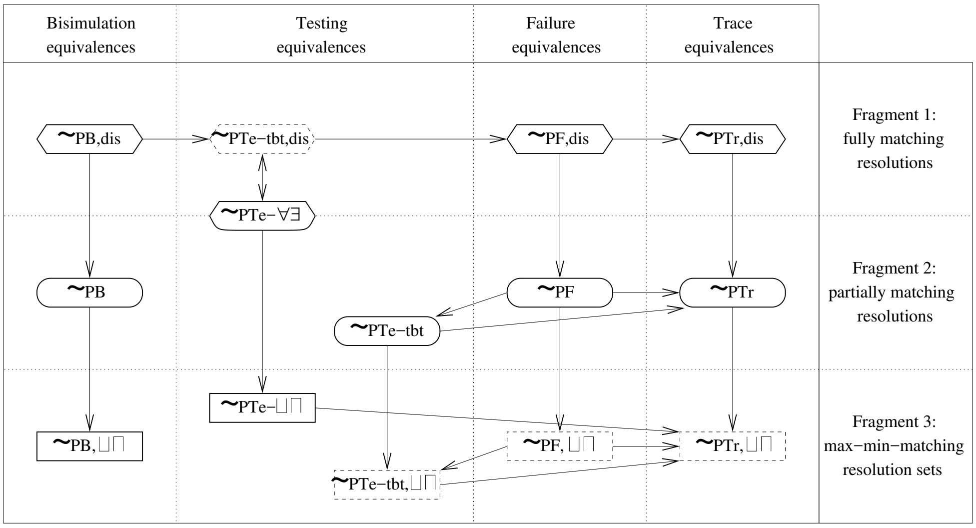

Figure 1: Reduced spectrum of strong behavioral equivalences for NPLTS models (deterministic schedulers)

We have proposed and analyzed three variants of trace, testing, failure, and bisimulation equivalences for NPLTS models. Their relationships are summarized in Fig. 1. In the spectrum, the absence of (chains of) arrows represents incomparability, bidirectional arrows connecting boxes indicate coincidence, and ordinary arrows stand for the strictly-more-discriminating-than relation. Continuous hexagonal boxes contain well known equivalences that compare probability distributions of all equivalence-specific events. In contrast, continuous rounded boxes contain more recent equivalences assigning a weaker role to schedulers that compare separately the probabilities of individual equivalence-specific events. Continuous rectangular boxes instead contain old and new equivalences based on extremal probabilities. The only hybrid box is the one containing $\sim_{\text{PTe-}\forall\exists}$, as this equivalence is half way between the first two definitional approaches. Dashed boxes contain equivalences that we have introduced to better assess the different impact of the approaches themselves.

In this companion paper, following [18] we enlarge the spectrum examined in [1] by additionally considering variants of trace equivalences (completed-trace equivalences), of decorated-trace equivalences (failure-trace, readiness, and ready-trace equivalences), and of bisimulation equivalences (kernels of simulation, completed-simulation, failure-simulation, and ready-simulation preorders). Finally, we show how the spectrum changes when using randomized schedulers in place of deterministic ones.

We refer the reader to [1] for motivations and for the description the three approaches to equivalence definition based on schedulers. Only to guarantee readability, we repeat here the background section of [1] that introduces the necessary terminology about NPLTS models and schedulers.

2. Nondeterministic and Probabilistic Processes

Processes combining nondeterminism and probability are typically described by means of extensions of the LTS model, in which every action-labeled transition goes from a source state to a *probability distribution over target states* rather than to a single target state. They are essentially Markov decision processes [4] and are representative of a number of slightly different probabilistic computational models including internal nondeterminism such as, e.g., concurrent Markov chains [19], alternating probabilistic models [7, 20, 13], probabilistic automata in the sense of [14], and the denotational probabilistic models in [9] (see [16] for an overview). We formalize them as a variant of simple probabilistic automata [14].

Definition 2.1. A *nondeterministic and probabilistic labeled transition system*, NPLTS for short, is a triple $(S, A, \longrightarrow)$ where:

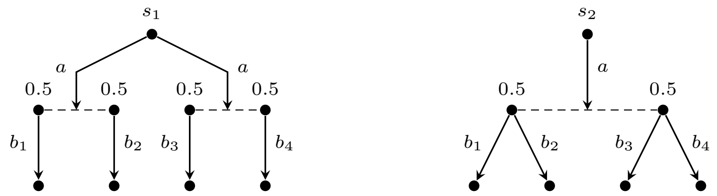

Figure 2: Graphical representation of NPLTS models: two examples

- S is an at most countable set of states.
- A is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times \mathit{Distr}(S)$ is a transition relation, where $\mathit{Distr}(S)$ is the set of discrete probability distributions over S. ■

A transition $(s, a, \mathcal{D})$ is written $s \xrightarrow{a} \mathcal{D}$. We say that $s' \in S$ is not reachable from s via that a-transition if $\mathcal{D}(s') = 0$, otherwise we say that it is reachable with probability $p = \mathcal{D}(s')$. The reachable states form the support of $\mathcal{D}$, i.e., $\mathit{supp}(\mathcal{D}) = \{s' \in S \mid \mathcal{D}(s') > 0\}$. We write $s \xrightarrow{a}$ to indicate that s has an a-transition. The choice among all the transitions departing from s is external and nondeterministic, while the choice of the target state for a specific transition is internal and probabilistic. An NPLTS represents (i) a *fully nondeterministic process* when every transition leads to a distribution that concentrates all the probability mass into a single target state or (ii) a *fully probabilistic process* when every state has at most one outgoing transition.

An NPLTS can be depicted as a directed graph-like structure in which vertices represent states and action-labeled edges represent action-labeled transitions. Given a transition $s \xrightarrow{a} \mathcal{D}$, the corresponding a-labeled edge goes from the vertex representing state s to a set of vertices linked by a dashed line, each of which represents a state $s' \in \mathit{supp}(\mathcal{D})$ and is labeled with $\mathcal{D}(s')$ – label omitted if $\mathcal{D}(s') = 1$. Figure 2 shows two NPLTS models: the one on the left mixes internal nondeterminism and probability, while the one on the right does not.

In this setting, a computation is a sequence of state-to-state steps, each denoted by $s \stackrel{a}{\longmapsto} s'$ and derived from a state-to-distribution transition $s \xrightarrow{a} \mathcal{D}$.

Definition 2.2. Let $\mathcal{L} = (S, A, \longrightarrow)$ be an NPLTS and $s, s' \in S$. We say that:

$$c \;\equiv\; s_0 \stackrel{a_1}{\longmapsto} s_1 \stackrel{a_2}{\longmapsto} s_2 \ldots s_{n-1} \stackrel{a_n}{\longmapsto} s_n$$

is a *computation* of $\mathcal{L}$ of length n from $s = s_0$ to $s' = s_n$ iff for all $i = 1, \ldots, n$ there exists a transition $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ such that $s_i \in \mathit{supp}(\mathcal{D}_i)$, with $\mathcal{D}_i(s_i)$ being the execution probability of step $s_{i-1} \stackrel{a_i}{\longmapsto} s_i$ conditioned on the selection of transition $s_{i-1} \xrightarrow{a_i} \mathcal{D}_i$ of $\mathcal{L}$ at state s_{i-1}. We say that c is *maximal* iff it is not a proper prefix of any other computation. We denote by *first*(c) and *last*(c) the initial state and the final state of c, respectively, and by $\mathcal{C}_{\mathrm{fin}}(s)$ the set of finite-length computations from s. ■

A resolution of a state s of an NPLTS $\mathcal{L}$ is the result of a possible way of resolving nondeterminism starting from s. A resolution is a tree-like structure whose branching points represent probabilistic choices. This is obtained by unfolding from s the graph structure underlying $\mathcal{L}$ and by selecting at each state a single transition of $\mathcal{L}$ (*deterministic scheduler*) or a convex combination of equally labeled transitions of $\mathcal{L}$ (*randomized scheduler*) among all the outgoing transitions of that state. Below, we introduce the notion of resolution arising from a deterministic scheduler as a fully probabilistic NPLTS (randomized schedulers are deferred to Sect. 4). Notice that, when $\mathcal{L}$ is fully nondeterministic, resolutions boil down to computations.

Definition 2.3. Let $\mathcal{L} = (S, A, \longrightarrow)$ be an NPLTS and $s \in S$. We say that an NPLTS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ is a *resolution* of s obtained *via a deterministic scheduler* iff there exists a state correspondence function $\mathit{corr}_{\mathcal{Z}} : Z \to S$ such that $s = \mathit{corr}_{\mathcal{Z}}(z_s)$, for some $z_s \in Z$, and for all $z \in Z$ it holds that:

- If $z \xrightarrow{a}_{\mathcal{Z}} \mathcal{D}$, then $corr_{\mathcal{Z}}(z) \xrightarrow{a} \mathcal{D}'$ with $\mathcal{D}(z') = \mathcal{D}'(corr_{\mathcal{Z}}(z'))$ for all $z' \in Z$.
- If $z \xrightarrow{a_1}_{\mathcal{Z}} \mathcal{D}_1$ and $z \xrightarrow{a_2}_{\mathcal{Z}} \mathcal{D}_2$, then $a_1 = a_2$ and $\mathcal{D}_1 = \mathcal{D}_2$.

We say that $\mathcal{Z}$ is *maximal* iff it cannot be further extended in accordance with the graph structure of $\mathcal{L}$ and the constraints above. We denote by $Res(s)$ the set of resolutions of s obtained via a deterministic scheduler and by $Res_{\max}(s)$ the set of maximal resolutions of s obtained via a deterministic scheduler. Moreover, we attach subscript $\alpha \in A^*$ to those two sets when we restrict attention to resolutions that have no maximal computations corresponding to proper prefixes of α-computations of $\mathcal{L}$. ■

Since $\mathcal{Z} \in Res(s)$ is fully probabilistic, the probability $prob(c)$ of executing $c \in \mathcal{C}_{\text{fin}}(z_s)$ can be defined as the product of the (no longer conditional) execution probabilities of the individual steps of c, with $prob(c)$ being always equal to 1 if $\mathcal{L}$ is fully nondeterministic. This notion is lifted to $C \subseteq \mathcal{C}_{\text{fin}}(z_s)$ by letting $prob(C) = \sum_{c \in C} prob(c)$ whenever none of the computations in C is a proper prefix of one of the others.

3. A Full Spectrum of Strong Behavioral Equivalences

In this section, following [18] we enlarge the spectrum by additionally considering variants of trace equivalences (completed-trace equivalences), further decorated-trace equivalences (failure-trace, readiness, and ready-trace equivalences), and variants of bisimulation equivalences (kernels of simulation, completed-simulation, failure-simulation, and ready-simulation preorders). Finally, we show how the spectrum changes when using randomized schedulers in place of deterministic ones.

3.1. Completed-Trace Equivalences

A variant of trace equivalence that additionally considers completed computations was introduced in the literature of fully nondeterministic models in order to equip trace equivalence with deadlock sensitivity. Given an NPLTS $\mathcal{L} = (S, A, \longrightarrow)$, $s \in S$, $\mathcal{Z} \in Res(s)$, and $\alpha \in A^*$, we recall that $\mathcal{CCC}(z_s, \alpha)$ denotes the set of *completed* α-compatible computations from z_s. In other words, each of these computations c belongs to the set $\mathcal{CC}(z_s, \alpha)$ of α-compatible computations from z_s and is such that $corr_{\mathcal{Z}}(last(c))$ has no outgoing transitions in $\mathcal{L}$.

Definition 3.1. (*Probabilistic completed-trace-distribution equivalence* – $\sim_{\text{PCTr,dis}}$)
$s_1 \sim_{\text{PCTr,dis}} s_2$ iff for each $\mathcal{Z}_1 \in Res(s_1)$ there exist $\mathcal{Z}_2, \mathcal{Z}_2' \in Res(s_2)$ such that <u>for all $\alpha \in A^*$</u>:

$$\begin{aligned} prob(\mathcal{CC}(z_{s_1}, \alpha)) &= prob(\mathcal{CC}(z_{s_2}, \alpha)) \\ prob(\mathcal{CCC}(z_{s_1}, \alpha)) &= prob(\mathcal{CCC}(z'_{s_2}, \alpha)) \end{aligned}$$

and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$. ■

Definition 3.2. (*Probabilistic completed-trace equivalence* – $\sim_{\text{PCTr}}$)
$s_1 \sim_{\text{PCTr}} s_2$ iff <u>for all $\alpha \in A^*$</u> it holds that for each $\mathcal{Z}_1 \in Res(s_1)$ there exist $\mathcal{Z}_2, \mathcal{Z}_2' \in Res(s_2)$ such that:

$$\begin{aligned} prob(\mathcal{CC}(z_{s_1}, \alpha)) &= prob(\mathcal{CC}(z_{s_2}, \alpha)) \\ prob(\mathcal{CCC}(z_{s_1}, \alpha)) &= prob(\mathcal{CCC}(z'_{s_2}, \alpha)) \end{aligned}$$

and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$. ■

Definition 3.3. (*Probabilistic $\sqcup\sqcap$-completed-trace equivalence* – $\sim_{\text{PCTr},\sqcup\sqcap}$)
$s_1 \sim_{\text{PCTr},\sqcup\sqcap} s_2$ iff for all $\alpha \in A^*$:

$$\begin{aligned} \bigsqcup_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{CC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{CC}(z_{s_2}, \alpha)) \\ \bigsqcap_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{CC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{CC}(z_{s_2}, \alpha)) \end{aligned}$$

and:

$$\begin{aligned} \bigsqcup_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{CCC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{CCC}(z_{s_2}, \alpha)) \\ \bigsqcap_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{CCC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{CCC}(z_{s_2}, \alpha)) \end{aligned}$$

■

We now investigate the relationships of the three completed-trace equivalences among themselves and with the various equivalences defined in [1]. As in the fully nondeterministic spectrum [18], completed-trace semantics is comprised between failure semantics and trace semantics. This holds in particular for the completed-trace equivalence based on fully matching resolutions, although completed-trace semantics coincides with trace semantics in the fully probabilistic spectrum [11, 8].

Theorem 3.4. It holds that:

1. $\sim_{\text{PCTr,dis}} \subseteq \sim_{\text{PCTr}} \subseteq \sim_{\text{PCTr},\sqcup\sqcap}$.
2. $\sim_{\text{PF,dis}} \subseteq \sim_{\text{PCTr,dis}} \subseteq \sim_{\text{PTr,dis}}$.
3. $\sim_{\text{PF}} \subseteq \sim_{\text{PCTr}} \subseteq \sim_{\text{PTr}}$.
4. $\sim_{\text{PF},\sqcup\sqcap} \subseteq \sim_{\text{PCTr},\sqcup\sqcap} \subseteq \sim_{\text{PTr},\sqcup\sqcap}$.

Proof Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1. Similar to the proof of Thm. 3.5 in [1].
2. Suppose that $s_1 \sim_{\text{PF,dis}} s_2$. Then we immediately derive that:
   - For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that for all $\alpha \in A^*$:
   $$\begin{aligned} \mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) &= \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, \emptyset))) = \\ &= \mathit{prob}(\mathcal{FCC}(z_{s_2}, (\alpha, \emptyset))) = \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha)) \\ \mathit{prob}(\mathcal{CCC}(z_{s_1}, \alpha)) &= \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, A))) = \\ &= \mathit{prob}(\mathcal{FCC}(z_{s_2}, (\alpha, A))) = \mathit{prob}(\mathcal{CCC}(z_{s_2}, \alpha)) \end{aligned}$$
   - Symmetrically for each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$.

   This means that $s_1 \sim_{\text{PCTr,dis}} s_2$.
   The fact that $s_1 \sim_{\text{PCTr,dis}} s_2$ implies $s_1 \sim_{\text{PTr,dis}} s_2$ is a straightforward consequence of the definition of the two equivalences.
3. Suppose that $s_1 \sim_{\text{PF}} s_2$. Then we immediately derive that for all $\alpha \in A^*$:
   - For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exist $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that:
   $$\begin{aligned} \mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) &= \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, \emptyset))) = \\ &= \mathit{prob}(\mathcal{FCC}(z_{s_2}, (\alpha, \emptyset))) = \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha)) \end{aligned}$$
   and $\mathcal{Z}'_2 \in \mathit{Res}(s_2)$ such that:
   $$\begin{aligned} \mathit{prob}(\mathcal{CCC}(z_{s_1}, \alpha)) &= \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, A))) = \\ &= \mathit{prob}(\mathcal{FCC}(z'_{s_2}, (\alpha, A))) = \mathit{prob}(\mathcal{CCC}(z'_{s_2}, \alpha)) \end{aligned}$$
   - Symmetrically for each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$.

   This means that $s_1 \sim_{\text{PCTr}} s_2$.
   The fact that $s_1 \sim_{\text{PCTr}} s_2$ implies $s_1 \sim_{\text{PTr}} s_2$ is a straightforward consequence of the definition of the two equivalences.
4. Suppose that $s_1 \sim_{\text{PF},\sqcup\sqcap} s_2$. Then we immediately derive that for all $\alpha \in A^*$:
   $$\begin{aligned} \bigsqcup_{\mathcal{Z}_1 \in \mathit{Res}_\alpha(s_1)} \mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathcal{Z}_1 \in \mathit{Res}_\alpha(s_1)} \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, \emptyset))) \\ &= \bigsqcup_{\mathcal{Z}_2 \in \mathit{Res}_\alpha(s_2)} \mathit{prob}(\mathcal{FCC}(z_{s_2}, (\alpha, \emptyset))) \\ &= \bigsqcup_{\mathcal{Z}_2 \in \mathit{Res}_\alpha(s_2)} \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha)) \\ \bigsqcap_{\mathcal{Z}_1 \in \mathit{Res}_\alpha(s_1)} \mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathcal{Z}_1 \in \mathit{Res}_\alpha(s_1)} \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, \emptyset))) \\ &= \bigsqcap_{\mathcal{Z}_2 \in \mathit{Res}_\alpha(s_2)} \mathit{prob}(\mathcal{FCC}(z_{s_2}, (\alpha, \emptyset))) \\ &= \bigsqcap_{\mathcal{Z}_2 \in \mathit{Res}_\alpha(s_2)} \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha)) \end{aligned}$$

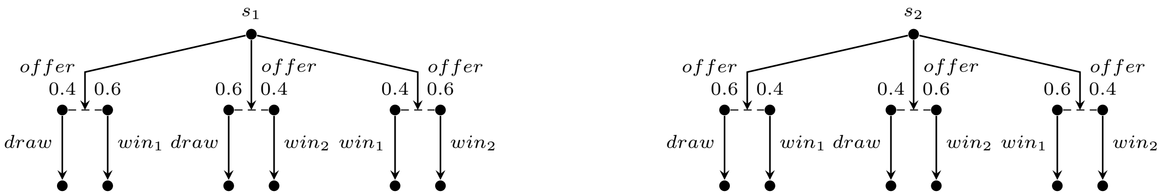

Figure 3: Two NPLTS models distinguished by equivalences in fragment 1 and identified by those in fragment 2

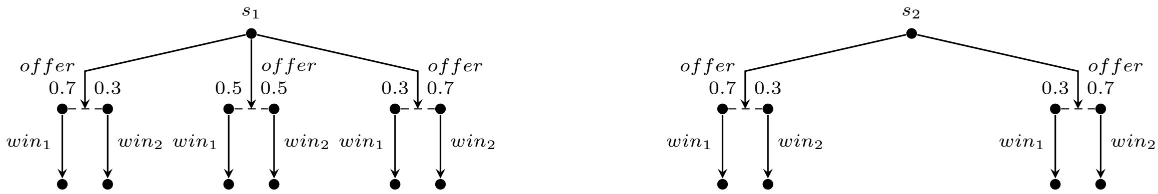

Figure 4: Two NPLTS models distinguished by equivalences in fragment 2 and identified by those in fragment 3

and:

$$\begin{aligned}
\bigsqcup_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{CCC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{FCC}(z_{s_1}, (\alpha, A))) \\
&= \bigsqcup_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{FCC}(z_{s_2}, (\alpha, A))) \\
&= \bigsqcup_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{CCC}(z_{s_2}, \alpha)) \\
\bigsqcap_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{CCC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{FCC}(z_{s_1}, (\alpha, A))) \\
&= \bigsqcap_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{FCC}(z_{s_2}, (\alpha, A))) \\
&= \bigsqcap_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{CCC}(z_{s_2}, \alpha))
\end{aligned}$$

This means that $s_1 \sim_{\mathrm{PCTr},\sqcup\sqcap} s_2$.
The fact that $s_1 \sim_{\mathrm{PCTr},\sqcup\sqcap} s_2$ implies $s_1 \sim_{\mathrm{PTr},\sqcup\sqcap} s_2$ is a straightforward consequence of the definition of the two equivalences. ■

All the inclusions in Thm. 3.4 are strict:

- Figures 3 and 4 respectively show that $\sim_{\mathrm{PCTr,dis}}$ is strictly finer than $\sim_{\mathrm{PCTr}}$ and $\sim_{\mathrm{PCTr}}$ is strictly finer than $\sim_{\mathrm{PCTr},\sqcup\sqcap}$.
- Figure 5 shows that $\sim_{\mathrm{PF,dis}}$, $\sim_{\mathrm{PF}}$, and $\sim_{\mathrm{PF},\sqcup\sqcap}$ are strictly finer than $\sim_{\mathrm{PCTr,dis}}$, $\sim_{\mathrm{PCTr}}$, and $\sim_{\mathrm{PCTr},\sqcup\sqcap}$, respectively. Indeed, for each resolution of s_1 (resp. s_2) there exists a resolution of s_2 (resp. s_1) such that both resolutions have precisely the same trace distribution and the same completed-trace distribution, thus s_1 and s_2 are identified by $\sim_{\mathrm{PCTr,dis}}$ (and hence by $\sim_{\mathrm{PCTr}}$ and $\sim_{\mathrm{PCTr},\sqcup\sqcap}$). In contrast, the leftmost a-computation of s_1 is compatible with the failure pair $(a, \{c\})$ while s_2 has no computation compatible with that failure pair, thus s_1 and s_2 are distinguished by $\sim_{\mathrm{PF},\sqcup\sqcap}$ (and hence by $\sim_{\mathrm{PF}}$ and $\sim_{\mathrm{PF,dis}}$).
- Figure 6 shows that $\sim_{\mathrm{PCTr,dis}}$, $\sim_{\mathrm{PCTr}}$, and $\sim_{\mathrm{PCTr},\sqcup\sqcap}$ are strictly finer than $\sim_{\mathrm{PTr,dis}}$, $\sim_{\mathrm{PTr}}$, and $\sim_{\mathrm{PTr},\sqcup\sqcap}$, respectively. Indeed, for each resolution of s_1 (resp. s_2) there exists a resolution of s_2 (resp. s_1) such that both resolutions have precisely the same trace distribution, thus s_1 and s_2 are identified by $\sim_{\mathrm{PTr,dis}}$ (and hence by $\sim_{\mathrm{PTr}}$ and $\sim_{\mathrm{PTr},\sqcup\sqcap}$). In contrast, the rightmost a-computation of s_1 is completed while

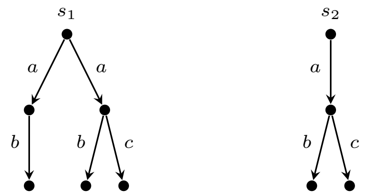

Figure 5: Two NPLTS models distinguished by $\sim_{\rm PF,dis}/\sim_{\rm PF}/\sim_{\rm PF,\sqcup\sqcap}$ and identified by $\sim_{\rm PCTr,dis}/\sim_{\rm PCTr}/\sim_{\rm PCTr,\sqcup\sqcap}$

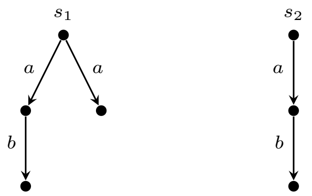

Figure 6: Two NPLTS models distinguished by $\sim_{\rm PCTr,dis}/\sim_{\rm PCTr}/\sim_{\rm PCTr,\sqcup\sqcap}$ and identified by $\sim_{\rm PTr,dis}/\sim_{\rm PTr}/\sim_{\rm PTr,\sqcup\sqcap}$

s_2 has no completed a-compatible computation, thus s_1 and s_2 are distinguished by $\sim_{\rm PCTr,\sqcup\sqcap}$ (and hence by $\sim_{\rm PCTr}$ and $\sim_{\rm PCTr,dis}$).

Moreover:

- $\sim_{\rm PB}$ and $\sim_{\rm PB,\sqcup\sqcap}$ are incomparable with the three completed-trace equivalences. Indeed, in Fig. 7 it holds that $s_1 \sim_{\rm PB} s_2$ (and hence $s_1 \sim_{\rm PB,\sqcup\sqcap} s_2$) – as can be seen by taking the equivalence relation that pairs states having equally labeled transitions leading to the same distribution – and $s_1 \not\sim_{\rm PCTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PCTr} s_2$ and $s_1 \not\sim_{\rm PCTr,dis} s_2$) – due to the trace $a\,b\,c$ having maximum probability 0.68 in the first process and 0.61 in the second process. In contrast, in Fig. 8 it holds that $s_1 \not\sim_{\rm PB,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PB} s_2$) – as the leftmost state with outgoing b-transitions reachable from s_2 is not $\sqcup\sqcap$-bisimilar to the two states with outgoing b-transitions reachable from s_1 – and $s_1 \sim_{\rm PCTr,dis} s_2$ (and hence $s_1 \sim_{\rm PCTr} s_2$ and $s_1 \sim_{\rm PCTr,\sqcup\sqcap} s_2$).

- $\sim_{\rm PTe\text{-}\sqcup\sqcap}$ is incomparable with the three completed-trace equivalences. Indeed, in Fig. 4 it holds that $s_1 \sim_{\rm PTe\text{-}\sqcup\sqcap} s_2$ and $s_1 \not\sim_{\rm PCTr} s_2$ (and hence $s_1 \not\sim_{\rm PCTr,dis}$). In contrast, in Fig. 10 it holds that $s_1 \not\sim_{\rm PTe\text{-}\sqcup\sqcap} s_2$ – due to the test shown in the figure – and $s_1 \sim_{\rm PCTr,dis} s_2$ (and hence $s_1 \sim_{\rm PCTr} s_2$). Likewise, in Fig. 9 it holds that $s_1 \sim_{\rm PTe\text{-}\sqcup\sqcap} s_2$ – as there is no test that results in an interaction system having a maximal resolution with differently labeled successful computations of the same length and hence no possibility of summing up their success probabilities – and $s_1 \not\sim_{\rm PCTr,\sqcup\sqcap} s_2$ – due to the completed trace $a\,b$ whose maximum probability is 0.24 in the first process and 0.21 in the second process. In contrast, in Fig. 3 it holds that $s_1 \not\sim_{\rm PTe\text{-}\sqcup\sqcap} s_2$ and $s_1 \sim_{\rm PCTr,\sqcup\sqcap} s_2$.

- $\sim_{\rm PTe\text{-}tbt}$ and $\sim_{\rm PTe\text{-}tbt,\sqcup\sqcap}$ are incomparable with the three completed-trace equivalences, because in Fig. 11 it holds that $s_1 \sim_{\rm PTe\text{-}tbt} s_2$ (and hence $s_1 \sim_{\rm PTe\text{-}tbt,\sqcup\sqcap} s_2$) and $s_1 \not\sim_{\rm PCTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PCTr} s_2$ and $s_1 \not\sim_{\rm PCTr,dis} s_2$), while in Fig. 12 it holds that $s_1 \not\sim_{\rm PTe\text{-}tbt,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PTe\text{-}tbt} s_2$) and $s_1 \sim_{\rm PCTr,dis} s_2$ (and hence $s_1 \sim_{\rm PCTr} s_2$ and $s_1 \sim_{\rm PCTr,\sqcup\sqcap} s_2$).

- $\sim_{\rm PF}$ and $\sim_{\rm PF,\sqcup\sqcap}$ are incomparable with $\sim_{\rm PCTr,dis}$, because in Fig. 3 it holds that $s_1 \sim_{\rm PF} s_2$ (and hence $s_1 \sim_{\rm PF,\sqcup\sqcap} s_2$) and $s_1 \not\sim_{\rm PCTr,dis} s_2$, while in Fig. 5 it holds that $s_1 \not\sim_{\rm PF,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PF} s_2$) and $s_1 \sim_{\rm PCTr,dis} s_2$.

- $\sim_{\rm PF,\sqcup\sqcap}$ is incomparable with $\sim_{\rm PCTr}$, because in Fig. 4 it holds that $s_1 \sim_{\rm PF,\sqcup\sqcap} s_2$ and $s_1 \not\sim_{\rm PCTr} s_2$, while in Fig. 5 it holds that $s_1 \not\sim_{\rm PF,\sqcup\sqcap} s_2$ and $s_1 \sim_{\rm PCTr} s_2$.

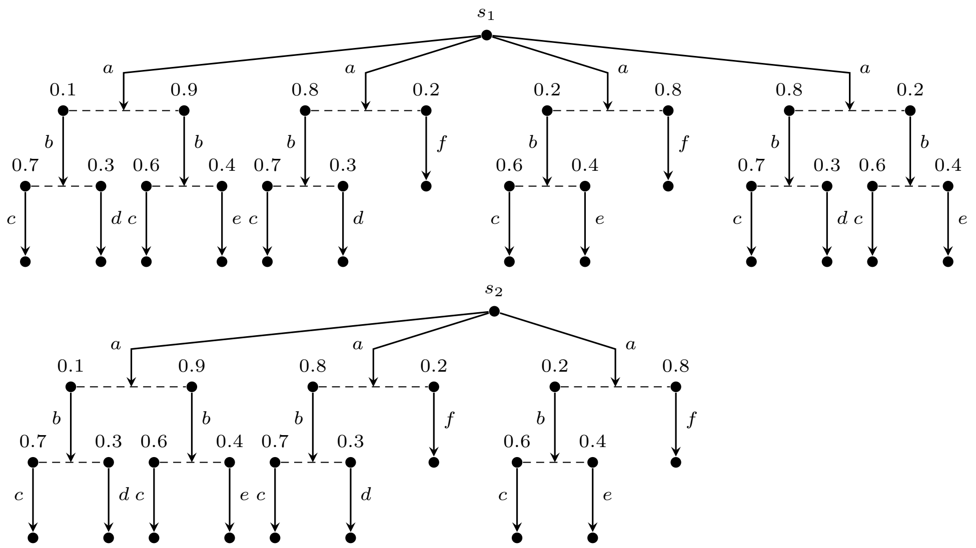

Figure 7: Two NPLTS models distinguished by $\sim_{\mathrm{PCTr,dis}}/\sim_{\mathrm{PCTr}}/\sim_{\mathrm{PCTr},\sqcup\sqcap}$ and identified by $\sim_{\mathrm{PB}}/\sim_{\mathrm{PB},\sqcup\sqcap}$

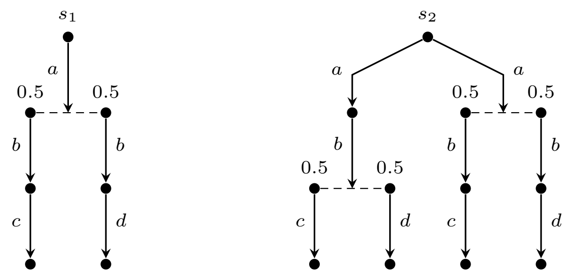

Figure 8: Two NPLTS models distinguished by $\sim_{\mathrm{PB}}/\sim_{\mathrm{PB},\sqcup\sqcap}$ and identified by $\sim_{\mathrm{PCTr,dis}}/\sim_{\mathrm{PCTr}}/\sim_{\mathrm{PCTr},\sqcup\sqcap}$

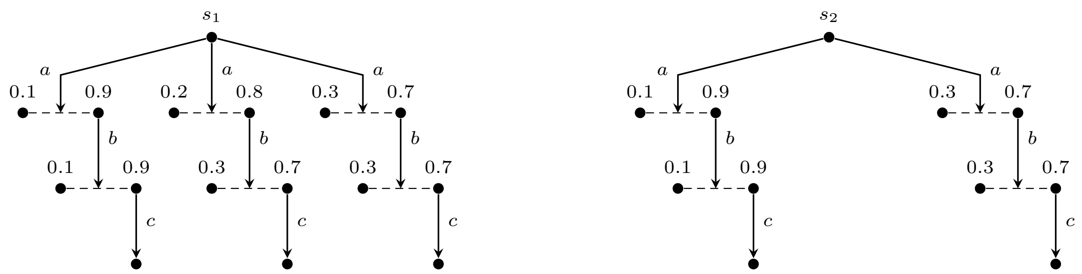

Figure 9: Two NPLTS models distinguished by $\sim_{\mathrm{PCTr},\sqcup\sqcap}$ and identified by $\sim_{\mathrm{PTe}\text{-}\sqcup\sqcap}$

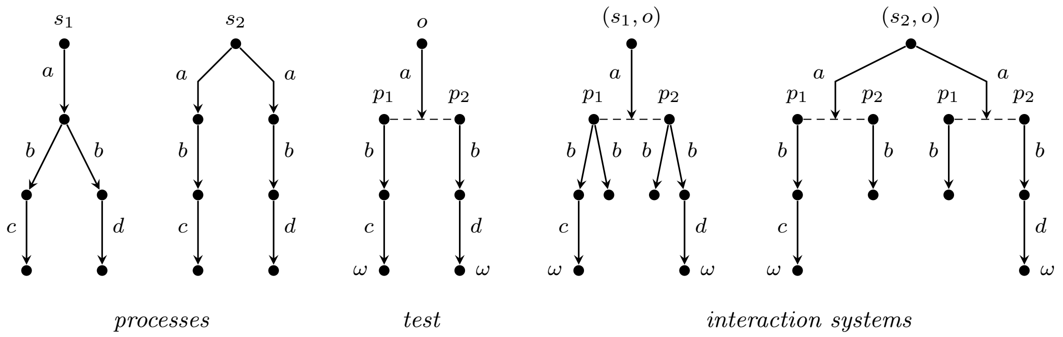

Figure 10: Two NPLTS models distinguished by $\sim_{\mathrm{PTe}\text{-}\sqcup\sqcap}$ and identified by $\sim_{\mathrm{PCTr,dis}}/\sim_{\mathrm{PCTr}}$

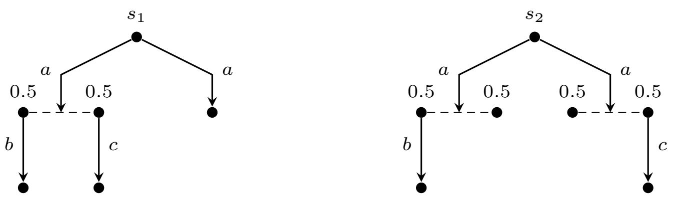

Figure 11: Two NPLTS models distinguished by $\sim_{\mathrm{PCTr,dis}}/\sim_{\mathrm{PCTr}}/\sim_{\mathrm{PCTr},\sqcup\sqcap}$ and identified by $\sim_{\mathrm{PTe\text{-}tbt}}/\sim_{\mathrm{PTe\text{-}tbt},\sqcup\sqcap}$

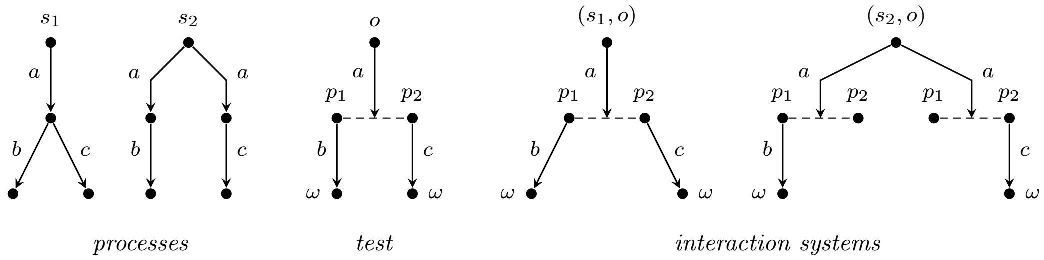

Figure 12: Two NPLTS models distinguished by $\sim_{\mathrm{PTe\text{-}tbt}}/\sim_{\mathrm{PTe\text{-}tbt},\sqcup\sqcap}$ and identified by $\sim_{\mathrm{PCTr,dis}}/\sim_{\mathrm{PCTr}}/\sim_{\mathrm{PCTr},\sqcup\sqcap}$

- $\sim_{\rm PCTr}$ and $\sim_{\rm PCTr,\sqcup\sqcap}$ are incomparable with $\sim_{\rm PTr,dis}$, because in Fig. 3 it holds that $s_1 \sim_{\rm PCTr} s_2$ (and hence $s_1 \sim_{\rm PCTr,\sqcup\sqcap} s_2$) and $s_1 \not\sim_{\rm PTr,dis} s_2$, while in Fig. 6 it holds that $s_1 \not\sim_{\rm PCTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PCTr} s_2$) and $s_1 \sim_{\rm PTr,dis} s_2$.
- $\sim_{\rm PCTr,\sqcup\sqcap}$ is incomparable with $\sim_{\rm PTr}$, because in Fig. 4 it holds that $s_1 \sim_{\rm PCTr,\sqcup\sqcap} s_2$ and $s_1 \not\sim_{\rm PTr} s_2$, while in Fig. 6 it holds that $s_1 \not\sim_{\rm PCTr,\sqcup\sqcap} s_2$ and $s_1 \sim_{\rm PTr} s_2$.

3.2. Failure-Trace, Readiness, and Ready-Trace Equivalences

Failure semantics generalizes completed-trace equivalence towards arbitrary safety properties. An extension of failure semantics is failure-trace semantics. We call *failure trace* an element $\phi \in (A \times 2^A)^*$ given by a sequence of $n \in \mathbb{N}$ pairs of the form (a_i, F_i). We say that $c \in \mathcal{C}_{\rm fin}(z_s)$ is *compatible* with ϕ iff $c \in \mathcal{CC}(z_s, a_1 \dots a_n)$ and, denoting by z_i the state reached by c after the i-th step for all $i = 1, \dots, n$, $corr_{\mathcal{Z}}(z_i)$ has no outgoing transitions in $\mathcal{L}$ labeled with an action in F_i. We denote by $\mathcal{FTCC}(z_s, \phi)$ the set of ϕ-compatible computations from z_s.

Definition 3.5. (*Probabilistic failure-trace-distribution equivalence* – $\sim_{\rm PFTr,dis}$)
$s_1 \sim_{\rm PFTr,dis} s_2$ iff for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that $\underline{\text{for all } \phi \in (A \times 2^A)^*}$:
$$prob(\mathcal{FTCC}(z_{s_1}, \phi)) \; = \; prob(\mathcal{FTCC}(z_{s_2}, \phi))$$
and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$. ∎

Definition 3.6. (*Probabilistic failure-trace equivalence* – $\sim_{\rm PFTr}$)
$s_1 \sim_{\rm PFTr} s_2$ iff $\underline{\text{for all } \phi \in (A \times 2^A)^*}$ it holds that for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that:
$$prob(\mathcal{FTCC}(z_{s_1}, \phi)) \; = \; prob(\mathcal{FTCC}(z_{s_2}, \phi))$$
and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$. ∎

Definition 3.7. (*Probabilistic $\sqcup\sqcap$-failure-trace equivalence* – $\sim_{\rm PFTr,\sqcup\sqcap}$)
$s_1 \sim_{\rm PF,\sqcup\sqcap} s_2$ iff for all $\phi \in (A \times 2^A)^*$:
$$\bigsqcup_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{FTCC}(z_{s_1}, \phi)) \; = \; \bigsqcup_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{FTCC}(z_{s_2}, \phi))$$
$$\bigsqcap_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{FTCC}(z_{s_1}, \phi)) \; = \; \bigsqcap_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{FTCC}(z_{s_2}, \phi))$$
∎

A different generalization towards liveness properties is readiness semantics, which considers the set of actions that can be accepted after performing a trace. We call *ready pair* an element $\varrho \in A^* \times 2^A$ formed by a trace α and a decoration R called *ready set*. We say that c is *compatible* with ϱ iff $c \in \mathcal{CC}(z_s, \alpha)$ and the set of actions labeling the transitions in $\mathcal{L}$ departing from $corr_{\mathcal{Z}}(last(c))$ is precisely R. We denote by $\mathcal{RCC}(z_s, \varrho)$ the set of ϱ-compatible computations from z_s.

Definition 3.8. (*Probabilistic readiness-distribution equivalence* – $\sim_{\rm PR,dis}$)
$s_1 \sim_{\rm PR,dis} s_2$ iff for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that $\underline{\text{for all } \varrho \in A^* \times 2^A}$:
$$prob(\mathcal{RCC}(z_{s_1}, \varrho)) \; = \; prob(\mathcal{RCC}(z_{s_2}, \varrho))$$
and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$. ∎

Definition 3.9. (*Probabilistic readiness equivalence* – $\sim_{\rm PR}$)
$s_1 \sim_{\rm PR} s_2$ iff $\underline{\text{for all } \varrho \in A^* \times 2^A}$ it holds that for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that:
$$prob(\mathcal{RCC}(z_{s_1}, \varrho)) \; = \; prob(\mathcal{RCC}(z_{s_2}, \varrho))$$
and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$. ∎

Definition 3.10. (*Probabilistic $\sqcup\sqcap$-readiness equivalence* – $\sim_{\rm PR,\sqcup\sqcap}$)
$s_1 \sim_{\rm PR,\sqcup\sqcap} s_2$ iff for all $\varrho = (\alpha, R) \in A^* \times 2^A$:
$$\bigsqcup_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{RCC}(z_{s_1}, \varrho)) \; = \; \bigsqcup_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{RCC}(z_{s_2}, \varrho))$$
$$\bigsqcap_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{RCC}(z_{s_1}, \varrho)) \; = \; \bigsqcap_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{RCC}(z_{s_2}, \varrho))$$
∎

Moreover, we call *ready trace* an element $\rho \in (A \times 2^A)^*$ given by a sequence of $n \in \mathbb{N}$ pairs of the form (a_i, R_i). We say that $c \in \mathcal{C}_{\text{fin}}(z_s)$ is *compatible* with ρ iff $c \in \mathcal{CC}(z_s, a_1 \dots a_n)$ and, denoting by z_i the state reached by c after the i-th step for all $i = 1, \dots, n$, the set of actions labeling the transitions in $\mathcal{L}$ departing from $corr_{\mathcal{Z}}(z_i)$ is precisely R_i. We denote by $\mathcal{RTCC}(z_s, \rho)$ the set of ρ-compatible computations from z_s.

Definition 3.11. (*Probabilistic ready-trace-distribution equivalence* – $\sim_{\text{PRTr,dis}}$)
$s_1 \sim_{\text{PRTr,dis}} s_2$ iff for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that $\underline{\text{for all } \rho \in (A \times 2^A)^*}$:

$$prob(\mathcal{RTCC}(z_{s_1}, \rho)) \; = \; prob(\mathcal{RTCC}(z_{s_2}, \rho))$$

and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$. ■

Definition 3.12. (*Probabilistic ready-trace equivalence* – $\sim_{\text{PRTr}}$)
$s_1 \sim_{\text{PRTr}} s_2$ iff $\underline{\text{for all } \rho \in (A \times 2^A)^*}$ it holds that for each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that:

$$prob(\mathcal{RTCC}(z_{s_1}, \rho)) \; = \; prob(\mathcal{RTCC}(z_{s_2}, \rho))$$

and symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$. ■

Definition 3.13. (*Probabilistic $\sqcup\sqcap$-ready-trace equivalence* – $\sim_{\text{PRTr},\sqcup\sqcap}$)
$s_1 \sim_{\text{PRTr},\sqcup\sqcap} s_2$ iff for all $\rho \in (A \times 2^A)^*$:

$$\bigsqcup_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{RTCC}(z_{s_1}, \rho)) \; = \; \bigsqcup_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{RTCC}(z_{s_2}, \rho))$$

$$\bigsqcap_{\mathcal{Z}_1 \in Res_\alpha(s_1)} prob(\mathcal{RTCC}(z_{s_1}, \rho)) \; = \; \bigsqcap_{\mathcal{Z}_2 \in Res_\alpha(s_2)} prob(\mathcal{RTCC}(z_{s_2}, \rho))$$

■

We now investigate the relationships of the nine additional decorated-trace equivalences among themselves and with the various equivalences defined in [1] and in this paper. As in the fully probabilistic spectrum [11, 8], for the decorated-trace equivalences based on fully matching resolutions it holds that readiness semantics coincides with failure semantics, and this extends to ready-trace semantics and failure-trace semantics. In contrast, for the other decorated-trace equivalences based on partially matching resolutions or extremal probabilities, unlike the fully nondeterministic spectrum [18] it turns out that ready-trace semantics and readiness semantic are incomparable with most of the other semantics.

Theorem 3.14. It holds that:

1. $\sim_{\pi,\text{dis}} \subseteq \sim_{\pi} \subseteq \sim_{\pi,\sqcup\sqcap}$ for all $\pi \in \{\text{PRTr}, \text{PFTr}, \text{PR}\}$.
2. $\sim_{\text{PTe-tbt,dis}} \subseteq \sim_{\text{PRTr,dis}}$.
3. $\sim_{\text{PRTr,dis}} = \sim_{\text{PFTr,dis}}$ over finitely-branching NPLTS models.
4. $\sim_{\text{PR,dis}} = \sim_{\text{PF,dis}}$ over finitely-branching NPLTS models.
5. $\sim_{\text{PFTr,dis}} \subseteq \sim_{\text{PF,dis}}$.
6. $\sim_{\text{PFTr}} \subseteq \sim_{\text{PF}}$.
7. $\sim_{\text{PFTr},\sqcup\sqcap} \subseteq \sim_{\text{PF},\sqcup\sqcap}$.

Proof Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1. Similar to the proof of Thm. 3.5 in [1].
2. We show that $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ implies $s_1 \sim_{\text{PRTr,dis}} s_2$ by building a test that permits to reason about all ready traces at once for each resolution of s_1 and s_2. We start by deriving a new NPLTS $(S_{\text{r}}, A_{\text{r}}, \longrightarrow_{\text{r}})$ that is isomorphic to the given one up to transition labels and terminal states. A transition $s \xrightarrow{a} \mathcal{D}$ becomes $s_{\text{r}} \xrightarrow{a \triangleleft R}_{\text{r}} \mathcal{D}_{\text{r}}$ where $R \subseteq A$ is the set of actions labeling the outgoing transitions of s and $\mathcal{D}_{\text{r}}(s_{\text{r}}) = \mathcal{D}(s)$ for all $s \in S$. If s is a terminal state, i.e., it has no outgoing transitions, then we add a transition $s_{\text{r}} \xrightarrow{\circ \triangleleft \emptyset}_{\text{r}} \delta_{s_{\text{r}}}$ where $\delta_{s_{\text{r}}}(s_{\text{r}}) = 1$ and $\delta_{s_{\text{r}}}(s'_{\text{r}}) = 0$ for all $s' \in S \setminus \{s\}$. Transition relabeling preserves $\sim_{\text{PTe-tbt,dis}}$, i.e., $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ implies $s_{1,\text{r}} \sim_{\text{PTe-tbt,dis}} s_{2,\text{r}}$, because $\sim_{\text{PTe-tbt,dis}}$ is able to distinguish a state that has a single α-compatible computation reaching a state with a nondeterministic branching formed by a b-transition and a c-transition, from a state that has two α-compatible computations such that one of them reaches a state with only one outgoing transition labeled with b

and the other one reaches a state with only one outgoing transition labeled with c (e.g., use a test that has a single α-compatible computation whose last step leads to a distribution whose support contains only a state with only one outgoing transition labeled with b that reaches success and a state with only one outgoing transition labeled with c that reaches success).

For each $\alpha_{\rm r} \in (A_{\rm r})^*$ and $R \subseteq A$, we build an NPT $\mathcal{T}_{\alpha_{\rm r}, R} = (O_{\alpha_{\rm r}, R}, A_{\rm r}, \longrightarrow_{\alpha_{\rm r}, R})$ having a single $\alpha_{\rm r}$-compatible computation that goes from the initial state $o_{\alpha_{\rm r}, R}$ to a state having a single transition to ω labeled with (i) $\circ \triangleleft \emptyset$ if $R = \emptyset$ or (ii) $\_ \triangleleft R$ if $R \neq \emptyset$. Since we compare individual states (like s_1 and s_2) rather than state distributions, the distinguishing power of $\sim_{\rm PTe\text{-}tbt,dis}$ does not change if we additionally consider tests starting with a single τ-transition that can initially evolve autonomously in any interaction system. We thus build a further NPT $\mathcal{T} = (O, A_{\rm r}, \longrightarrow_{\mathcal{T}})$ that has an initial τ-transition and then behaves as one of the tests $\mathcal{T}_{\alpha_{\rm r}, R}$, i.e., its initial τ-transition goes from the initial state o to a state distribution whose support is the set $\{o_{\alpha_{\rm r}, R} \mid \alpha_{\rm r} \in (A_{\rm r})^* \wedge R \subseteq A\}$, with the probability $p_{\alpha_{\rm r}, R}$ associated with $o_{\alpha_{\rm r}, R}$ being taken from the distribution whose values are of the form $1/2^i$, $i \in \mathbb{N}_{>0}$. Note that $\mathcal{T}$ is not finite state, but this affects only the initial step, whose only purpose is to internally select a specific ready trace.

After this step, $\mathcal{T}$ interacts with the process under test. Let $\rho \in (A \times 2^A)^*$ be a ready trace of the form $(a_1, R_1) \dots (a_n, R_n)$, where $n \in \mathbb{N}$. Given $s \in S$, consider the trace $\alpha_{\rho, \rm r} \in (A_{\rm r})^*$ of length $n + 1$ in which the first element is $a_1 \triangleleft R$, with $R \subseteq A$ being the set of actions labeling the outgoing transitions of s, the subsequent elements are of the form $a_i \triangleleft R_{i-1}$ for $i = 2, \dots, n$, and the last element is (i) $\circ \triangleleft \emptyset$ if $R_n = \emptyset$ or (ii) $\_ \triangleleft R_n$ if $R_n \neq \emptyset$. Then for all $\mathcal{Z} \in Res(s)$ it holds that:

$$prob(\mathcal{RTCC}(z_s, \rho)) \; = \; 0$$

if there is no $a_1 \dots a_n$-compatible computation from z_s, otherwise:

$$prob(\mathcal{RTCC}(z_s, \rho)) \; = \; prob(\mathcal{SCC}(z_{s_{\rm r}, o}, \alpha_{\rho, \rm r})) / p_{\alpha'_{\rho, \rm r}, R_n}$$

where $\alpha'_{\rho, \rm r}$ is $\alpha_{\rho, \rm r}$ without its last element.

Suppose that $s_1 \sim_{\rm PTe\text{-}tbt,dis} s_2$, which implies that s_1 and s_2 have the same set R of actions labeling their outgoing transitions and $s_{1, \rm r} \sim_{\rm PTe\text{-}tbt,dis} s_{2, \rm r}$. Then:

- For each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that for all ready traces $\rho = (a_1, R_1) \dots (a_n, R_n) \in (A \times 2^A)^*$ either:
$$prob(\mathcal{RTCC}(z_{s_1}, \rho)) \; = \; 0 \; = \; prob(\mathcal{RTCC}(z_{s_2}, \rho))$$
or:
$$\begin{aligned} prob(\mathcal{RTCC}(z_{s_1}, \rho)) \; &= \; prob(\mathcal{SCC}(z_{s_{1, \rm r}, o}, \alpha_{\rho, \rm r})) / p_{\alpha'_{\rho, \rm r}, R_n} \; = \\ &= \; prob(\mathcal{SCC}(z_{s_{2, \rm r}, o}, \alpha_{\rho, \rm r})) / p_{\alpha'_{\rho, \rm r}, R_n} \; = \; prob(\mathcal{RTCC}(z_{s_2}, \rho)) \end{aligned}$$
- Symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$.

This means that $s_1 \sim_{\rm PRTr,dis} s_2$.

3. We preliminarily observe that for all $s \in S$, $\mathcal{Z} \in Res(s)$, $n \in \mathbb{N}$, $\alpha = a_1 \dots a_n \in A^*$, and $F_1, \dots, F_n$, $R_1, \dots, R_n \in 2^A$ it holds that:
$$\begin{aligned} & prob(\mathcal{FTCC}(z_s, (a_1, F_1) \dots (a_n, F_n))) \; = \\ & \quad = \; \textstyle\sum_{R'_1, \dots, R'_n \in 2^A \text{ s.t. } R'_i \cap F_i = \emptyset \text{ for all } i = 1, \dots, n} prob(\mathcal{RTCC}(z_s, (a_1, R'_1) \dots (a_n, R'_n))) \\ & prob(\mathcal{RTCC}(z_s, (a_1, R_1) \dots (a_n, R_n))) \; = \; prob(\mathcal{FTCC}(z_s, (a_1, \overline{R}_1) \dots (a_n, \overline{R}_n))) \\ & \quad - \textstyle\sum_{R'_1, \dots, R'_n \in 2^A \text{ s.t. } R'_i \subseteq R_i \text{ for all } i = 1, \dots, n}^{R'_j \subset R_J \text{ for some } j = 1, \dots, n} prob(\mathcal{RTCC}(z_s, (a_1, R'_1) \dots (a_n, R'_n))) \end{aligned}$$
where $\overline{R}_i = A \setminus R_i$ for all $i = 1, \dots, n$.
Suppose that $s_1 \sim_{\rm PRTr,dis} s_2$. Then we immediately derive that:

- For each $\mathcal{Z}_1 \in Res(s_1)$ there exists $\mathcal{Z}_2 \in Res(s_2)$ such that for all $(a_1, F_1) \dots (a_n, F_n) \in (A \times 2^A)^*$:
$$\begin{aligned} & prob(\mathcal{FTCC}(z_{s_1}, (a_1, F_1) \dots (a_n, F_n))) \; = \\ & \quad = \; \textstyle\sum_{R'_1, \dots, R'_n \in 2^A \text{ s.t. } R'_i \cap F_i = \emptyset \text{ for all } i = 1, \dots, n} prob(\mathcal{RTCC}(z_{s_1}, (a_1, R'_1) \dots (a_n, R'_n))) \\ & \quad = \; \textstyle\sum_{R'_1, \dots, R'_n \in 2^A \text{ s.t. } R'_i \cap F_i = \emptyset \text{ for all } i = 1, \dots, n} prob(\mathcal{RTCC}(z_{s_2}, (a_1, R'_1) \dots (a_n, R'_n))) \\ & \quad = \; prob(\mathcal{FTCC}(z_{s_2}, (a_1, F_1) \dots (a_n, F_n))) \end{aligned}$$
- Symmetrically for each $\mathcal{Z}_2 \in Res(s_2)$.

This means that $s_1 \sim_{\mathrm{PFTr,dis}} s_2$.
Suppose now that $s_1 \sim_{\mathrm{PFTr,dis}} s_2$. For each ready trace $\rho \in (A \times 2^A)^*$ including at least one infinite ready set, it trivially holds that for all $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ and $\mathcal{Z}_2 \in \mathit{Res}(s_2)$:

$$\mathit{prob}(\mathcal{RTCC}(z_{s_1}, \rho)) \;=\; 0 \;=\; \mathit{prob}(\mathcal{RTCC}(z_{s_2}, \rho))$$

whenever the considered NPLTS is finitely branching. Thus, in order to prove that $s_1 \sim_{\mathrm{PRTr,dis}} s_2$, we can restrict ourselves to ready traces including only finite ready sets. Given an arbitrary $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ that is matched by some $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ according to $\sim_{\mathrm{PFTr,dis}}$, we show that the matching holds also under $\sim_{\mathrm{PRTr,dis}}$ by proceeding by induction on the sum $k \in \mathbb{N}$ of the cardinalities of the ready sets occurring in ready traces including only finite ready sets:

- Let $k = 0$, i.e., consider ready traces whose ready sets are all empty. Then for all $\alpha = a_1 \dots a_n \in A^*$:
$$\begin{aligned} \mathit{prob}(\mathcal{RTCC}(z_{s_1}, (a_1, \emptyset) \dots (a_n, \emptyset))) \;&=\; \mathit{prob}(\mathcal{FTCC}(z_{s_1}, (a_1, A) \dots (a_n, A))) \\ &=\; \mathit{prob}(\mathcal{FTCC}(z_{s_2}, (a_1, A) \dots (a_n, A))) \\ &=\; \mathit{prob}(\mathcal{RTCC}(z_{s_2}, (a_1, \emptyset) \dots (a_n, \emptyset))) \end{aligned}$$

- Let $k \in \mathbb{N}_{>0}$ and suppose that the result holds for all ready traces for which the sum of the cardinalities of the ready sets is less than k. Then for all $(a_1, R_1) \dots (a_n, R_n) \in (A \times 2^A)^*$ such that $\sum_{1 \le i \le n} |R_i| = k$:
$$\begin{aligned} \mathit{prob}(\mathcal{RTCC}(z_{s_1}, (a_1, R_1) \dots (a_n, R_n))) \;&=\; \mathit{prob}(\mathcal{FTCC}(z_{s_1}, (a_1, \overline{R}_1) \dots (a_n, \overline{R}_n))) \\ &\quad - \textstyle\sum_{R'_1, \dots, R'_n \in 2^A \text{ s.t. } R'_i \subseteq R_i \text{ for all } i = 1, \dots, n}^{R'_j \subset R_J \text{ for some } j = 1, \dots, n} \mathit{prob}(\mathcal{RTCC}(z_{s_1}, (a_1, R'_1) \dots (a_n, R'_n))) \\ &=\; \mathit{prob}(\mathcal{FTCC}(z_{s_2}, (a_1, \overline{R}_1) \dots (a_n, \overline{R}_n))) \\ &\quad - \textstyle\sum_{R'_1, \dots, R'_n \in 2^A \text{ s.t. } R'_i \subseteq R_i \text{ for all } i = 1, \dots, n}^{R'_j \subset R_J \text{ for some } j = 1, \dots, n} \mathit{prob}(\mathcal{RTCC}(z_{s_2}, (a_1, R'_1) \dots (a_n, R'_n))) \\ &=\; \mathit{prob}(\mathcal{RTCC}(z_{s_2}, (a_1, R_1) \dots (a_n, R_n))) \end{aligned}$$

A similar result holds also starting from an arbitrary $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ that is matched by some $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ according to $\sim_{\mathrm{PFTr,dis}}$. Therefore, we can conclude that $s_1 \sim_{\mathrm{PRTr,dis}} s_2$.

4. We preliminarily observe that for all $s \in S$, $\mathcal{Z} \in \mathit{Res}(s)$, $\alpha \in A^*$, and $F, R \in 2^A$ it holds that:
$$\begin{aligned} \mathit{prob}(\mathcal{FCC}(z_s, (\alpha, F))) \;&=\; \textstyle\sum_{R' \in 2^A \text{ s.t. } R' \cap F = \emptyset} \mathit{prob}(\mathcal{RCC}(z_s, (\alpha, R'))) \\ \mathit{prob}(\mathcal{RCC}(z_s, (\alpha, R))) \;&=\; \mathit{prob}(\mathcal{FCC}(z_s, (\alpha, A \setminus R))) - \textstyle\sum_{R' \subset R} \mathit{prob}(\mathcal{RCC}(z_s, (\alpha, R'))) \end{aligned}$$
Suppose that $s_1 \sim_{\mathrm{PR,dis}} s_2$. Then we immediately derive that:

- For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that for all $(\alpha, F) \in A^* \times 2^A$:
$$\begin{aligned} \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, F))) \;&=\; \textstyle\sum_{R' \in 2^A \text{ s.t. } R' \cap F = \emptyset} \mathit{prob}(\mathcal{RCC}(z_{s_1}, (\alpha, R'))) \\ &=\; \textstyle\sum_{R' \in 2^A \text{ s.t. } R' \cap F = \emptyset} \mathit{prob}(\mathcal{RCC}(z_{s_2}, (\alpha, R'))) \\ &=\; \mathit{prob}(\mathcal{FCC}(z_{s_2}, (\alpha, F))) \end{aligned}$$

- Symmetrically for each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$.

This means that $s_1 \sim_{\mathrm{PF,dis}} s_2$.
Suppose now that $s_1 \sim_{\mathrm{PF,dis}} s_2$. For each ready pair $(\alpha, R) \in A^* \times 2^A$ such that R is infinite, it trivially holds that for all $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ and $\mathcal{Z}_2 \in \mathit{Res}(s_2)$:

$$\mathit{prob}(\mathcal{RCC}(z_{s_1}, (\alpha, R))) \;=\; 0 \;=\; \mathit{prob}(\mathcal{RCC}(z_{s_2}, (\alpha, R)))$$

whenever the considered NPLTS is finitely branching. Thus, in order to prove that $s_1 \sim_{\mathrm{PR,dis}} s_2$, we can restrict ourselves to ready pairs whose ready set is finite. Given an arbitrary $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ that is matched by some $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ according to $\sim_{\mathrm{PF,dis}}$, we show that the matching holds also under $\sim_{\mathrm{PR,dis}}$ by proceeding by induction on the cardinality $k \in \mathbb{N}$ of the ready set of ready pairs whose ready set is finite:

- Let $k = 0$, i.e., consider ready pairs whose ready set is empty. Then for all $\alpha \in A^*$:
$$\begin{aligned} \mathit{prob}(\mathcal{RCC}(z_{s_1}, (\alpha, \emptyset))) \;&=\; \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, A))) \;= \\ &=\; \mathit{prob}(\mathcal{FCC}(z_{s_2}, (\alpha, A))) \;=\; \mathit{prob}(\mathcal{RCC}(z_{s_2}, (\alpha, \emptyset))) \end{aligned}$$

- Let $k \in \mathbb{N}_{>0}$ and suppose that the result holds for all ready pairs whose ready set has cardinality less than k. Then for all $(\alpha, R) \in A^* \times 2^A$ such that $|R| = k$:
$$\begin{aligned} \mathit{prob}(\mathcal{RCC}(z_{s_1}, (\alpha, R))) &= \mathit{prob}(\mathcal{FCC}(z_{s_1}, (\alpha, A \setminus R))) - \textstyle\sum_{R' \subset R} \mathit{prob}(\mathcal{RCC}(z_{s_1}, (\alpha, R'))) \\ &= \mathit{prob}(\mathcal{FCC}(z_{s_2}, (\alpha, A \setminus R))) - \textstyle\sum_{R' \subset R} \mathit{prob}(\mathcal{RCC}(z_{s_2}, (\alpha, R'))) \\ &= \mathit{prob}(\mathcal{RCC}(z_{s_2}, (\alpha, R))) \end{aligned}$$

A similar result holds also starting from an arbitrary $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ that is matched by some $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ according to $\sim_{\mathrm{PF,dis}}$. Therefore, we can conclude that $s_1 \sim_{\mathrm{PR,dis}} s_2$.

5. Suppose that $s_1 \sim_{\mathrm{PFTr,dis}} s_2$. Then we immediately derive that:

   - For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that for all $(a_1 \dots a_n, F) \in A^* \times 2^A$:
$$\begin{aligned} \mathit{prob}(\mathcal{FCC}(z_{s_1}, (a_1 \dots a_n, F))) &= \mathit{prob}(\mathcal{FTCC}(z_{s_1}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= \mathit{prob}(\mathcal{FTCC}(z_{s_2}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= \mathit{prob}(\mathcal{FCC}(z_{s_2}, (a_1 \dots a_n, F))) \end{aligned}$$

   - Symmetrically for each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$.

   This means that $s_1 \sim_{\mathrm{PF,dis}} s_2$.

6. Suppose that $s_1 \sim_{\mathrm{PFTr}} s_2$. Then we immediately derive that for all $(a_1 \dots a_n, F) \in A^* \times 2^A$:

   - For each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that:
$$\begin{aligned} \mathit{prob}(\mathcal{FCC}(z_{s_1}, (a_1 \dots a_n, F))) &= \mathit{prob}(\mathcal{FTCC}(z_{s_1}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= \mathit{prob}(\mathcal{FTCC}(z_{s_2}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= \mathit{prob}(\mathcal{FCC}(z_{s_2}, (a_1 \dots a_n, F))) \end{aligned}$$

   - Symmetrically for each $\mathcal{Z}_2 \in \mathit{Res}(s_2)$.

   This means that $s_1 \sim_{\mathrm{PF}} s_2$.

7. Suppose that $s_1 \sim_{\mathrm{PFTr},\sqcup\sqcap} s_2$. Then we immediately derive that for all $\varphi = (\alpha, F) \in A^* \times 2^A$:
$$\begin{aligned} \bigsqcup_{\mathcal{Z}_1 \in \mathit{Res}_\alpha(s_1)} \mathit{prob}(\mathcal{FCC}(z_{s_1}, \varphi)) &= \bigsqcup_{\mathcal{Z}_1 \in \mathit{Res}_\alpha(s_1)} \mathit{prob}(\mathcal{FTCC}(z_{s_1}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= \bigsqcup_{\mathcal{Z}_2 \in \mathit{Res}_\alpha(s_2)} \mathit{prob}(\mathcal{FTCC}(z_{s_2}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= \bigsqcup_{\mathcal{Z}_2 \in \mathit{Res}_\alpha(s_2)} \mathit{prob}(\mathcal{FCC}(z_{s_2}, \varphi)) \\ \bigsqcap_{\mathcal{Z}_1 \in \mathit{Res}_\alpha(s_1)} \mathit{prob}(\mathcal{FCC}(z_{s_1}, \varphi)) &= \bigsqcap_{\mathcal{Z}_1 \in \mathit{Res}_\alpha(s_1)} \mathit{prob}(\mathcal{FTCC}(z_{s_1}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= \bigsqcap_{\mathcal{Z}_2 \in \mathit{Res}_\alpha(s_2)} \mathit{prob}(\mathcal{FTCC}(z_{s_2}, (a_1, \emptyset) \dots (a_{n-1}, \emptyset)(a_n, F))) \\ &= \bigsqcap_{\mathcal{Z}_2 \in \mathit{Res}_\alpha(s_2)} \mathit{prob}(\mathcal{FCC}(z_{s_2}, \varphi)) \end{aligned}$$
   where $a_1 \dots a_n = \alpha$. This means that $s_1 \sim_{\mathrm{PF},\sqcup\sqcap} s_2$. ■

All the inclusions in Thm. 3.14 are strict:

- Figures 3 and 4 respectively show that for all $\pi \in \{\mathrm{PRTr}, \mathrm{PFTr}, \mathrm{PR}\}$ it holds that $\sim_{\pi,\mathrm{dis}}$ is strictly finer than $\sim_\pi$ and $\sim_\pi$ is strictly finer than $\sim_{\pi,\sqcup\sqcap}$.

- Figure 10 shows that $\sim_{\text{PTe-tbt,dis}}$ is strictly finer than $\sim_{\mathrm{PRTr,dis}}$. It holds that $s_1 \not\sim_{\text{PTe-tbt,dis}} s_2$ because $\sim_{\text{PTe-tbt,dis}}$ coincides with $\sim_{\text{PTe-}\forall\exists}$ and the test in the considered figure distinguishes the two processes with respect to $\sim_{\text{PTe-}\forall\exists}$. In contrast, $s_1 \sim_{\mathrm{PRTr,dis}} s_2$ because for each resolution of s_1 (resp. s_2) there exists a resolution of s_2 (resp. s_1) having precisely the same ready-trace distribution.

- Figure 13 shows that $\sim_{\mathrm{PRTr,dis}}$ and $\sim_{\mathrm{PFTr,dis}}$ are strictly finer than $\sim_{\mathrm{PR,dis}}$ and $\sim_{\mathrm{PF,dis}}$, respectively. It holds that $s_1 \not\sim_{\mathrm{PRTr,dis}} s_2$ (and hence $s_1 \not\sim_{\mathrm{PFTr,dis}} s_2$) because the ready-trace distribution of the leftmost maximal resolution of s_1 in which the choice between b and d is resolved in favor of b is not

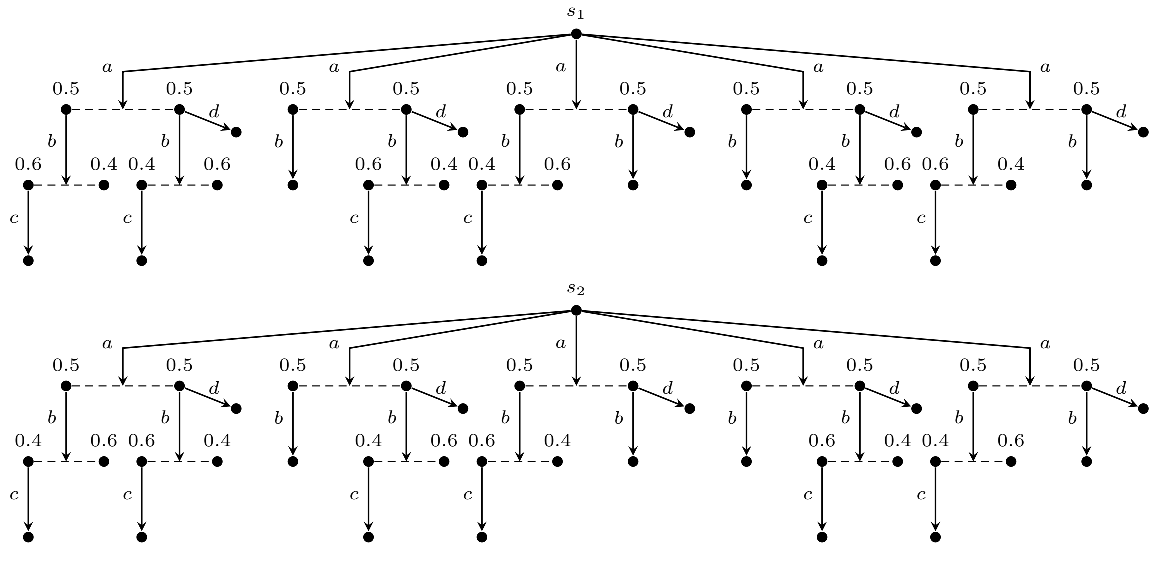

Figure 13: Two NPLTS models distinguished by $\sim_{\rm PRTr,dis}/\sim_{\rm PFTr,dis}$ and identified by $\sim_{\rm PR,dis}/\sim_{\rm PF,dis}$

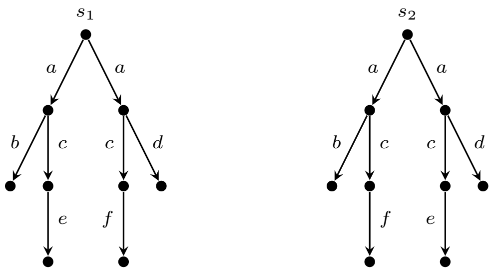

Figure 14: Two NPLTS models distinguished by $\sim_{\rm PFTr}/\sim_{\rm PFTr,\sqcup\sqcap}$ and identified by $\sim_{\rm PF}/\sim_{\rm PF,\sqcup\sqcap}$

matched by the ready-trace distribution of any of the resolutions of s_2. In contrast, $s_1 \sim_{\rm PR,dis} s_2$ (and hence $s_1 \sim_{\rm PF,dis} s_2$) because for each resolution of s_1 (resp. s_2) there exists a resolution of s_2 (resp. s_1) having precisely the same readiness distribution. In particular, the readiness distribution of the leftmost maximal resolution of s_1 considered before is matched by the readiness distribution of the leftmost maximal resolution of s_2 in which the choice between b and d is resolved as before, because when dealing with ready pairs instead of ready traces the probabilities of performing the two b-transitions in those resolutions can be summed up in the case of traces of length greater than 1. Likewise, the readiness distribution of the leftmost maximal resolution of s_1 in which the choice between b and d is resolved in favor of d is matched by the readiness distribution of the central maximal resolution of s_2 in which the choice between b and d is resolved in the same way.

- Figure 14 shows that $\sim_{\rm PFTr}$ and $\sim_{\rm PFTr,\sqcup\sqcap}$ are strictly finer than $\sim_{\rm PF}$ and $\sim_{\rm PF,\sqcup\sqcap}$, respectively. It holds that $s_1 \not\sim_{\rm PFTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PFTr} s_2$) because s_1 has a computation compatible with the failure trace $(a, A \setminus \{b, c\})\,(c, A \setminus \{e\})\,(e, A)$ while s_2 has no computation compatible with that failure trace. In contrast, $s_1 \sim_{\rm PF} s_2$ (and hence $s_1 \sim_{\rm PF,\sqcup\sqcap} s_2$) because, given an arbitrary failure pair, for each resolution of s_1 (resp. s_2) there exists a resolution of s_2 (resp. s_1) having the same probability of performing a computation compatible with that failure pair.

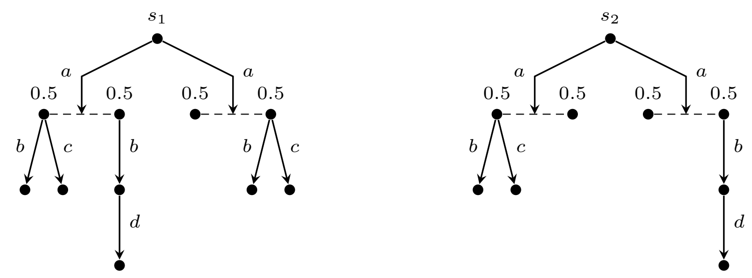

Figure 15: Two NPLTS models distinguished by $\sim_{\rm PTr,\sqcup\sqcap}$ and identified by $\sim_{\rm PRTr}/\sim_{\rm PR}$

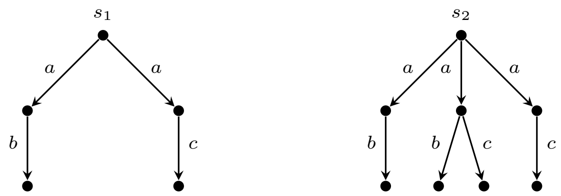

Figure 16: Two NPLTS models distinguished by $\sim_{\rm PRTr,\sqcup\sqcap}/\sim_{\rm PR,\sqcup\sqcap}$ and identified by $\sim_{\rm PCTr,dis}/\sim_{\rm PFTr}$

Moreover:

- $\sim_{\rm PB}$ and $\sim_{\rm PB,\sqcup\sqcap}$ are incomparable with the nine decorated-trace equivalences introduced in this section. Indeed, in Fig. 7 it holds that $s_1 \sim_{\rm PB} s_2$ (and hence $s_1 \sim_{\rm PB,\sqcup\sqcap} s_2$) – as can be seen by taking the equivalence relation that pairs states having equally labeled transitions leading to the same distribution – $s_1 \not\sim_{\rm PRTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PRTr} s_2$ and $s_1 \not\sim_{\rm PRTr,dis} s_2$) – due to the ready trace $(a, \{b\})\,(b, \{c\})\,(c, \emptyset)$ having maximum probability 0.68 in the first process and 0.61 in the second process – $s_1 \not\sim_{\rm PR,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PR} s_2$ and $s_1 \not\sim_{\rm PR,dis} s_2$) – due to the ready pair $(a\,b\,c, \emptyset)$ having maximum probability 0.68 in the first process and 0.61 in the second process – and $s_1 \not\sim_{\rm PFTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PFTr} s_2$ and $s_1 \not\sim_{\rm PFTr,dis} s_2$) – due to the failure trace $(a, A \setminus \{b\})\,(b, A \setminus \{c\})\,(c, A)$ having maximum probability 0.68 in the first process and 0.61 in the second process. In contrast, in Fig. 8 it holds that $s_1 \not\sim_{\rm PB,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PB} s_2$) – as the leftmost state with outgoing b-transitions reachable from s_2 is not $\sqcup\sqcap$-bisimilar to the two states with outgoing b-transitions reachable from s_1 – and $s_1 \sim_{\rm PTe\text{-}tbt,dis} s_2$ (and hence s_1 and s_2 are also identified by the nine decorated-trace equivalences).

- $\sim_{\rm PTe\text{-}\sqcup\sqcap}$ is incomparable with the nine decorated-trace equivalences introduced in this section. Indeed, in Fig. 4 it holds that $s_1 \sim_{\rm PTe\text{-}\sqcup\sqcap} s_2$, $s_1 \not\sim_{\rm PRTr} s_2$ (and hence $s_1 \not\sim_{\rm PRTr,dis}$), $s_1 \not\sim_{\rm PR} s_2$ (and hence $s_1 \not\sim_{\rm PR,dis}$), and $s_1 \not\sim_{\rm PFTr} s_2$ (and hence $s_1 \not\sim_{\rm PFTr,dis}$). In contrast, in Fig. 10 it holds that $s_1 \not\sim_{\rm PTe\text{-}\sqcup\sqcap} s_2$ and $s_1 \sim_{\rm PRTr,dis} s_2$ (and hence $s_1 \sim_{\rm PR,dis} s_2$, $s_1 \sim_{\rm PFTr,dis} s_2$, $s_1 \sim_{\rm PRTr} s_2$, $s_1 \sim_{\rm PR} s_2$, and $s_1 \sim_{\rm PFTr} s_2$). Likewise, in Fig. 9 it holds that $s_1 \sim_{\rm PTe\text{-}\sqcup\sqcap} s_2$ – as there is no test that results in an interaction system having a maximal resolution with differently labeled successful computations of the same length and hence no possibility of summing up their success probabilities – $s_1 \not\sim_{\rm PRTr,\sqcup\sqcap} s_2$ – due to the ready trace $(a, \{b\})\,(b, \emptyset)$ whose maximum probability is 0.24 in the first process and 0.21 in the second process – $s_1 \not\sim_{\rm PR,\sqcup\sqcap} s_2$ – due to the ready pair $(a\,b, \emptyset)$ whose maximum probability is 0.24 in the first process and 0.21 in the second process – and $s_1 \not\sim_{\rm PFTr,\sqcup\sqcap} s_2$ – due to the failure trace $(a, A \setminus \{b\})\,(b, A)$ whose maximum probability is 0.24 in the first process and 0.21 in the second process. In contrast, in Fig. 3 it holds that $s_1 \not\sim_{\rm PTe\text{-}\sqcup\sqcap} s_2$, $s_1 \sim_{\rm PRTr,\sqcup\sqcap} s_2$, $s_1 \sim_{\rm PR,\sqcup\sqcap} s_2$, and $s_1 \sim_{\rm PFTr,\sqcup\sqcap} s_2$.

- $\sim_{\rm PRTr}$, $\sim_{\rm PR}$, $\sim_{\rm PRTr,\sqcup\sqcap}$, and $\sim_{\rm PR,\sqcup\sqcap}$ are incomparable with $\sim_{\rm PCTr,dis}$, $\sim_{\rm PTr,dis}$, $\sim_{\rm PFTr}$, $\sim_{\rm PF}$, $\sim_{\rm PTe\text{-}tbt}$, $\sim_{\rm PCTr}$, $\sim_{\rm PTr}$, $\sim_{\rm PFTr,\sqcup\sqcap}$, $\sim_{\rm PF,\sqcup\sqcap}$, $\sim_{\rm PTe\text{-}tbt,\sqcup\sqcap}$, $\sim_{\rm PCTr,\sqcup\sqcap}$, and $\sim_{\rm PTr,\sqcup\sqcap}$. Indeed, in Fig. 15 it holds that

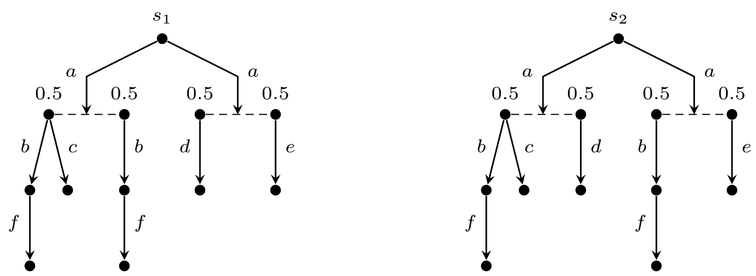

Figure 17: Two NPLTS models distinguished by $\sim_{\rm PR}/\sim_{\rm PR,\sqcup\sqcap}$ and identified by $\sim_{\rm PRTr}/\sim_{\rm PRTr,\sqcup\sqcap}$

$s_1 \sim_{\rm PRTr} s_2$ (and hence $s_1 \sim_{\rm PRTr,\sqcup\sqcap} s_2$), $s_1 \sim_{\rm PR} s_2$ (and hence $s_1 \sim_{\rm PR,\sqcup\sqcap} s_2$), and $s_1 \not\sim_{\rm PTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PCTr,\sqcup\sqcap} s_2$, $s_1 \not\sim_{\rm PTe\text{-}tbt,\sqcup\sqcap} s_2$, $s_1 \not\sim_{\rm PF,\sqcup\sqcap} s_2$, $s_1 \not\sim_{\rm PFTr,\sqcup\sqcap} s_2$, $s_1 \not\sim_{\rm PTr} s_2$, $s_1 \not\sim_{\rm PCTr} s_2$, $s_1 \not\sim_{\rm PTe\text{-}tbt} s_2$, $s_1 \not\sim_{\rm PF} s_2$, $s_1 \not\sim_{\rm PFTr} s_2$, $s_1 \not\sim_{\rm PTr,dis} s_2$, and $s_1 \not\sim_{\rm PCTr,dis} s_2$) – due to the trace $a\,b$ having maximum probability 1 in the first process and 0.5 in the second process. In contrast, in Fig. 16 it holds that $s_1 \not\sim_{\rm PRTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PRTr} s_2$), $s_1 \not\sim_{\rm PR,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PR} s_2$), $s_1 \sim_{\rm PCTr,dis} s_2$ (and hence $s_1 \sim_{\rm PTr,dis} s_2$), and $s_1 \sim_{\rm PFTr} s_2$ (and hence $s_1 \sim_{\rm PF} s_2$, $s_1 \sim_{\rm PTe\text{-}tbt} s_2$, $s_1 \sim_{\rm PCTr} s_2$, $s_1 \sim_{\rm PTr} s_2$, $s_1 \sim_{\rm PFTr,\sqcup\sqcap} s_2$, $s_1 \sim_{\rm PF,\sqcup\sqcap} s_2$, $s_1 \sim_{\rm PTe\text{-}tbt,\sqcup\sqcap} s_2$, $s_1 \sim_{\rm PCTr,\sqcup\sqcap} s_2$, and $s_1 \sim_{\rm PTr,\sqcup\sqcap} s_2$).

- $\sim_{\rm PRTr}$ and $\sim_{\rm PRTr,\sqcup\sqcap}$ are incomparable with $\sim_{\rm PR}$ and $\sim_{\rm PR,\sqcup\sqcap}$. Indeed, in Fig. 17 it holds that $s_1 \sim_{\rm PRTr} s_2$ (and hence $s_1 \sim_{\rm PRTr,\sqcup\sqcap} s_2$) and $s_1 \not\sim_{\rm PR,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PR} s_2$) – due to the ready pair $(a\,b\,f, \emptyset)$ having maximum probability 1 in the first process and 0.5 in the second process. In contrast, in Fig. 14 it holds that $s_1 \not\sim_{\rm PRTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PRTr} s_2$) – due to the ready trace $(a, \{b, c\})\,(c, \{e\})\,(e, \emptyset)$ having maximum probability 1 in the first process and 0 in the second process – and $s_1 \sim_{\rm PR} s_2$ (and hence $s_1 \sim_{\rm PR,\sqcup\sqcap} s_2$).

- $\sim_{\rm PFTr}$ and $\sim_{\rm PFTr,\sqcup\sqcap}$ are incomparable with $\sim_{\rm PCTr,dis}$ and $\sim_{\rm PTr,dis}$, because in Fig. 3 it holds that $s_1 \sim_{\rm PFTr} s_2$ (and hence $s_1 \sim_{\rm PFTr,\sqcup\sqcap} s_2$) and $s_1 \not\sim_{\rm PTr,dis} s_2$ (and hence $s_1 \not\sim_{\rm PCTr,dis} s_2$), while in Fig. 5 it holds that $s_1 \not\sim_{\rm PFTr,\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\rm PFTr} s_2$) and $s_1 \sim_{\rm PCTr,dis} s_2$ (and hence $s_1 \sim_{\rm PTr,dis} s_2$).

- $\sim_{\rm PFTr,\sqcup\sqcap}$ is incomparable with $\sim_{\rm PCTr}$ and $\sim_{\rm PTr}$, because in Fig. 4 it holds that $s_1 \sim_{\rm PFTr,\sqcup\sqcap} s_2$ and $s_1 \not\sim_{\rm PTr} s_2$ (and hence $s_1 \not\sim_{\rm PCTr} s_2$), while in Fig. 5 it holds that $s_1 \not\sim_{\rm PFTr,\sqcup\sqcap} s_2$ and $s_1 \sim_{\rm PCTr} s_2$ (and hence $s_1 \sim_{\rm PTr} s_2$).

- $\sim_{\rm PFTr,\sqcup\sqcap}$ is incomparable with $\sim_{\rm PF}$ and $\sim_{\rm PTe\text{-}tbt}$ too. Indeed, in Fig. 4 it holds that $s_1 \sim_{\rm PFTr,\sqcup\sqcap} s_2$ and $s_1 \not\sim_{\rm PTe\text{-}tbt} s_2$ (and hence $s_1 \not\sim_{\rm PF} s_2$). In contrast, in Fig. 14 it holds that $s_1 \not\sim_{\rm PFTr,\sqcup\sqcap} s_2$ and $s_1 \sim_{\rm PF} s_2$. Likewise, in Fig. 11 it holds that $s_1 \not\sim_{\rm PFTr,\sqcup\sqcap} s_2$ and $s_1 \sim_{\rm PTe\text{-}tbt} s_2$.

3.3. Simulation, Completed-Simulation, Failure-Simulation, and Ready-Simulation Equivalences

The variant of bisimulation equivalence in which only one direction is considered is called simulation preorder, which is a refinement of trace inclusion. Simulation equivalence is defined as the kernel of simulation preorder. In the probabilistic setting, simulation equivalence was defined by means of weight functions in [10]. Here we shall follow an alternative characterization introduced in [5], which relies on preorders as well as on closed sets. Given an NPLTS $(S, A, \longrightarrow)$, a relation $\mathcal{S}$ over S, and $S' \subseteq S$, we say that S' is an *$\mathcal{S}$-closed set* iff $\mathcal{S}(S') = \{s'' \in S \mid \exists s' \in S'.\, (s', s'') \in \mathcal{S}\}$ is contained in S'. Notice that, if $\mathcal{S}$ is an equivalence relation, then an $\mathcal{S}$-closed set is a group of equivalence classes.

Definition 3.15. (*Probabilistic set-distribution similarity* – $\sim_{\rm PS,dis}$ – [15])
$s_1 \sim_{\rm PS,dis} s_2$ iff $s_1 \sqsubseteq_{\rm PS,dis} s_2$ and $s_2 \sqsubseteq_{\rm PS,dis} s_1$, where $\sqsubseteq_{\rm PS,dis}$ is the largest probabilistic set-distribution simulation. A preorder $\mathcal{S}$ over S is a *probabilistic set-distribution simulation* iff, whenever $(s_1, s_2) \in \mathcal{S}$, then for each $s_1 \xrightarrow{a} \mathcal{D}_1$ there exists $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\underline{\text{for all } \mathcal{S}\text{-closed } S' \subseteq S}$ it holds that $\mathcal{D}_1(S') \leq \mathcal{D}_2(S')$. ■

Definition 3.16. (*Probabilistic similarity* – $\sim_{\rm PS}$ – [17])
$s_1 \sim_{\rm PS} s_2$ iff $s_1 \sqsubseteq_{\rm PS} s_2$ and $s_2 \sqsubseteq_{\rm PS} s_1$, where $\sqsubseteq_{\rm PS}$ is the largest probabilistic simulation. A preorder $\mathcal{S}$ over S is a *probabilistic simulation* iff, whenever $(s_1, s_2) \in \mathcal{S}$, then $\underline{\text{for all } \mathcal{S}\text{-closed } S' \subseteq S}$ it holds that for each $s_1 \xrightarrow{a} \mathcal{D}_1$ there exists $s_2 \xrightarrow{a} \mathcal{D}_2$ such that $\mathcal{D}_1(S') \leq \mathcal{D}_2(S')$. ■

Definition 3.17. (*Probabilistic* $\sqcup$*-similarity* – $\sim_{\rm PS,\sqcup}$)
$s_1 \sim_{\rm PS,\sqcup} s_2$ iff $s_1 \sqsubseteq_{\rm PS,\sqcup} s_2$ and $s_2 \sqsubseteq_{\rm PS,\sqcup} s_1$, where $\sqsubseteq_{\rm PS,\sqcup}$ is the largest probabilistic $\sqcup$-simulation. A preorder $\mathcal{S}$ over S is a *probabilistic* $\sqcup$*-simulation* iff, whenever $(s_1, s_2) \in \mathcal{S}$, then for all $\mathcal{S}$-closed $S' \subseteq S$ and $a \in A$ it holds that $s_1 \xrightarrow{a}$ implies $s_2 \xrightarrow{a}$ and:

$$\bigsqcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(S') \leq \bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(S')$$

■

Similar to trace semantics, a number of variants of simulation semantics can be defined in which the sets of actions that can be refused or accepted by states are also considered. Given $s \in S$, in the following we let $init(s) = \{a \in A \mid s \xrightarrow{a}\}$. Observing that $init(s_1) \subseteq init(s_2)$ whenever s_1 and s_2 are related by a simulation semantics, the additional constraints are the following, where the names of the obtained variants are reported in parentheses:

- $init(s_1) = \emptyset \Longrightarrow init(s_2) = \emptyset$, for *completed simulation* ($\sim_{\rm PCS,dis}$, $\sim_{\rm PCS}$, $\sim_{\rm PCS,\sqcup}$).
- $init(s_1) \cap F = \emptyset \Longrightarrow init(s_2) \cap F = \emptyset$ for all $F \in 2^A$, for *failure simulation* ($\sim_{\rm PFS,dis}$, $\sim_{\rm PFS}$, $\sim_{\rm PFS,\sqcup}$).
- $init(s_1) = init(s_2)$, for *ready simulation* ($\sim_{\rm PRS,dis}$, $\sim_{\rm PRS}$, $\sim_{\rm PRS,\sqcup}$).

Of the variants mentioned above, only $\sim_{\rm PFS,dis}$ has appeared in the literature of nondeterministic and probabilistic processes [3, 2].

We now investigate the relationships of the twelve simulation-based equivalences among themselves and with the various equivalences defined in [1] and in this paper. First of all, it turns out that every simulation-based equivalence relying on partially matching transitions coincides with the corresponding simulation-based equivalence relying on extremal probabilities. Moreover, ready-simulation semantics coincides with failure-simulation semantics, but the various simulation-based semantics do not collapse to bisimulation semantics as in the case of fully probabilistic processes [10]. Each of the simulation-based equivalences relying on fully matching transitions is comprised between bisimilarity and the corresponding trace equivalence, as in the fully nondeterministic spectrum [18]. In contrast, the simulation-based equivalences relying on partially matching transitions or extremal probabilities are incomparable with most of the other equivalences.

Theorem 3.18. It holds that:

1. $\sim_{\pi,\rm dis} \subseteq \sim_\pi = \sim_{\pi,\sqcup}$ for all $\pi \in \{\rm PS, PCS, PFS, PRS\}$ over image-finite NPLTS models.
2. $\sim_{{\rm PB},\sigma'} \subseteq \sim_{{\rm PRS},\sigma} = \sim_{{\rm PFS},\sigma} \subseteq \sim_{{\rm PCS},\sigma} \subseteq \sim_{{\rm PS},\sigma}$ for all $\sigma \in \{{\rm dis}, \varepsilon, \sqcup\}$ and $\sigma' \in \{{\rm dis}, \varepsilon, \sqcup\sqcap\}$ related to σ.
3. $\sim_{\rm PS,dis} \subseteq \sim_{\rm PTr,dis}$.
4. $\sim_{\rm PCS,dis} \subseteq \sim_{\rm PCTr,dis}$.
5. $\sim_{\rm PRS,dis} \subseteq \sim_{\rm PTe\text{-}tbt,dis}$.

Proof Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1. The proof of the fact that $\sim_{\pi,\rm dis} \subseteq \sim_\pi \subseteq \sim_{\pi,\sqcup}$ for all $\pi \in \{\rm PS, PCS, PFS, PRS\}$ is similar to the proof of Thm. 6.5(1) in [1]. Moreover, it holds that $\sim_{\pi,\sqcup} \subseteq \sim_\pi$ (and hence $\sim_\pi = \sim_{\pi,\sqcup}$) when the NPLTS is image finite. In fact, supposing that $s_1 \sim_{\pi,\sqcup} s_2$, given a $\sim_{\pi,\sqcup}$-closed set $S' \subseteq S$ image finiteness guarantees that the following two sets:
$$\bigcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \{\mathcal{D}_1(S')\} \quad \text{and} \quad \bigcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \{\mathcal{D}_2(S')\}$$
are finite. In turn, the finiteness of those two sets ensures that their suprema respectively belong to the two sets themselves. As a consequence, starting from:

$$\bigsqcup_{s_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(S') \;\leq\; \bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(S')$$

when both s_1 and s_2 have at least one outgoing a-transition, it holds that for each $s_1 \xrightarrow{a} \mathcal{D}'_1$ there exists $s_2 \xrightarrow{a} \mathcal{D}'_2$ such that $\mathcal{D}'_1(S') \leq \mathcal{D}'_2(S')$ because we can take $\mathcal{D}'_2$ such that $\mathcal{D}'_2(S') = \bigsqcup_{s_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(S')$. This means that $s_1 \sim_\pi s_2$.

2. The fact that $\sim_{\mathrm{PB},\sigma'} \subseteq \sim_{\mathrm{PRS},\sigma} \subseteq \sim_{\mathrm{PFS},\sigma} \subseteq \sim_{\mathrm{PCS},\sigma} \subseteq \sim_{\mathrm{PS},\sigma}$ for all $\sigma \in \{\mathrm{dis}, \varepsilon, \sqcup\}$ and $\sigma' \in \{\mathrm{dis}, \varepsilon, \sqcup\sqcap\}$ related to σ is a straightforward consequence of the definition of the various equivalences. Moreover, it holds that $\sim_{\mathrm{PFS},\sigma} \subseteq \sim_{\mathrm{PRS},\sigma}$ (and hence $\sim_{\mathrm{PRS},\sigma} = \sim_{\mathrm{PFS},\sigma}$). In fact, supposing that s_1 and s_2 are related by a simulation semantics so that $\mathit{init}(s_1) \subseteq \mathit{init}(s_2)$, if $\mathit{init}(s_1) \neq \mathit{init}(s_2)$ because of some $a \in A$ such that $a \notin \mathit{init}(s_1)$ and $a \in \mathit{init}(s_2)$ – which means that $s_1 \not\sim_{\mathrm{PRS},\sigma} s_2$ – then $\mathit{init}(s_1) \cap \{a\} = \emptyset$ but $\mathit{init}(s_2) \cap \{a\} \neq \emptyset$ – which means that $s_1 \not\sim_{\mathrm{PFS},\sigma} s_2$.
3. We show that $s_1 \sqsubseteq_{\mathrm{PS,dis}} s_2 \Longrightarrow s_1 \sqsubseteq_{\mathrm{PTr,dis}} s_2$ from which the result will follow, where $s_1 \sqsubseteq_{\mathrm{PTr,dis}} s_2$ means that for each $\mathcal{Z}_1 \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(s_2)$ such that for all $\alpha \in A^*$ it holds that $\mathit{prob}(\mathcal{CC}(z_{s_1}, \alpha)) = \mathit{prob}(\mathcal{CC}(z_{s_2}, \alpha))$.
Suppose that $s_1 \sqsubseteq_{\mathrm{PS,dis}} s_2$. This means that $(s_1, s_2) \in \mathcal{S}$ for some probabilistic set-distribution simulation $\mathcal{S}$ over S. In turn, this induces projections of $\mathcal{S}$ that are fpr-simulations over pairs of matching resolutions and, since resolutions are fully probabilistic, we derive from [10] that such projections are actually fpr-bisimulations [6]. As a consequence, whenever $(r_1, r_2) \in \mathcal{S}$, then for each $\mathcal{Z}_1 \in \mathit{Res}(r_1)$ there exists $\mathcal{Z}_2 \in \mathit{Res}(r_2)$ such that the preorder $\mathcal{S}_{1,2}$ over $Z = Z_1 \cup Z_2$ corresponding to $\mathcal{S}$ projected onto $Z \times Z$ is an fpr-bisimulation, i.e., it is an equivalence relation and, whenever $(z_{s'_1}, z_{s'_2}) \in \mathcal{S}_{1,2}$, then for each $z_{s'_1} \xrightarrow{a} \mathcal{D}_1$ there exists $z_{s'_2} \xrightarrow{a} \mathcal{D}_2$ such that for all equivalence classes $C \in Z/\mathcal{S}_{1,2}$ it holds that $\mathcal{D}_1(C) = \mathcal{D}_2(C)$.
Given $s'_1, s'_2 \in S$ such that $(s'_1, s'_2) \in \mathcal{S}$ and given $\mathcal{Z}_1 \in \mathit{Res}(s'_1)$ and $\mathcal{Z}_2 \in \mathit{Res}(s'_2)$ such that $z_{s'_1}$ and $z_{s'_2}$ are related by one of the projections of $\mathcal{S}$, we prove that for all $\alpha \in A^*$ it holds that:
$$\mathit{prob}(\mathcal{CC}(z_{s'_1}, \alpha)) \;=\; \mathit{prob}(\mathcal{CC}(z_{s'_2}, \alpha))$$
by proceeding by induction on the length n of α:
   - If $n = 0$, i.e., $\alpha = \varepsilon$, then:
   $$\mathit{prob}(\mathcal{CC}(z_{s'_1}, \alpha)) \;=\; 1 \;=\; \mathit{prob}(\mathcal{CC}(z_{s'_2}, \alpha))$$
   - Let $n \in \mathbb{N}_{>0}$ and suppose that the result holds for all traces of length $m = 0, \dots, n-1$ that label computations starting from pairs of states of Z related by one of the projections of $\mathcal{S}$. Assume that $\alpha = a\,\alpha'$. Given $s \in S$ and $\mathcal{Z} \in \mathit{Res}(s)$, it holds that, whenever $z_s \xrightarrow{a} \mathcal{D}$, then:
   $$\mathit{prob}(\mathcal{CC}(z_s, \alpha)) \;=\; \sum_{z_{s'} \in Z} \mathcal{D}(z_{s'}) \cdot \mathit{prob}(\mathcal{CC}(z_{s'}, \alpha')) \;=\; \sum_{[z_{s'}] \in Z/\mathcal{S}'} \mathcal{D}([z_{s'}]) \cdot \mathit{prob}(\mathcal{CC}(z_{s'}, \alpha'))$$
   where $\mathcal{S}'$ is a projection of $\mathcal{S}$ and the factorization of $\mathit{prob}(\mathcal{CC}(z_{s'}, \alpha'))$ with respect to the specific representative $z_{s'}$ of the equivalence class $[z_{s'}]$ stems from the application of the induction hypothesis on α' to all states of that equivalence class. Since $z_{s'_1}$ and $z_{s'_2}$ are related by a projection $\mathcal{S}_{1,2}$ of $\mathcal{S}$, it follows that, whenever $z_{s'_1} \xrightarrow{a} \mathcal{D}_1$, then $z_{s'_2} \xrightarrow{a} \mathcal{D}_2$ and:
   $$\begin{aligned} \mathit{prob}(\mathcal{CC}(z_{s'_1}, \alpha)) \;&=\; \sum_{[z_{s'}] \in Z/\mathcal{S}_{1,2}} \mathcal{D}_1([z_{s'}]) \cdot \mathit{prob}(\mathcal{CC}(z_{s'}, \alpha')) \;= \\ &=\; \sum_{[z_{s'}] \in Z/\mathcal{S}_{1,2}} \mathcal{D}_2([z_{s'}]) \cdot \mathit{prob}(\mathcal{CC}(z_{s'}, \alpha')) \;=\; \mathit{prob}(\mathcal{CC}(z_{s'_2}, \alpha)) \end{aligned}$$

   Therefore $s_1 \sqsubseteq_{\mathrm{PTr,dis}} s_2$.
4. The proof that $s_1 \sqsubseteq_{\mathrm{PCS,dis}} s_2 \Longrightarrow s_1 \sqsubseteq_{\mathrm{PCTr,dis}} s_2$, from which the result follows, is similar to the proof of the previous result. We note that:
   - For fully probabilistic models like resolutions, fpr-completed simulations are coarser than fpr-bisimulations and finer than fpr-simulations. Since fpr-simulations are fpr-bisimulations over these models [10], also fpr-completed simulations are fpr-bisimulations.
   - In the base case of the induction, it additionally holds that:
   $$\mathit{prob}(\mathcal{CCC}(z_{s'_1}, \alpha)) \;=\; \mathit{prob}(\mathcal{CCC}(z_{s'_2}, \alpha)) \;=\; \begin{cases} 1 & \text{if } \mathit{init}(s'_1) = \mathit{init}(s'_2) = \emptyset \\ 0 & \text{if } \mathit{init}(s'_1) \neq \emptyset \neq \mathit{init}(s'_2) \end{cases}$$

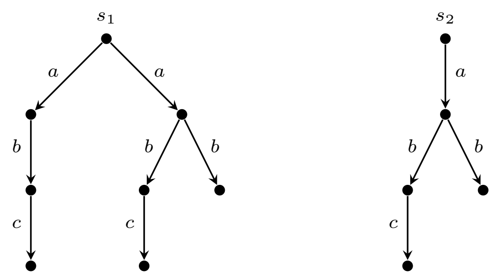

Figure 18: Two NPLTS models distinguished by $\sim_{\mathrm{PB,dis}}/\sim_{\mathrm{PB}}/\sim_{\mathrm{PB},\sqcup\sqcap}$ and identified by $\sim_{\mathrm{PRS,dis}}/\sim_{\mathrm{PRS}}/\sim_{\mathrm{PRS},\sqcup}$

where $\mathit{init}(s'_1) = \emptyset$ iff $\mathit{init}(s'_2) = \emptyset$ because $(s'_1, s'_2) \in \mathcal{S}$ and $\mathcal{S}$ is a probabilistic set-distribution completed simulation.

- In the general case of the induction, $\mathit{prob}(\mathcal{CCC}(z_s, \alpha))$ is expressed recursively in the same way as $\mathit{prob}(\mathcal{CC}(z_s, \alpha))$.

5. The proof that $s_1 \sqsubseteq_{\mathrm{PRS,dis}} s_2 \Longrightarrow s_1 \sqsubseteq_{\mathrm{PTe\text{-}tbt,dis}} s_2$, from which the result follows, is similar to the proof of Thm. 6.5(2) in [1]. We note that:

   - We exploit the fact that states related by $\sqsubseteq_{\mathrm{PRS,dis}}$ have the same set of actions labeling their outgoing transitions to establish a connection among resolutions of the interaction systems that are maximal (remember that states not enjoying that property are trivially distinguished by $\sqsubseteq_{\mathrm{PTe\text{-}tbt,dis}}$).
   - For fully probabilistic models like maximal resolutions, fpr-ready simulations are coarser than fpr-bisimulations and finer than fpr-simulations. Since fpr-simulations are fpr-bisimulations over these models [10], also fpr-ready simulations are fpr-bisimulations. ■

All the inclusions in Thm. 3.18 are strict:

- Figures 3 and 4 respectively show that for all $\pi \in \{\mathrm{PS}, \mathrm{PCS}, \mathrm{PFS}, \mathrm{PRS}\}$ it holds that $\sim_{\pi,\mathrm{dis}}$ is strictly finer than $\sim_\pi$ and $\sim_\pi$ is strictly finer than $\sim_{\pi,\sqcup}$.

- Figure 18 shows that $\sim_{\mathrm{PB},\sigma'}$ is strictly finer than $\sim_{\mathrm{PRS},\sigma}$ for all $\sigma \in \{\mathrm{dis}, \varepsilon, \sqcup\}$ and $\sigma' \in \{\mathrm{dis}, \varepsilon, \sqcup\sqcap\}$ related to σ. In particular, s_1 and s_2 are not bisimilar because the leftmost state with outgoing b-transitions reachable from s_1 is not bisimilar to the only state with outgoing b-transitions reachable from s_2.

- Figure 5 shows that $\sim_{\mathrm{PRS},\sigma}$ is strictly finer than $\sim_{\mathrm{PCS},\sigma}$ for all $\sigma \in \{\mathrm{dis}, \varepsilon, \sqcup\}$. In particular, s_1 and s_2 are not ready similar because the leftmost state with outgoing b-transitions reachable from s_1 is not ready similar to the only state with outgoing b-transitions reachable from s_2.

- Figure 6 shows that $\sim_{\mathrm{PCS},\sigma}$ is strictly finer than $\sim_{\mathrm{PS},\sigma}$ for all $\sigma \in \{\mathrm{dis}, \varepsilon, \sqcup\}$. In particular, s_1 and s_2 are not completed similar because the rightmost state reachable from s_1 after performing a is not completed similar to the only state reachable from s_2 after performing a.

- Figure 8 shows that $\sim_{\mathrm{PS,dis}}$, $\sim_{\mathrm{PCS,dis}}$, and $\sim_{\mathrm{PRS,dis}}$ are strictly finer than $\sim_{\mathrm{PTr,dis}}$, $\sim_{\mathrm{PCTr,dis}}$, and $\sim_{\mathrm{PTe\text{-}tbt,dis}}$, respectively. It holds that $s_1 \not\sim_{\mathrm{PS,dis}} s_2$ (and hence $s_1 \not\sim_{\mathrm{PCS,dis}} s_2$ and $s_1 \not\sim_{\mathrm{PRS,dis}} s_2$) because the leftmost state with outgoing b-transitions reachable from s_2 is not set-distribution similar to the two states with outgoing b-transitions reachable from s_1. In contrast, $s_1 \sim_{\mathrm{PTe\text{-}tbt,dis}} s_2$ (and hence $s_1 \sim_{\mathrm{PCTr,dis}} s_2$ and $s_1 \sim_{\mathrm{PTr,dis}} s_2$) because success probabilities are computed in a trace-by-trace fashion without adding up over different traces.

Moreover:

- $\sim_{\mathrm{PCS,dis}}$ is incomparable with the five testing equivalences and the twelve decorated-trace equivalences. Indeed, in Fig. 5 it holds that $s_1 \sim_{\mathrm{PCS,dis}} s_2$ – as can be seen by taking the preorder that pairs states having at least one equally labeled transition – $s_1 \not\sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$ (and hence s_1 and s_2 are also distinguished by the other four testing equivalences, the three failure equivalences, and the three failure-trace equivalences) – due to the test having an a-transition followed by a c-transition leading to success, which results in a maximal resolution with completed trace a when interacting with the first process and no maximal resolution with completed trace a when interacting with the second process – and $s_1 \not\sim_{\mathrm{PR},\sqcup\sqcap} s_2$ and $s_1 \not\sim_{\mathrm{PRTr},\sqcup\sqcap} s_2$ (and hence s_1 and s_2 are also distinguished by the other two readiness equivalences and the other two ready-trace equivalences) – due to the ready pair and ready trace $(a, \{b\})$ having maximum probability 1 in the first process and 0 in the second process. In contrast, in Fig. 8 it holds that $s_1 \not\sim_{\mathrm{PCS,dis}} s_2$ – as the leftmost state with outgoing b-transitions reachable from s_2 is not set-distribution completed similar to the two states with outgoing b-transitions reachable from s_1 – and $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ (and hence s_1 and s_2 are also identified by the other four testing equivalences and the twelve decorated-trace equivalences).

- $\sim_{\mathrm{PS,dis}}$ is incomparable with the five testing equivalences, the twelve decorated-trace equivalences, and the three completed-trace equivalences. Indeed, in Fig. 6 it holds that $s_1 \sim_{\mathrm{PS,dis}} s_2$ – as can be seen by taking the preorder that pairs states having equally labeled transitions – $s_1 \not\sim_{\mathrm{PCTr},\sqcup\sqcap} s_2$ (and hence s_1 and s_2 are also distinguished by the other two completed-trace equivalences, the three failure equivalences, and the three failure-trace equivalences) – due to the completed trace a having maximum probability 1 in the first process and 0 in the second process – $s_1 \not\sim_{\text{PTe-tbt},\sqcup\sqcap} s_2$ (and hence s_1 and s_2 are also distinguished by the other four testing equivalences) – due to the test having an a-transition followed by a b-transition leading to success, which results in a maximal resolution with completed trace a when interacting with the first process and no maximal resolution with completed trace a when interacting with the second process – and $s_1 \not\sim_{\mathrm{PR},\sqcup\sqcap} s_2$ and $s_1 \not\sim_{\mathrm{PRTr},\sqcup\sqcap} s_2$ (and hence s_1 and s_2 are also distinguished by the other two readiness equivalences and the other two ready-trace equivalences) – due to the ready pair and ready trace $(a, \emptyset)$ having maximum probability 1 in the first process and 0 in the second process. In contrast, in Fig. 8 it holds that $s_1 \not\sim_{\mathrm{PS,dis}} s_2$ – as the leftmost state with outgoing b-transitions reachable from s_2 is not set-distribution similar to the two states with outgoing b-transitions reachable from s_1 – and $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ (and hence s_1 and s_2 are also identified by the other four testing equivalences, the twelve decorated-trace equivalences, and the three completed-trace equivalences).

- $\sim_\pi$ and $\sim_{\pi,\sqcup}$ are incomparable with the five testing equivalences and the eighteen trace-based equivalences for all $\pi \in \{\mathrm{PS}, \mathrm{PCS}, \mathrm{PFS}, \mathrm{PRS}\}$. Indeed, in Fig. 7 it holds that $s_1 \sim_{\mathrm{PRS}} s_2$ (and hence s_1 and s_2 are also identified by the other seven simulation-based equivalences) – as can be seen by taking the preorder that pairs states having equally labeled transitions leading to the same distribution – and $s_1 \not\sim_{\mathrm{PTr},\sqcup\sqcap} s_2$, $s_1 \not\sim_{\mathrm{PR},\sqcup\sqcap} s_2$, and $s_1 \not\sim_{\mathrm{PRTr},\sqcup\sqcap} s_2$ (and hence s_1 and s_2 are also distinguished by the other fifteen trace-based equivalences and the five testing equivalences) – due to the trace $a\,b\,c$, the ready pair $(a\,b\,c, \emptyset)$, and the ready trace $(a, \{b\})\,(b, \{c\})\,(c, \emptyset)$ having maximum probability 0.68 in the first process and 0.61 in the second process. In contrast, in Fig. 8 it holds that $s_1 \not\sim_{\mathrm{PS},\sqcup\sqcap} s_2$ (and hence s_1 and s_2 are also distinguished by the other seven simulation-based equivalences) – as the leftmost state with outgoing b-transitions reachable from s_2 is not $\sqcup$-similar to the two states with outgoing b-transitions reachable from s_1 – and $s_1 \sim_{\text{PTe-tbt,dis}} s_2$ (and hence s_1 and s_2 are also identified by the other four testing equivalences and the eighteen trace-based equivalences).

- $\sim_{\mathrm{PB}}$ and $\sim_{\mathrm{PB},\sqcup\sqcap}$ are incomparable with $\sim_{\pi,\mathrm{dis}}$ for all $\pi \in \{\mathrm{PS}, \mathrm{PCS}, \mathrm{PFS}, \mathrm{PRS}\}$, because in Fig. 3 it holds that $s_1 \sim_{\mathrm{PB}} s_2$ (and hence $s_1 \sim_{\mathrm{PB},\sqcup\sqcap} s_2$) and $s_1 \not\sim_{\mathrm{PS,dis}} s_2$ (and hence $s_1 \not\sim_{\mathrm{PCS,dis}} s_2$, $s_1 \not\sim_{\mathrm{PFS,dis}} s_2$, and $s_1 \not\sim_{\mathrm{PRS,dis}} s_2$), while in Fig. 18 it holds that $s_1 \not\sim_{\mathrm{PB},\sqcup\sqcap} s_2$ (and hence $s_1 \not\sim_{\mathrm{PB}} s_2$) and $s_1 \sim_{\mathrm{PRS,dis}} s_2$ (and hence $s_1 \sim_{\mathrm{PFS,dis}} s_2$, $s_1 \sim_{\mathrm{PCS,dis}} s_2$, and $s_1 \sim_{\mathrm{PS,dis}} s_2$).

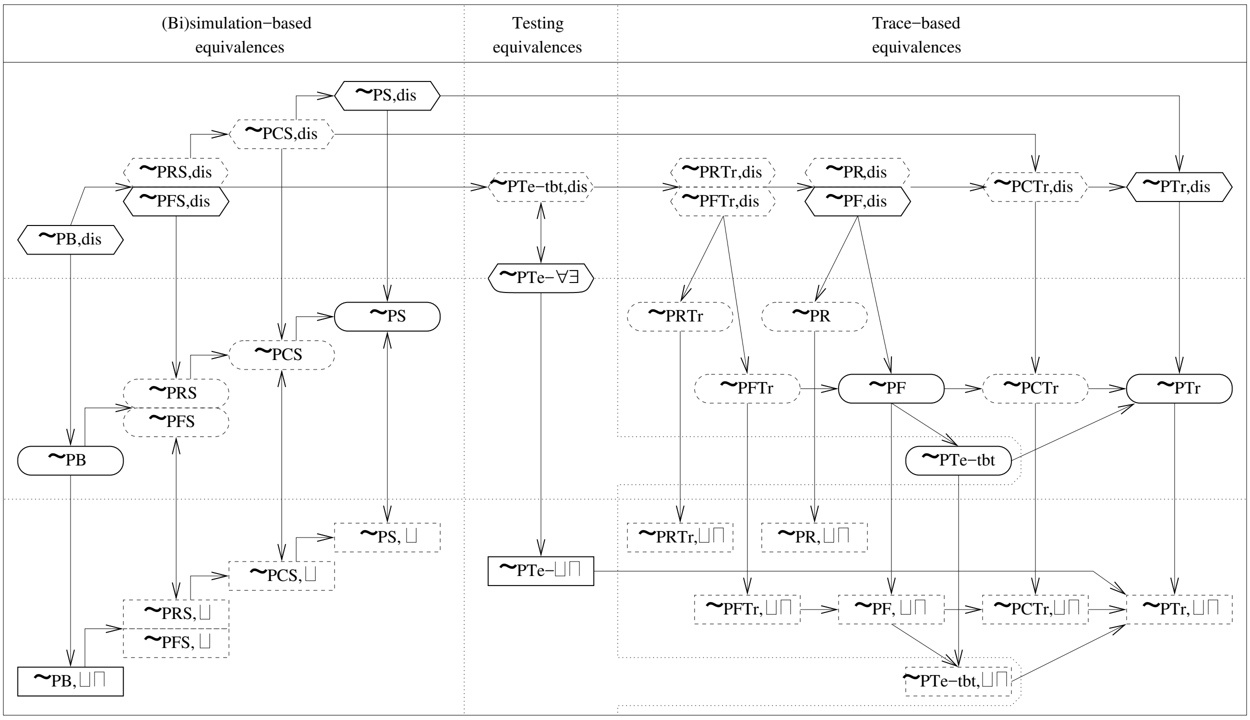

Figure 19: Full spectrum of strong behavioral equivalences for finitely-branching NPLTS models (deterministic schedulers)

- $\sim_{\mathrm{PRS}}$, $\sim_{\mathrm{PFS}}$, $\sim_{\mathrm{PRS},\sqcup}$, and $\sim_{\mathrm{PFS},\sqcup}$ are incomparable with $\sim_{\mathrm{PCS,dis}}$ and $\sim_{\mathrm{PS,dis}}$, because in Fig. 3 it holds that $s_1 \sim_{\mathrm{PRS}} s_2$ (and hence $s_1 \sim_{\mathrm{PFS}} s_2$, $s_1 \sim_{\mathrm{PRS},\sqcup} s_2$, and $s_1 \sim_{\mathrm{PFS},\sqcup} s_2$) and $s_1 \not\sim_{\mathrm{PS,dis}} s_2$ (and hence $s_1 \not\sim_{\mathrm{PCS,dis}} s_2$), while in Fig. 5 it holds that $s_1 \not\sim_{\mathrm{PFS},\sqcup} s_2$ (and hence $s_1 \not\sim_{\mathrm{PRS},\sqcup} s_2$, $s_1 \not\sim_{\mathrm{PFS}} s_2$, and $s_1 \not\sim_{\mathrm{PRS}} s_2$) and $s_1 \sim_{\mathrm{PCS,dis}} s_2$ (and hence $s_1 \sim_{\mathrm{PS,dis}} s_2$).

- $\sim_{\mathrm{PCS}}$ and $\sim_{\mathrm{PCS},\sqcup}$ are incomparable with $\sim_{\mathrm{PS,dis}}$, because in Fig. 3 it holds that $s_1 \sim_{\mathrm{PCS}} s_2$ (and hence $s_1 \sim_{\mathrm{PCS},\sqcup} s_2$) and $s_1 \not\sim_{\mathrm{PS,dis}} s_2$, while in Fig. 6 it holds that $s_1 \sim_{\mathrm{PS,dis}} s_2$ and $s_1 \not\sim_{\mathrm{PCS},\sqcup} s_2$ (and hence $s_1 \not\sim_{\mathrm{PCS}} s_2$).

3.4. A Full Spectrum

The spectum of all the considered equivalences is depicted in Fig. 19. We have followed the same graphical conventions mentioned at the beginning of Sect. 3, with adjacency of boxes within the same fragment having the same meaning as bidirectional arrows connecting boxes of different fragments, i.e., coincidence. Note that there are many more dashed boxes (corresponding to equivalences introduced in this paper) than in Fig. 1.

The top fragment of the spectrum in Fig. 19 refers to the considered equivalences that are based on fully matching resolutions. Similar to the spectrum for fully probabilistic processes in [11, 8], many equivalences collapse into a single one; in particular, ready-simulation semantics coincides with failure-simulation semantics, ready-trace semantics coincides with failure-trace semantics, and readiness semantics coincides with failure semantics. Different from the fully probabilistic spectrum, in the top fragment we have that the various simulation-based semantics do not coincide with bisimulation semantics [10] and that completed-trace semantics does not coincide with trace semantics [11, 8]. Moreover, testing semantics turns out to be finer than failure semantics.

The central fragment and the bottom fragment of the spectrum in Fig. 19 instead refer to the considered equivalences that are based on partially matching resolutions and extremal probabilities, respectively. These equivalences are coarser than those in the top fragment and do not flatten the specificity of the intuition behind the original definition of the behavioral equivalences for LTS models. Therefore, the two fragments at hand preserve much of the original spectrum in [18] for fully nondeterministic processes, with testing semantics being coarser than failure semantics. It is worth noting the coincidence of corresponding simulation-based equivalences in the two fragments (due to the fact that the comparison operator $\leq$ is used in their definitions), whereas this is not the case for the two bisimulation equivalences (as the comparison operator $=$ is used instead in their definitions). We finally stress the isolation of bisimulation semantics, simulation semantics, ready-trace semantics, and readiness semantics in the two fragments, as well as the partial isolation of $\sim_{\text{PTe-}\sqcup\sqcap}$.

4. Deterministic Schedulers vs. Randomized Schedulers

So far, we have considered strong equivalences for NPLTS models that compare probabilities calculated after resolving nondeterminism by means of deterministic schedulers. We now examine the case of randomized schedulers. Each of them selects at each state a convex combination of equally labeled transitions, which is called a *combined transition* [14]. Notice that a deterministic scheduler is a special case of randomized scheduler in which every selected combination involves a single ordinary transition.

Definition 4.1. Let $\mathcal{L} = (S, A, \longrightarrow)$ be an NPLTS and $s \in S$. We say that an NPLTS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ is a *resolution* of s obtained *via a randomized scheduler* iff there exists a state correspondence function $corr_{\mathcal{Z}} : Z \to S$ such that $s = corr_{\mathcal{Z}}(z_s)$, for some $z_s \in Z$, and for all $z \in Z$ it holds that:

- If $z \xrightarrow{a}_{\mathcal{Z}} \mathcal{D}$, then there exist $n \in \mathbb{N}_{>0}$, $\{p_i \in \mathbb{R}_{]0,1]} \mid 1 \leq i \leq n\}$, and $\{corr_{\mathcal{Z}}(z) \xrightarrow{a} \mathcal{D}_i \mid 1 \leq i \leq n\}$ such that $\sum_{i=1}^{n} p_i = 1$ and $\mathcal{D}(z') = \sum_{i=1}^{n} p_i \cdot \mathcal{D}_i(corr_{\mathcal{Z}}(z'))$ for all $z' \in Z$.
- If $z \xrightarrow{a_1}_{\mathcal{Z}} \mathcal{D}_1$ and $z \xrightarrow{a_2}_{\mathcal{Z}} \mathcal{D}_2$, then $a_1 = a_2$ and $\mathcal{D}_1 = \mathcal{D}_2$. ■

For each strong behavioral equivalence $\sim$ introduced in [1] and in this paper, we denote by $\sim^{\text{ct}}$ the corresponding equivalence based on combined transitions (ct-equivalence for short), i.e., in which nondeterminism is resolved by means of randomized schedulers. For the eighteen trace-based equivalences and the five testing equivalences, the only modification in their definitions is the use of Res^{ct} in place of Res, where Res^{ct} is the set of resolutions of a state obtained via a randomized scheduler. For the fifteen (bi)simulation-based equivalences, the only modification in their definitions is the direct use of combined transitions (denoted by $\longrightarrow_{\text{c}}$) instead of ordinary transitions.

All the results connecting the various equivalences and the counterexamples showing strict inclusion or incomparability are still valid for the ct-equivalences. A notable exception is given by the counterexamples based on Fig. 4, as the central *offer*-transition of s_1 can now be obtained as a convex combination of the two *offer*-transitions of s_2 with both coefficients equal to 0.5. Indeed, no ct-equivalence can be finer than the corresponding equivalence arising from deterministic schedulers, as matching ordinary transitions induce matching combined transitions.

While every ct-equivalence based on fully matching resolutions is still strictly finer than the corresponding ct-equivalences based on partially matching resolutions or extremal probabilities (the counterexample provided by Fig. 3 is still valid), it turns out that every ct-equivalence based on partially matching resolutions coincides with the corresponding ct-equivalence based on extremal probabilities. As far as $\sim_{\text{PTe-}\sqcup\sqcap}$ and $\sim_{\text{PTe-}\forall\exists}$ are concerned, their ct-variants coincide as well. In other words, when moving to randomized schedulers, the central fragment and the bottom fragment of the spectrum in Fig. 19 collapse. Pictorially, all the ordinary arrows in Fig. 19 going from the central fragment to the bottom one become bidirectional in the presence of randomized schedulers. Moreover, it holds that every ct-equivalence based on extremal probabilities coincides with the corresponding equivalence in the bottom fragment of the spectrum in Fig. 19.

Theorem 4.2. It holds that:

1. $\sim \, \subseteq \, \sim^{\text{ct}}$ for every considered equivalence $\sim$.
2. $\sim^{\text{ct}}_{\pi,\text{dis}} \subseteq \sim^{\text{ct}}_{\pi} = \sim^{\text{ct}}_{\pi,\sqcup\sqcap} = \sim_{\pi,\sqcup\sqcap}$ for all $\pi \in \{\text{PB}, \text{PTe-tbt}, \text{PRTr}, \text{PFTr}, \text{PR}, \text{PF}, \text{PCTr}, \text{PTr}\}$ over image-finite NPLTS models.

3. $\sim^{\rm ct}_{\pi,\rm dis} \subseteq \sim^{\rm ct}_{\pi} = \sim^{\rm ct}_{\pi,\sqcup} = \sim_{\pi,\sqcup}$ for all $\pi \in \{\rm PS, PCS, PFS, PRS\}$ over image-finite NPLTS models.
4. $\sim_{\rm PTe\text{-}\sqcup\sqcap} = \sim^{\rm ct}_{\rm PTe\text{-}\sqcup\sqcap} = \sim^{\rm ct}_{\rm PTe\text{-}\forall\exists} = \sim^{\rm ct}_{\rm PTe\text{-}tbt,dis}$ over image-finite NPLTS models.

Proof Let $(S, A, \longrightarrow)$ be an NPLTS and $s_1, s_2 \in S$:

1. Since matching ordinary transitions induce matching combined transitions, $\sim^{\rm ct}$ performs at least the same identifications as $\sim$.
2. The proof of the fact that $\sim^{\rm ct}_{\pi,\rm dis} \subseteq \sim^{\rm ct}_{\pi} \subseteq \sim^{\rm ct}_{\pi,\sqcup\sqcap}$ is similar to the proof of Thm. 6.5(1) in [1] when $\pi = \rm PB$, to the proof of Thm. 5.9(2) in [1] when $\pi = \rm PTe\text{-}tbt$, and to the proof of Thm. 3.5 in [1] in all the other cases. Moreover, it holds that $\sim^{\rm ct}_{\pi,\sqcup\sqcap} \subseteq \sim^{\rm ct}_{\pi}$ (and hence $\sim^{\rm ct}_{\pi} = \sim^{\rm ct}_{\pi,\sqcup\sqcap}$) when the NPLTS is image finite.
Consider the case $\pi = \rm PB$ and suppose that $s_1 \sim^{\rm ct}_{\rm PB,\sqcup\sqcap} s_2$. This means that there exists a ct-probabilistic $\sqcup\sqcap$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. Given $\mathcal{G} \in 2^{S/\mathcal{B}}$ and $a \in A$, assume that there exists $s_1 \xrightarrow{a} \mathcal{D}_1$ such that $\mathcal{D}_1(\bigcup \mathcal{G}) = p$. Since $(s_1, s_2) \in \mathcal{B}$ and the NPLTS is image finite, there exist $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}'_2$ such that $\mathcal{D}'_2(\bigcup \mathcal{G}) = p' \leq p$ and $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}''_2$ such that $\mathcal{D}''_2(\bigcup \mathcal{G}) = p'' \geq p$. If $p' = p$ (resp. $p'' = p$), then $s_1 \xrightarrow{a} \mathcal{D}_1$ is trivially matched by $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}'_2$ (resp. $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}''_2$) with respect to $\sim^{\rm ct}_{\rm PB}$ when considering $\mathcal{G}$. Assume that $p' < p < p''$ and note that $s_2 \xrightarrow{a}_{\rm c} (x \cdot \mathcal{D}'_2 + y \cdot \mathcal{D}''_2)$ for all $x, y \in \mathbb{R}_{]0,1]}$ such that $x + y = 1$. Indeed, directly from the definition of combined transition, we have that:
   - Since $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}'_2$, there exist $n \in \mathbb{N}_{>0}$, $\{p'_i \in \mathbb{R}_{]0,1]} \mid 1 \leq i \leq n\}$, and $\{s_2 \xrightarrow{a} \hat{\mathcal{D}}'_i \mid 1 \leq i \leq n\}$ such that $\sum_{i=1}^{n} p'_i = 1$ and $\sum_{i=1}^{n} p'_i \cdot \hat{\mathcal{D}}'_i = \mathcal{D}'_2$.
   - Since $s_2 \xrightarrow{a}_{\rm c} \mathcal{D}''_2$, there exist $m \in \mathbb{N}_{>0}$, $\{p''_j \in \mathbb{R}_{]0,1]} \mid 1 \leq j \leq m\}$, and $\{s_2 \xrightarrow{a} \hat{\mathcal{D}}''_j \mid 1 \leq j \leq m\}$ such that $\sum_{j=1}^{m} p''_j = 1$ and $\sum_{j=1}^{m} p''_j \cdot \hat{\mathcal{D}}''_j = \mathcal{D}''_2$.

   Hence, $(x \cdot \mathcal{D}'_2 + y \cdot \mathcal{D}''_2)$ can be obtained from the appropriate combination of:
$$\{s_2 \xrightarrow{a} \hat{\mathcal{D}}'_i \mid 1 \leq i \leq n\} \cup \{s_2 \xrightarrow{a} \hat{\mathcal{D}}''_j \mid 1 \leq j \leq m\}$$
   with coefficients:
$$\{x \cdot p'_i \in \mathbb{R}_{]0,1]} \mid 1 \leq i \leq n\} \cup \{y \cdot p''_j \in \mathbb{R}_{]0,1]} \mid 1 \leq j \leq m\}$$
   If we take $x = \frac{p''-p}{p''-p'}$ and $y = \frac{p-p'}{p''-p'}$, then $s_2 \xrightarrow{a}_{\rm c} \left(\frac{p''-p}{p''-p'} \cdot \mathcal{D}'_2 + \frac{p-p'}{p''-p'} \cdot \mathcal{D}''_2\right)$ with:
$$\begin{aligned}\left(\tfrac{p''-p}{p''-p'} \cdot \mathcal{D}'_2 + \tfrac{p-p'}{p''-p'} \cdot \mathcal{D}''_2\right)(\textstyle\bigcup \mathcal{G}) &= \tfrac{p''-p}{p''-p'} \cdot \mathcal{D}'_2(\textstyle\bigcup \mathcal{G}) + \tfrac{p-p'}{p''-p'} \cdot \mathcal{D}''_2(\textstyle\bigcup \mathcal{G}) \\ &= \tfrac{p''-p}{p''-p'} \cdot p' + \tfrac{p-p'}{p''-p'} \cdot p'' \;=\; \tfrac{p' \cdot p'' - p \cdot p' + p \cdot p'' - p' \cdot p''}{p''-p'} \\ &= p \cdot \tfrac{p''-p'}{p''-p'} \;=\; p \;=\; \mathcal{D}_1(\textstyle\bigcup \mathcal{G})\end{aligned}$$
   Due to the generality of $(s_1, s_2) \in \mathcal{B}$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$, it turns out that $\mathcal{B}$ is also a ct-probabilistic bisimulation, i.e., $s_1 \sim^{\rm ct}_{\rm PB} s_2$.
   The proof for the other seven cases is similar, with actions being replaced by traces and transitions being replaced by resolutions. For instance, suppose that $s_1 \sim^{\rm ct}_{\rm PTr,\sqcup\sqcap} s_2$. Given $\alpha \in A^*$, assume that there exists $\mathcal{Z}_1 \in Res^{\rm ct}_{\alpha}(s_1)$ such that $prob(\mathcal{CC}(z_{s_1}, \alpha)) = p$. Since $s_1 \sim^{\rm ct}_{\rm PTr,\sqcup\sqcap} s_2$ and the NPLTS is image finite, there exist $\mathcal{Z}'_2, \mathcal{Z}''_2 \in Res^{\rm ct}_{\alpha}(s_2)$ such that $prob(\mathcal{CC}(z'_{s_2}, \alpha)) = p' \leq p$ and $prob(\mathcal{CC}(z''_{s_2}, \alpha)) = p'' \geq p$. If $p' = p$ (resp. $p'' = p$), then $\mathcal{Z}_1$ is trivially matched by $\mathcal{Z}'_2$ (resp. $\mathcal{Z}''_2$) with respect to $\sim^{\rm ct}_{\rm PTr}$ when considering α. Assume that $p' < p < p''$ and consider the resolution $\mathcal{Z}_2 = x \cdot \mathcal{Z}'_2 + y \cdot \mathcal{Z}''_2$ of s_2 defined as follows for $x, y \in \mathbb{R}_{]0,1]}$ such that $x + y = 1$. Since $p' \neq p''$ and they both refer to the probability of performing an α-compatible computation from s_2, the two resolutions $\mathcal{Z}'_2$ and $\mathcal{Z}''_2$ of s_2 differ at least in one point in which the nondeterministic choice between two transitions labeled with the same action occurring in α has been resolved differently. We obtain $\mathcal{Z}_2$ from $\mathcal{Z}'_2$ and $\mathcal{Z}''_2$ by combining the two different transitions into a single one with coefficients x and y for their target distributions, respectively, in the first of those points. If we take $x = \frac{p''-p}{p''-p'}$ and $y = \frac{p-p'}{p''-p'}$, then:
$$\begin{aligned} prob(\mathcal{CC}(z_{s_2}, \alpha)) &= \tfrac{p''-p}{p''-p'} \cdot prob(\mathcal{CC}(z'_{s_2}, \alpha)) + \tfrac{p-p'}{p''-p'} \cdot prob(\mathcal{CC}(z''_{s_2}, \alpha)) \\ &= \tfrac{p''-p}{p''-p'} \cdot p' + \tfrac{p-p'}{p''-p'} \cdot p'' \;=\; \tfrac{p' \cdot p'' - p \cdot p' + p \cdot p'' - p' \cdot p''}{p''-p'} \\ &= p \cdot \tfrac{p''-p'}{p''-p'} \;=\; p \;=\; prob(\mathcal{CC}(z_{s_1}, \alpha))\end{aligned}$$

Due to the generality of $\alpha \in A^*$, it turns out that $s_1 \sim_{\mathrm{PTr}}^{\mathrm{ct}} s_2$.
The fact that $\sim_{\pi,\sqcup\sqcap} \subseteq \sim_{\pi,\sqcup\sqcap}^{\mathrm{ct}}$ stems from the first result of this theorem. Moreover, it holds that $\sim_{\pi,\sqcup\sqcap}^{\mathrm{ct}} \subseteq \sim_{\pi,\sqcup\sqcap}$ (and hence $\sim_{\pi,\sqcup\sqcap}^{\mathrm{ct}} = \sim_{\pi,\sqcup\sqcap}$) when the NPLTS is image finite.
Consider the case $\pi = \mathrm{PB}$ and suppose that $s_1 \sim_{\mathrm{PB},\sqcup\sqcap}^{\mathrm{ct}} s_2$. This means that there exists a ct-probabilistic $\sqcup\sqcap$-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. In other words, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ and $a \in A$ it holds that $s'_1 \xrightarrow{a}$ iff $s'_2 \xrightarrow{a}$ and:

$$\begin{aligned}
\bigsqcup_{s'_1 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) &= \bigsqcup_{s'_2 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G}) \\
\bigsqcap_{s'_1 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) &= \bigsqcap_{s'_2 \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})
\end{aligned}$$

Since the NPLTS is image finite, given $\mathcal{G} \in 2^{S/\mathcal{B}}$, $a \in A$, and $s \in S$ having at least one outgoing a-transition, it holds that:

$$\begin{aligned}
\bigsqcup_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}} \mathcal{D}(\bigcup \mathcal{G}) &= \bigsqcup_{s \xrightarrow{a} \mathcal{D}} \mathcal{D}(\bigcup \mathcal{G}) \\
\bigsqcap_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}} \mathcal{D}(\bigcup \mathcal{G}) &= \bigsqcap_{s \xrightarrow{a} \mathcal{D}} \mathcal{D}(\bigcup \mathcal{G})
\end{aligned}$$

because the supremum and the infimum on the left are respectively achieved by two ordinary a-transitions of s. In fact, let $\mathcal{D}_\sqcup$ (resp. $\mathcal{D}_\sqcap$) be the target of an a-transition of s assigning the maximum (resp. minimum) value to $\bigcup \mathcal{G}$ among all the a-transitions of s and consider an arbitrary convex combination of a subset $\{s \xrightarrow{a} \mathcal{D}_i \mid 1 \le i \le n\}$ of those transitions, with coefficients $p_1, \dots, p_n$ and $n \in \mathbb{N}_{>0}$. Then:

$$\begin{aligned}
\sum_{i=1}^{n} p_i \cdot \mathcal{D}_i(\bigcup \mathcal{G}) \;&\le\; \sum_{i=1}^{n} p_i \cdot \mathcal{D}_\sqcup(\bigcup \mathcal{G}) \;=\; \mathcal{D}_\sqcup(\bigcup \mathcal{G}) \\
\sum_{i=1}^{n} p_i \cdot \mathcal{D}_i(\bigcup \mathcal{G}) \;&\ge\; \sum_{i=1}^{n} p_i \cdot \mathcal{D}_\sqcap(\bigcup \mathcal{G}) \;=\; \mathcal{D}_\sqcap(\bigcup \mathcal{G})
\end{aligned}$$

As a consequence, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $s'_1 \xrightarrow{a}$ iff $s'_2 \xrightarrow{a}$ and:

$$\begin{aligned}
\bigsqcup_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) &= \bigsqcup_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G}) \\
\bigsqcap_{s'_1 \xrightarrow{a} \mathcal{D}_1} \mathcal{D}_1(\bigcup \mathcal{G}) &= \bigsqcap_{s'_2 \xrightarrow{a} \mathcal{D}_2} \mathcal{D}_2(\bigcup \mathcal{G})
\end{aligned}$$

This means that $\mathcal{B}$ is also a probabilistic $\sqcup\sqcap$-bisimulation, i.e., $s_1 \sim_{\mathrm{PB},\sqcup\sqcap} s_2$.
The proof for the other seven cases is similar, with actions being replaced by traces and transitions being replaced by resolutions. For instance, suppose that $s_1 \sim_{\mathrm{PTr},\sqcup\sqcap}^{\mathrm{ct}} s_2$. This means that for all $\alpha \in A^*$:

$$\begin{aligned}
\bigsqcup_{\mathcal{Z}_1 \in Res_\alpha^{\mathrm{ct}}(s_1)} prob(\mathcal{CC}(z_{s_1}, \alpha)) &= \bigsqcup_{\mathcal{Z}_2 \in Res_\alpha^{\mathrm{ct}}(s_2)} prob(\mathcal{CC}(z_{s_2}, \alpha)) \\
\bigsqcap_{\mathcal{Z}_1 \in Res_\alpha^{\mathrm{ct}}(s_1)} prob(\mathcal{CC}(z_{s_1}, \alpha)) &= \bigsqcap_{\mathcal{Z}_2 \in Res_\alpha^{\mathrm{ct}}(s_2)} prob(\mathcal{CC}(z_{s_2}, \alpha))
\end{aligned}$$

Given $\alpha \in A^*$ and $s \in S$, it holds that:

$$\begin{aligned}
\bigsqcup_{\mathcal{Z} \in Res_\alpha^{\mathrm{ct}}(s)} prob(\mathcal{CC}(z_s, \alpha)) &= \bigsqcup_{\mathcal{Z} \in Res_\alpha(s)} prob(\mathcal{CC}(z_s, \alpha)) \\
\bigsqcap_{\mathcal{Z} \in Res_\alpha^{\mathrm{ct}}(s)} prob(\mathcal{CC}(z_s, \alpha)) &= \bigsqcap_{\mathcal{Z} \in Res_\alpha(s)} prob(\mathcal{CC}(z_s, \alpha))
\end{aligned}$$

In fact, observing that:

$$\bigsqcup_{\mathcal{Z} \in Res_\alpha^{\mathrm{ct}}(s)} prob(\mathcal{CC}(z_s, \alpha)) \ge \bigsqcup_{\mathcal{Z} \in Res_\alpha(s)} prob(\mathcal{CC}(z_s, \alpha))$$

because the set of probabilities on the left contains the set of probabilities on the right (a dual property based on $\le$ holds for infima), we prove that:

$$\bigsqcup_{\mathcal{Z} \in Res_\alpha^{\mathrm{ct}}(s)} prob(\mathcal{CC}(z_s, \alpha)) \le \bigsqcup_{\mathcal{Z} \in Res_\alpha(s)} prob(\mathcal{CC}(z_s, \alpha))$$

by proceeding by induction on the length n of α (a dual property based on $\ge$ can be proved for infima):

- If $n = 0$, i.e., $\alpha = \varepsilon$, then:

$$\bigsqcup_{\mathcal{Z}\in Res^{\mathrm{ct}}_{\alpha}(s)} prob(\mathcal{CC}(z_s,\alpha)) \;=\; 1 \;=\; \bigsqcup_{\mathcal{Z}\in Res_{\alpha}(s)} prob(\mathcal{CC}(z_s,\alpha))$$

- Let $n \in \mathbb{N}_{>0}$ and suppose that the property holds for all traces of length $m = 0,\dots,n-1$. Assume that $\alpha = a\,\alpha'$. If s has no outgoing a-transitions (an outgoing non-a-transition in the case of infima), then:
$$\bigsqcup_{\mathcal{Z}\in Res^{\mathrm{ct}}_{\alpha}(s)} prob(\mathcal{CC}(z_s,\alpha)) \;=\; 0 \;=\; \bigsqcup_{\mathcal{Z}\in Res_{\alpha}(s)} prob(\mathcal{CC}(z_s,\alpha))$$
otherwise, indicating with $s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}$ a combined transition from s with $\mathcal{D}_{\mathrm{c}} = \sum_{i=1}^{m} p_i \cdot \mathcal{D}_i$, we have that:
$$\begin{aligned}
&\bigsqcup_{\mathcal{Z}\in Res^{\mathrm{ct}}_{\alpha}(s)} prob(\mathcal{CC}(z_s,\alpha)) \;=\\
&= \bigsqcup_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{s'\in S} \left(\mathcal{D}_{\mathrm{c}}(s') \cdot \bigsqcup_{\mathcal{Z}'\in Res^{\mathrm{ct}}_{\alpha'}(s')} prob(\mathcal{CC}(z'_{s'},\alpha'))\right)\\
&\le \bigsqcup_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{s'\in S} \left(\mathcal{D}_{\mathrm{c}}(s') \cdot \bigsqcup_{\mathcal{Z}'\in Res_{\alpha'}(s')} prob(\mathcal{CC}(z'_{s'},\alpha'))\right)\\
&= \bigsqcup_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{s'\in S} \left(\sum_{i=1}^{m}(p_i \cdot \mathcal{D}_i(s')) \cdot \bigsqcup_{\mathcal{Z}'\in Res_{\alpha'}(s')} prob(\mathcal{CC}(z'_{s'},\alpha'))\right)\\
&= \bigsqcup_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{i=1}^{m} p_i \cdot \left(\sum_{s'\in S}\left(\mathcal{D}_i(s') \cdot \bigsqcup_{\mathcal{Z}'\in Res_{\alpha'}(s')} prob(\mathcal{CC}(z'_{s'},\alpha'))\right)\right)\\
&\le \bigsqcup_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{i=1}^{m} p_i \cdot \bigsqcup_{i=1}^{m} \left(\sum_{s'\in S}\left(\mathcal{D}_i(s') \cdot \bigsqcup_{\mathcal{Z}'\in Res_{\alpha'}(s')} prob(\mathcal{CC}(z'_{s'},\alpha'))\right)\right)\\
&= \bigsqcup_{s \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \bigsqcup_{i=1}^{m} \left(\sum_{s'\in S}\left(\mathcal{D}_i(s') \cdot \bigsqcup_{\mathcal{Z}'\in Res_{\alpha'}(s')} prob(\mathcal{CC}(z'_{s'},\alpha'))\right)\right)\\
&= \bigsqcup_{s \xrightarrow{a} \mathcal{D}} \sum_{s'\in S} \left(\mathcal{D}(s') \cdot \bigsqcup_{\mathcal{Z}'\in Res_{\alpha'}(s')} prob(\mathcal{CC}(z'_{s'},\alpha'))\right)\\
&= \bigsqcup_{\mathcal{Z}\in Res_{\alpha}(s)} prob(\mathcal{CC}(z_s,\alpha))
\end{aligned}$$
where in the third line we have exploited the induction hypothesis and in the seventh line the fact that $\sum_{i=1}^{m} p_i = 1$.

As a consequence, for all $\alpha \in A^*$:
$$\bigsqcup_{\mathcal{Z}_1\in Res_{\alpha}(s_1)} prob(\mathcal{CC}(z_{s_1},\alpha)) \;=\; \bigsqcup_{\mathcal{Z}_2\in Res_{\alpha}(s_2)} prob(\mathcal{CC}(z_{s_2},\alpha))$$
$$\bigsqcap_{\mathcal{Z}_1\in Res_{\alpha}(s_1)} prob(\mathcal{CC}(z_{s_1},\alpha)) \;=\; \bigsqcap_{\mathcal{Z}_2\in Res_{\alpha}(s_2)} prob(\mathcal{CC}(z_{s_2},\alpha))$$
This means that $s_1 \sim_{\mathrm{PTr},\sqcup\sqcap} s_2$.

3. The proof of the fact that $\sim^{\mathrm{ct}}_{\pi,\mathrm{dis}} \subseteq \sim^{\mathrm{ct}}_{\pi} = \sim^{\mathrm{ct}}_{\pi,\sqcup}$ is similar to the proof of Thm. 3.18(1).
The proof of the fact that $\sim^{\mathrm{ct}}_{\pi,\sqcup} = \sim_{\pi,\sqcup}$ is similar to the proof of the corresponding part of the second result of this theorem for bisimulation semantics.
4. We start by proving that $\sim_{\mathrm{PTe}\text{-}\sqcup\sqcap} = \sim^{\mathrm{ct}}_{\mathrm{PTe}\text{-}\sqcup\sqcap}$. Given an arbitrary state $s \in S$ and an arbitrary NPT $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$, it holds that:
$$\bigsqcup_{\mathcal{Z}\in Res^{\mathrm{ct}}_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o})) \;=\; \bigsqcup_{\mathcal{Z}\in Res_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o}))$$
$$\bigsqcap_{\mathcal{Z}\in Res^{\mathrm{ct}}_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o})) \;=\; \bigsqcap_{\mathcal{Z}\in Res_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o}))$$
In fact, first of all we note that:
$$\bigsqcup_{\mathcal{Z}\in Res^{\mathrm{ct}}_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o})) \;\ge\; \bigsqcup_{\mathcal{Z}\in Res_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o}))$$
because a deterministic scheduler is a special case of randomized scheduler and hence the set of probabilities on the left contains the set of probabilities on the right (a dual property based on $\le$ holds for

infima). Therefore, it suffices to show that:

$$\bigsqcup_{\mathcal{Z} \in Res^{\mathrm{ct}}_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o})) \;\leq\; \bigsqcup_{\mathcal{Z} \in Res_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o}))$$

as we prove below by proceeding by induction on the length n of the longest successful computation from (s, o), which is finite because $\mathcal{T}$ is finite (a dual property based on $\geq$ can be established for infima):

- If $n = 0$, i.e., $o = \omega$, then:

$$\bigsqcup_{\mathcal{Z} \in Res^{\mathrm{ct}}_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o})) \;=\; 1 \;=\; \bigsqcup_{\mathcal{Z} \in Res_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o}))$$

- Let $n \in \mathbb{N}_{>0}$ and suppose that the property holds for all configurations from which the longest successful computation has length $m = 0, \ldots, n-1$. Indicating with $(s, o) \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}$ a combined transition from (s, o) with $\mathcal{D}_{\mathrm{c}} = \sum_{i=1}^{m} p_i \cdot \mathcal{D}_i$, we have that:

$$\begin{aligned}
&\bigsqcup_{\mathcal{Z} \in Res^{\mathrm{ct}}_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o})) \;=\\
&\quad= \bigsqcup_{(s,o) \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{(s',o') \in S \times O} \Big(\mathcal{D}_{\mathrm{c}}(s', o') \cdot \bigsqcup_{\mathcal{Z}' \in Res^{\mathrm{ct}}_{\max}(s',o')} prob(\mathcal{SC}(z'_{s',o'})) \Big)\\
&\quad\leq \bigsqcup_{(s,o) \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{(s',o') \in S \times O} \Big(\mathcal{D}_{\mathrm{c}}(s', o') \cdot \bigsqcup_{\mathcal{Z}' \in Res_{\max}(s',o')} prob(\mathcal{SC}(z'_{s',o'})) \Big)\\
&\quad= \bigsqcup_{(s,o) \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{(s',o') \in S \times O} \Big(\sum_{i=1}^{m} (p_i \cdot \mathcal{D}_i(s', o')) \cdot \bigsqcup_{\mathcal{Z}' \in Res_{\max}(s',o')} prob(\mathcal{SC}(z'_{s',o'})) \Big)\\
&\quad= \bigsqcup_{(s,o) \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{i=1}^{m} p_i \cdot \Big(\sum_{(s',o') \in S \times O} \Big(\mathcal{D}_i(s', o') \cdot \bigsqcup_{\mathcal{Z}' \in Res_{\max}(s',o')} prob(\mathcal{SC}(z'_{s',o'})) \Big) \Big)\\
&\quad\leq \bigsqcup_{(s,o) \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \sum_{i=1}^{m} p_i \cdot \bigsqcup_{i=1}^{m} \Big(\sum_{(s',o') \in S \times O} \Big(\mathcal{D}_i(s', o') \cdot \bigsqcup_{\mathcal{Z}' \in Res_{\max}(s',o')} prob(\mathcal{SC}(z'_{s',o'})) \Big) \Big)\\
&\quad= \bigsqcup_{(s,o) \xrightarrow{a}_{\mathrm{c}} \mathcal{D}_{\mathrm{c}}} \bigsqcup_{i=1}^{m} \Big(\sum_{(s',o') \in S \times O} \Big(\mathcal{D}_i(s', o') \cdot \bigsqcup_{\mathcal{Z}' \in Res_{\max}(s',o')} prob(\mathcal{SC}(z'_{s',o'})) \Big) \Big)\\
&\quad= \bigsqcup_{(s,o) \xrightarrow{a} \mathcal{D}} \sum_{(s',o') \in S \times O} \Big(\mathcal{D}(s', o') \cdot \bigsqcup_{\mathcal{Z}' \in Res_{\max}(s',o')} prob(\mathcal{SC}(z'_{s',o'})) \Big)\\
&\quad= \bigsqcup_{\mathcal{Z} \in Res_{\max}(s,o)} prob(\mathcal{SC}(z_{s,o}))
\end{aligned}$$

  where in the third line we have exploited the induction hypothesis and in the seventh line the fact that $\sum_{i=1}^{m} p_i = 1$.

We now prove that $\sim_{\text{PTe-}\sqcup\sqcap} = \sim^{\mathrm{ct}}_{\text{PTe-}\forall\exists}$. Suppose that $s_1 \sim_{\text{PTe-}\sqcup\sqcap} s_2$ and consider an arbitrary NPT $\mathcal{T} = (O, A, \longrightarrow_{\mathcal{T}})$ with initial state $o \in O$, so that:

$$\bigsqcup_{\mathcal{Z}_1 \in Res_{\max}(s_1,o)} prob(\mathcal{SC}(z_{s_1,o})) \;=\; p_{\sqcup} \;=\; \bigsqcup_{\mathcal{Z}_2 \in Res_{\max}(s_2,o)} prob(\mathcal{SC}(z_{s_2,o}))$$

$$\bigsqcap_{\mathcal{Z}_1 \in Res_{\max}(s_1,o)} prob(\mathcal{SC}(z_{s_1,o})) \;=\; p_{\sqcap} \;=\; \bigsqcap_{\mathcal{Z}_2 \in Res_{\max}(s_2,o)} prob(\mathcal{SC}(z_{s_2,o}))$$

If $p_{\sqcup} = p_{\sqcap}$, then all the maximal resolutions of (s_1, o) and (s_2, o) have the same success probability, from which it trivially follows that $s_1 \sim_{\text{PTe-}\forall\exists} s_2$ and hence $s_1 \sim^{\mathrm{ct}}_{\text{PTe-}\forall\exists} s_2$.
Recalling that the NPLTS is image finite and the test is finite so that $Res_{\max}(s_1, o)$ and $Res_{\max}(s_2, o)$ are both finite, if $p_{\sqcup} > p_{\sqcap}$, then $p_{\sqcup}$ must be achieved on $\mathcal{Z}_{1,\sqcup} \in Res_{\max}(s_1, o)$ and $\mathcal{Z}_{2,\sqcup} \in Res_{\max}(s_2, o)$ exhibiting the same successful traces, otherwise – observing that both resolutions must have at least one successful trace, otherwise it would be $p_{\sqcup} = 0$ thus violating $p_{\sqcup} > p_{\sqcap}$ – states s_1 and s_2 would be distinguished with respect to $\sim_{\text{PTe-}\sqcup\sqcap}$ by a test obtained from $\mathcal{T}$ by making success reachable only along the successful traces of the one of $\mathcal{Z}_{1,\sqcup}$ and $\mathcal{Z}_{2,\sqcup}$ having a successful trace not possessed by the other, unless that resolution also contains all the successful traces of the other resolution, in which

case success must be made reachable only along the successful traces of the other resolution in order to contradict $s_1 \sim_{\mathrm{PTe}\text{-}\sqcup\sqcap} s_2$.

Likewise, $p_\sqcap$ must be achieved on $\mathcal{Z}_{1,\sqcap} \in \mathit{Res}_{\max}(s_1, o)$ and $\mathcal{Z}_{2,\sqcap} \in \mathit{Res}_{\max}(s_2, o)$ exhibiting the same unsuccessful maximal traces, otherwise – observing that both resolutions must have at least one unsuccessful maximal trace, otherwise it would be $p_\sqcap = 1$ thus violating $p_\sqcup > p_\sqcap$ – states s_1 and s_2 would be distinguished with respect to $\sim_{\mathrm{PTe}\text{-}\sqcup\sqcap}$ by a test obtained from $\mathcal{T}$ by making success reachable also along an unsuccessful maximal trace occurring only in either $\mathcal{Z}_{1,\sqcap}$ or $\mathcal{Z}_{2,\sqcap}$.

By reasoning on the dual test $\mathcal{T}'$ in which the final states of $\mathcal{T}$ that are successful (resp. unsuccessful) are made unsuccessful (resp. successful), it turns out that $\mathcal{Z}_{1,\sqcup}$ and $\mathcal{Z}_{2,\sqcup}$ must also exhibit the same unsuccessful maximal traces and that $\mathcal{Z}_{1,\sqcap}$ and $\mathcal{Z}_{2,\sqcap}$ must also exhibit the same successful traces.

If $\mathcal{Z}_{1,\sqcup}$ and $\mathcal{Z}_{2,\sqcup}$ do not have sequences of initial transitions in common with $\mathcal{Z}_{1,\sqcap}$ and $\mathcal{Z}_{2,\sqcap}$, then $\mathcal{Z}_{1,\sqcup}$ and $\mathcal{Z}_{1,\sqcap}$ on one side and $\mathcal{Z}_{2,\sqcup}$ and $\mathcal{Z}_{2,\sqcap}$ on the other side cannot generate via convex combinations any new resolution that would arise from a randomized scheduler, otherwise they can generate all such resolutions having a certain sequence of initial transitions, thus covering all the intermediate success probabilities between $p_\sqcup$ and $p_\sqcap$ for that sequence of initial transitions. This shows that for each $\mathcal{Z}_1 \in \mathit{Res}^{\mathrm{ct}}_{\max}(s_1, o)$ with that sequence of initial transitions there exists $\mathcal{Z}_2 \in \mathit{Res}^{\mathrm{ct}}_{\max}(s_2, o)$ with that sequence of initial transitions such that $\mathit{prob}(\mathcal{SC}(z_{s_1,o})) = \mathit{prob}(\mathcal{SC}(z_{s_2,o}))$, and vice versa.

The same procedure can now be applied to the remaining resolutions in $\mathit{Res}_{\max}(s_1, o)$ and $\mathit{Res}_{\max}(s_2, o)$ that are not convex combinations of previously considered resolutions, starting from those among the remaining resolutions on which the maximal and minimal success probabilities are achieved. We can thus conclude that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PTe}\text{-}\forall\exists} s_2$.

The fact that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PTe}\text{-}\forall\exists} s_2$ implies $s_1 \sim_{\mathrm{PTe}\text{-}\sqcup\sqcap} s_2$ follows from the fact that $s_1 \sim^{\mathrm{ct}}_{\mathrm{PTe}\text{-}\forall\exists} s_2$ implies $s_1 \sim^{\mathrm{ct}}_{\mathrm{PTe}\text{-}\sqcup\sqcap} s_2$ (the proof is similar to that of Thm. 5.9(1) in [1]) and from $\sim^{\mathrm{ct}}_{\mathrm{PTe}\text{-}\sqcup\sqcap} = \sim_{\mathrm{PTe}\text{-}\sqcup\sqcap}$.

Finally, $\sim^{\mathrm{ct}}_{\mathrm{PTe}\text{-}\forall\exists} = \sim^{\mathrm{ct}}_{\mathrm{PTe}\text{-tbt,dis}}$ is a straightforward consequence of Thm. 5.9(2) in [1]. ■

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