ULTraS at Work: Compositionality Metaresults for Bisimulation and Trace Semantics

Marco Bernardo

Dipartimento di Scienze Pure e Applicate, Università di Urbino, Italy

Abstract

The ULTraS metamodel can be instantiated to a large number of well established models, including labeled transition systems together with their probabilistic, deterministic timed, and stochastic timed extensions. Several metaequivalences have been defined on the ULTraS metamodel, which can be instantiated to well known behavioral equivalences for those specific models. However, new equivalences pop up, instead of the widely accepted ones, in the case of processes featuring probabilities and internal nondeterminism. Most importantly, the properties of the metaequivalences have not been investigated yet. Focusing on bisimulation and trace semantics, we first show that, by simply introducing the notion of resolution in the ULTraS theory, and exchanging the order of certain universal quantifiers in the definition of the metaequivalences, it is possible to retrieve the behavioral equivalences not captured before, as well as to keep the new ones. We then study the compositionality, with respect to typical process algebraic operators, of the two bisimulation metaequivalences and of the two trace metaequivalences respectively arising from the two different quantification orders. The congruence metaresults for parallel composition confirm the existence of a foundational difference in the compositionality of bisimulation and trace semantics when internal nondeterminism is present, which had recently emerged in the specific setting of probabilistic and nondeterministic processes.

Keywords: behavioral metamodels, compositionality, bisimulation semantics, trace semantics

1. Introduction

The purpose of *behavioral metamodels* is, on the one hand, to offer a uniform view of models that have appeared in the literature (unifying theories) and, on the other hand, to provide general methodologies, results, and tools that can be instantiated to a wide range of specific behavioral models (reuse facilities). In particular, the activity of developing a new theory, model, calculus, or language accounting for certain behavioral aspects would naturally benefit from the existence of a suitable metamodel. This should reduce the effort needed for defining syntax, semantics, and behavioral relations, investigating compositionality as well as equational and logical characterizations, and designing verification algorithms.

Frameworks like operational semantic rule formats [1], Segala probabilistic automata [34], and weighted automata [16] can be viewed to some extent as behavioral metamodels, even though their emphasis is more on ensuring certain properties in a general setting or achieving a higher level of expressivity. Only recently metamodels have been developed with the *explicit* purpose of paving the way to unifying theories and reuse facilities in the field of concurrency, without resorting to abstract representations such as the categorical ones based, e.g., on coalgebras and bialgebras. Among these recent metamodels, we mention those based on the labeled transition system model [26] that are known as weighted labeled transition systems (WLTS), state-to-function labeled transition systems (FuTS), and uniform labeled transition systems (ULTraS).

The WLTS metamodel [27] relies on *commutative monoids* to express and combine weights attached to transition labels under a *weight determinacy condition*. Such a condition guarantees the uniqueness of the weight distribution associated with the transitions labeled with the same action that depart from the same state, thereby excluding internal nondeterminism. This metamodel is equipped with a notion of weighted bisimilarity and a rule format ensuring the compositionality of bisimulation semantics.

In contrast, the other two metamodels do not impose any weight determinacy. They share the idea that an action-labeled transition should go from a state to a *function/distribution over states*, rather than to a single state, in the same spirit as the probabilistic models of [29, 34]. The function/distribution associates with every state its *degree of reachability* – taken from a suitably structured set – via that transition.

The FuTS metamodel [12] is a unifying framework for providing the operational semantics of process calculi possibly including various aspects of different nature. Algebraic structures in the form of *commutative semirings* are systematically employed for a compositional and compact definition of the operational semantic rules. This metamodel supports a precise understanding of similarities and differences among process calculi of the same class, a fact exploited in particular for the many stochastic process calculi that have appeared in the literature. The framework was subsequently enriched in [30] with combined and nested variants of the FuTS metamodel as well as a notion of bisimilarity addressed from a coalgebraic viewpoint.

The ULTraS metamodel, proposed in [4] and then extended in [8], does not address the linguistic level, but focuses instead on models and equivalences. By simply relying on *preordered sets equipped with minimum*, it is general enough to encompass labeled transition systems, action-labeled discrete-/continuous-time Markov chains, discrete-/continuous-time Markov decision processes possibly enriched with internal nondeterminism, timed automata, probabilistic timed automata, and Markov automata.

As far as behavioral equivalences are concerned, in [4] three definitions were presented for the ULTraS framework, which are respectively inspired by the well known bisimulation, testing, and trace approaches [19]. Such definitions rely on the concept of *measure function*, which measures the degree of reachability of a set of states when executing a certain sequence of actions from a given state, thereby lifting to multi-step reachability the notion of one-step reachability embodied in transitions. By suitably instantiating the measure function, the three metaequivalences, whose general properties have not been studied yet, can be reduced to behavioral equivalences that have appeared in the literature for specific models.

However, there is a notable exception, given by processes featuring probabilities and internal nondeterminism such as Segala simple probabilistic automata [34], for which *new* equivalences arise from the instantiation procedure. In the case of bisimulation semantics, so far within the ULTraS framework it is not possible to capture the probabilistic bisimilarity of Segala and Lynch [36]. Instead, a coarser bisimilarity emerges. As recently investigated in [7], this equivalence is characterized by a suitably reinterpreted version of the probabilistic modal logic of [29], but it is *not* a congruence with respect to parallel composition. Likewise, as far as trace semantics is concerned, the probabilistic trace-distribution equivalence of Segala [35] is not obtained. In contrast, a coarser trace equivalence is generated. As later studied in [5], this equivalence *is* a congruence with respect to parallel composition, as opposed to the negative compositionality results of [25, 35] for other probabilistic trace equivalences.

The objective of this paper is twofold. Firstly, we further generalize the definition of the bisimulation and trace metaequivalences, so to obtain also the widely accepted equivalences of [36] and [35] that were not captured in [4]. This is achieved by introducing in the ULTraS setting the notion of *resolution* – borrowed from testing theories for probabilistic and nondeterministic processes [42, 24, 23, 13] – which, from arbitrary ULTraS models, allows us to derive submodels in which every state has at most one outgoing transition.

Having this notion of resolution available, we can define bisimulation and trace metaequivalences in two different ways. In the first case, which we call *pre*-metaequivalence, the quantification over classes of equivalent states (for bisimulation semantics) or traces (for trace semantics) occurs *before* establishing a matching among resolutions, so that a resolution of the challenger model can be matched by *several* resolutions of the defender model with respect to *different* classes or traces. This captures the new equivalences emerged in [4]. In the second case, which we call *post*-metaequivalence, the quantification over classes or traces occurs *after* establishing a matching among resolutions, so that a resolution of the challenger model has to be matched by a *single* resolution of the defender model with respect to *all* classes or traces. This captures the well known equivalences of [36] and [35].

Incidentally, the introduction of resolutions has the side effect of reconciling the ULTraS metamodel with the FuTS metamodel of [12]. The reason is that working with resolutions causes measure functions employed in the definition of the metaequivalences to return always a *single* value, a fact not possible in [4] in the presence of internal nondeterminism, i.e., of several identically labeled transitions departing from the same state. In turn, this causes a semiring structure to emerge in the ULTraS framework, where the multiplicative operation enables the calculation of multi-step reachability from values of consecutive single-step reachability along the same trajectory, while the additive operation enables the aggregation of values of multi-step reachability along different trajectories starting from the same state.

The second, and most important, objective of this paper is to put ULTraS at work by investigating general properties of the newly defined bisimulation and trace metaequivalences. In particular, we concentrate on the compositionality of those metaequivalences with respect to generalizations of typical process algebraic operators such as action prefix, guarded choice, nondeterministic choice, and parallel composition.

The congruence metaresults for parallel composition, together with the accompanying counterexamples, confirm the existence of a foundational difference in the compositionality of bisimulation and trace semantics, which had recently emerged [7, 5] in the specific setting of Segala simple probabilistic automata [34]. The difference shows up in the presence of internal nondeterminism. For bisimulation semantics, only the post-metaequivalence is always compositional and, under certain conditions, it turns out to be the coarsest congruence with respect to parallel composition included in the pre-metaequivalence. In contrast, for trace semantics, only the pre-metaequivalence, which is coarser than the post-metaequivalence, is always a congruence with respect to parallel composition.

Paper outline: In Sect. 2, we recall the ULTraS metamodel and introduce the notion of resolution. In Sect. 3, we revisit measure functions as well as bisimulation and trace metaequivalences, by replacing the former with measure schemata relying on explicit semiring structures and the latter with pre- and post-metaequivalences. In Sect. 4, we exhibit congruence metaresults and counterexamples with respect to typical process algebraic operators for the four newly defined metaequivalences. Finally, Sect. 5 concludes the paper.

2. Importing Resolutions in the ULTraS Metamodel

In this section, after recalling the ULTraS metamodel, we enrich it by introducing the notion of resolution, which we borrow from testing theories for nondeterministic and probabilistic processes [42, 24, 23, 13]. As we will see in the next section, resolutions are instrumental to capture in the ULTraS framework some well known behavioral equivalences, such as the probabilistic bisimilarity of [36] and the probabilistic trace-distribution equivalence of [35], which could not be addressed in [4].

The definition of ULTraS has a discrete-state structure and is parameterized with respect to a set D. According to [4], D-values are interpreted as different degrees of *one-step reachability*, ordered based on a relation $\sqsubseteq_D$ that is equipped with minimum $\bot_D$ expressing *unreachability*. Let us denote by $(S \rightarrow D)$ the set of functions from a set S to D. When S is a set of states, every element Δ of $(S \rightarrow D)$ can be interpreted as a function that *distributes reachability* over all possible next states. We call *support* of Δ the set $supp(\Delta) = \{s \in S \mid \Delta(s) \neq \bot_D\}$ of states that are reachable according to the D-distribution Δ over S.

As a slight refinement with respect to the definitions in [4, 8], below we consider the set $(S \rightarrow D)_{\rm nefs}$ of D-distributions Δ over S such that $0 < |supp(\Delta)| < \omega$. The first constraint establishes that the target distribution of each transition has a nonempty support, so to avoid distributions always returning $\bot_D$ and hence transitions leading to nowhere. The second constraint ensures that the same distribution has a finite support, which will be exploited for a correct definition of measure functions for metaequivalences in Sect. 3.

Definition 2.1. Let $(D, \sqsubseteq_D, \bot_D)$ be a preordered set equipped with minimum. A *uniform labeled transition system* on it, or D-ULTraS for short, is a triple $\mathcal{U} = (S, A, \longrightarrow)$ where:

- $S \neq \emptyset$ is an at most countable set of states.
- $A \neq \emptyset$ is a countable set of transition-labeling actions.
- $\longrightarrow \subseteq S \times A \times (S \rightarrow D)_{\rm nefs}$ is a transition relation. ■

Every transition (s, a, Δ) of $\mathcal{U}$ is written $s \xrightarrow{a} \Delta$, where $\Delta(s')$ is a D-value quantifying – in a broad sense – the degree of reachability of s' from s via that transition, and $\Delta(s') = \bot_D$ means that s' is not reachable with that transition. In the directed graph description of $\mathcal{U}$ (see the forthcoming Figs. 1, 2, and 3), vertices represent states and action-labeled edges represent action-labeled transitions. Given a transition $s \xrightarrow{a} \Delta$, the corresponding a-labeled edge goes from the vertex representing state s to a set of vertices linked by a dashed line, each of which represents a state $s' \in supp(\Delta)$ and is labeled with $\Delta(s')$.

Example 2.2. As shown in [4, 8], we can use the set $\mathbb{B} = \{\bot, \top\}$ with $\bot \sqsubseteq_{\mathbb{B}} \top$ for capturing labeled transition systems [26] and timed automata [2], the set $\mathbb{R}_{[0,1]}$ with the usual $\leq$ for capturing action-labeled discrete-time Markov chains [39], Markov decision processes [14], probabilistic automata [34], probabilistic timed automata [28], and Markov automata [17], and the set $\mathbb{R}_{\geq 0}$ with the usual $\leq$ for capturing action-labeled continuous-time Markov chains [39] and continuous-time Markov decision processes [33]. ■

The presence of several transitions departing from the same state of $\mathcal{U}$ describes a choice among different behaviors. This choice is not necessarily nondeterministic, as shown by the ULTraS encoding of fully probabilistic models exhibited in [4]; however, it is surely nondeterministic in the case of identically labeled transitions (*internal nondeterminism*). We call *resolution* of a state s of $\mathcal{U}$ the result of a possible way of resolving choices starting from s, as if a deterministic scheduler were applied that, at the current state, selects one of its outgoing transitions or no transitions at all. Other classes of schedulers, such as the randomized ones (see, e.g., [34]), are not considered in this paper because their applicability depends on the parameter D of the ULTraS.

We formalize a resolution as a tree-like structure, whose branching points correspond to target distributions of transitions, obtained by unfolding from s the graph structure of $\mathcal{U}$ and by selecting at each reached state at most one of its outgoing transitions. Since $\mathcal{U}$ can be cyclic, as in [23, 13] we make use of a correspondence function from the unfolded state space to the original state space. This function must be injective over the support of the target distribution of each transition of the unfolded state space, so to guarantee that the transition simulates the corresponding one selected in the original state space.

Definition 2.3. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS. A D-ULTraS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$, with no cycles and Z disjoint from S, is a *resolution* of $s \in S$, written $\mathcal{Z} \in Res(s)$, iff there exists a state correspondence function $corr_{\mathcal{Z}} : Z \to S$ such that $s = corr_{\mathcal{Z}}(z_s)$ for some $z_s \in Z$, and for all $z \in Z$ it holds that:

- If $z \xrightarrow{a}_{\mathcal{Z}} \Delta$, then $corr_{\mathcal{Z}}(z) \xrightarrow{a} \Delta'$ with $corr_{\mathcal{Z}}$ being injective over $supp(\Delta)$ and $\Delta(z') = \Delta'(corr_{\mathcal{Z}}(z'))$ for all $z' \in supp(\Delta)$.
- State z has at most one outgoing transition. ■

We observe that the WLTS metamodel of [27] and the notion of resolution for the ULTraS metamodel share the idea of ruling out internal nondeterminism. However, the latter is more restrictive than the former, in that it has a tree-like structure and admits at most one transition from any of its states.

For bisimulation semantics, we need to resolve choices only at the first step or, more generally, only at the first k steps in case of a multi-step definition of bisimilarity. We thus introduce a notion of *partial* resolution – which has the same characteristics as a resolution in its initial part, i.e., states not in S for guaranteeing the absence of cycles and choices – whose states and transitions are identical to the original ones after the first k steps. This will be useful in the next section to easily identify the classes of equivalent states reached after performing a transition.

Definition 2.4. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS. A D-ULTraS $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}})$ is a *k-resolution* of $s \in S$ for $k \in \mathbb{N}_{\geq 1}$, written $\mathcal{Z} \in k\text{-}Res(s)$, iff there exists a state correspondence function $corr_{\mathcal{Z}} : Z \to S$ such that $s = corr_{\mathcal{Z}}(z_s)$ for some $z_s \in Z$, and for all $z \in Z$ it holds that:

- If $z \xrightarrow{a}_{\mathcal{Z}} \Delta$, then $corr_{\mathcal{Z}}(z) \xrightarrow{a} \Delta'$ with $corr_{\mathcal{Z}}$ being injective over $supp(\Delta)$ and $\Delta(z') = \Delta'(corr_{\mathcal{Z}}(z'))$ for all $z' \in supp(\Delta)$.
- If z is reachable from z_s with a sequence of less than k transitions, then:
  - $z \notin S$;
  - z cannot be part of a cycle;
  - z has at most one outgoing transition;

  otherwise z is equal to $corr_{\mathcal{Z}}(z) \in S$ and has the same outgoing transitions that it has in $\mathcal{U}$. ■

3. Revisiting ULTraS Behavioral Metaequivalences

Various metaequivalences were presented in the ULTraS framework of [4] whose definitions are parameterized with respect to a *measure function*. This expresses the degree of *multi-step reachability* (i.e., reachability after performing a sequence of actions) of a *set of states*, in terms of values taken from a preordered set equipped with minimum, which does not necessarily coincide with the one on which the ULTraS is based. When suitably instantiating the measure function, the resulting equivalences were shown to coincide, in most cases, with those that have appeared in the literature. However, in the case of probabilistic models admitting internal nondeterminism, it was not possible to capture well known equivalences such as the bisimulation equivalence of Segala and Lynch [36] and the trace-distribution equivalence of Segala [35]. In this section, we redefine bisimulation and trace metaequivalences for ULTraS by making explicit use of resolutions and show that this allows us to capture the two equivalences mentioned above.

3.1. From Preordered Sets to Reachability-Consistent Semirings

Before providing the new definitions, we recall that the measure functions instantiated in [4] for probabilistic models admitting internal nondeterminism return *sets* of values. The reason is that each of those functions is applied to the overall ULTraS, thereby resulting, in the presence of internal nondeterminism, in possibly different values, for different resolutions, with which a set of states is reached after performing a sequence of actions. The key observation is that applying measure functions to resolutions, instead of the overall ULTraS, causes *single* values to be returned.

This fact determines two important consequences. Firstly, the same preordered set equipped with minimum $(D, \sqsubseteq_D, \bot_D)$ can now be exploited for both one-step reachability – in the definition of the ULTraS metamodel – and multi-step reachability – in the definition of the metaequivalences. Secondly, with respect to [4], where it remained somehow implicit, a *semiring* structure can finally be imposed on D, thus reconciling the ULTraS metamodel with the FuTS metamodel of [12]. In other words, we can assume D to be equipped with two binary operations, respectively denoted by $\oplus$ and $\otimes$, such that:

- $\oplus$ is associative and commutative and admits neutral element 0_D.
- $\otimes$ is associative and admits neutral element 1_D and absorbing element 0_D.
- $\otimes$ distributes over $\oplus$.

Intuitively, $\oplus$ is useful for aggregating values of multi-step reachability along different trajectories starting from the same state, as well as for shorthands of the form $\Delta(S') = \bigoplus_{s' \in S'} \Delta(s')$ given a transition $s \xrightarrow{a} \Delta$ (note that, since $supp(\Delta)$ is finite, at most finitely many $s' \in S'$ will yield via Δ a D-value different from 0_D). On the other hand, $\otimes$ is useful for calculating multi-step reachability from values of consecutive single-step reachability along the same trajectory.[1]

Moreover, we assume that these two binary operations are *reachability consistent*, in the sense that they satisfy the following properties complying with the intuition behind the concept of reachability:

- $0_D = \bot_D$ (i.e., the zero of the semiring denotes unreachability as the minimum of the preordered set).
- $d_1 \otimes d_2 \neq 0_D$ whenever $d_1 \neq 0_D \neq d_2$ (i.e., two consecutive steps cannot result in unreachability).
- The sum via $\oplus$ of finitely many values 1_D is always different from 0_D (known as *characteristic zero*; it implies that two nonzero values sum up to zero only if they are one the inverse of the other w.r.t. $\oplus$, thus avoiding inappropriate zero results when aggregating values of trajectories from the same state).

An implication of these further constraints, in particular characteristic zero, is that the ring $(\mathbb{Z}_n, +_n, \times_n)$ of the classes of integer numbers that are congruent modulo $n \in \mathbb{N}_{\geq 2}$ cannot be used in the new ULTraS framework. This is especially true for $n = 2$, in which case $+_n$ corresponds to addition modulo 2 – note that $1 +_n 1 = 0$ – and $\times_n$ corresponds to multiplication over $\{0, 1\}$. Therefore, the only admitted structure with two elements is the Boolean lattice $(\mathbb{B}, \vee, \wedge)$, as $\top \vee \top = \top$.

[1] Note that the semiring structure does not override the preorder structure, as the latter is necessary, e.g., to define behavioral metapreorders in the ULTraS setting.

3.2. Upgrading Measure Functions to Measure Schemata

Based on the elicited reachability-consistent semiring structure, we now revise the definition of measure function introduced in [4] to measure the degree of reachability of a set of states when executing a certain sequence of actions from a given state.[2] The revision consists of providing a notion of measure schema for an ULTraS as a set of *homogeneous* measure functions, one for each ULTraS resolution. Let us denote by A^* the set of traces over an action set A, by ε the empty trace, and by $|\alpha|$ the length of a trace $\alpha \in A^*$.

Definition 3.1. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring and $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS. A *D-measure schema* $\mathcal{M}$ for $\mathcal{U}$ is a set of *measure functions* of the form $\mathcal{M}_{\mathcal{Z}} : Z \times A^* \times 2^Z \to D$, one for each $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in \mathit{Res}(s)$ and $s \in S$, that are inductively defined on the length of their second argument as follows:

$$\mathcal{M}_{\mathcal{Z}}(z, \alpha, Z') = \begin{cases} f_{\mathcal{Z}}\big(\bigoplus\limits_{z' \in \mathit{supp}(\Delta)} (\Delta(z') \otimes \mathcal{M}_{\mathcal{Z}}(z', \alpha', Z')), z, a, \Delta\big) & \text{if } \alpha = a\,\alpha' \text{ and } z \xrightarrow{a}_{\mathcal{Z}} \Delta \\ 1_D & \text{if } \alpha = \varepsilon \text{ and } z \in Z' \\ 0_D & \text{otherwise} \end{cases}$$

where $f_{\mathcal{Z}} : D \times Z \times A \times (Z \to D)_{\text{nefs}} \to D$. ■

In the first clause, the value of $\mathcal{M}_{\mathcal{Z}}(z, \alpha, Z')$ is built around a sum of products of D-values, with the summation being well defined because $\mathit{supp}(\Delta)$ is finite as established in Def. 2.1. To provide some degree of flexibility, further parameters, internal or external to $\mathcal{U}$, may be taken into account. On the one hand, the auxiliary function $f_{\mathcal{Z}}$ is introduced, which returns its first argument unless otherwise stated, but can also exploit information related to the source state, the action label, or the target distribution of the transition elicited in the first clause. On the other hand, with abuse of notation, when necessary we admit that $\mathcal{M}_{\mathcal{Z}}$ may depend on arguments external to $\mathcal{U}$, which are consistently inherited by $f_{\mathcal{Z}}$; their codomain remains D. The definition above applies to $\mathcal{Z} \in k\text{-}\mathit{Res}(s)$ by restricting to traces $\alpha \in A^*$ such that $|\alpha| \leq k$; note that $Z' \subseteq S$ when $|\alpha| = k$. For simplicity, we will indicate with the same name $\mathcal{M}$ both a measure schema $\mathcal{M}$ and any of the homogeneous measure functions $\mathcal{M}_{\mathcal{Z}}$ contained in it.

Example 3.2. When instantiating the ULTraS metamodel to the fully nondeterministic setting of labeled transition systems [26], the employed semiring is $(\mathbb{B}, \vee, \wedge)$ and we denote by $\mathcal{M}_{\text{nd}}$ the corresponding measure schema obtained from Def. 3.1 by replacing $\oplus$ with $\vee$ and $\otimes$ with $\wedge$. Following [8], this applies also to timed automata [2]. In the probabilistic case – be it generative [20] (action-labeled discrete-time Markov chains [39]), reactive [29] (Markov decision processes [14]), or including internal nondeterminism (Segala simple probabilistic automata [34]) – the employed semiring is $(\mathbb{R}_{\geq 0}, +, \times)$ and we denote by $\mathcal{M}_{\text{pb}}$ the corresponding measure schema obtained from Def. 3.1 by replacing $\oplus$ with $+$ and $\otimes$ with $\times$. Following [8], this applies also to probabilistic timed automata [28] and Markov automata [17].

As for the stochastic case (action-labeled continuous-time Markov chains [39] and continuous-time Markov decision processes [33]), the employed semiring is again $(\mathbb{R}_{\geq 0}, +, \times)$ and hence the corresponding measure schema is obtained from Def. 3.1 by replacing $\oplus$ with $+$ and $\otimes$ with $\times$. As shown in [4], the measure schema can be defined in two different ways, with two alternative auxiliary functions. In the following, given $z \xrightarrow{a}_{\mathcal{Z}} \Delta$, the real value $\mathrm{E}(z)$ denotes the sum of the values – representing rates of exponentially distributed delays[3] – associated with certain states; it is the sum of the values associated with the states in the support of the target of *all* transitions departing from the state s corresponding to z if nondeterminism is absent in the original model (action-labeled continuous-time Markov chain), otherwise it is $\Delta(Z)$. Moreover, given $z' \in \mathit{supp}(\Delta)$, the real value $\frac{\Delta(z')}{\mathrm{E}(z)}$ is the probability of selecting z' among all the states in $\mathit{supp}(\Delta)$.

The *end-to-end* option originates a measure schema $\mathcal{M}_{\text{ete}}$ such that $\mathcal{M}_{\mathcal{Z},\text{ete}}(z, \alpha, Z')(t)$ expresses the probability of performing within $t \in \mathbb{R}_{\geq 0}$ time units a computation from state z that is labeled with trace α and leads to a state in Z'. This value is computed as the convolution of two probability distributions.

[2] According to the semiring terminology, this function corresponds to a formal power series (see, e.g., [16]).

[3] In this continuous-time Markovian setting, from an operational viewpoint the *race policy* is applied. An exponentially distributed duration, uniquely characterized by its rate, is associated with every activity that is enabled at a certain point in time. The activity that is selected is the one that samples the least duration from its associated exponential distribution. The probability of selecting a specific activity is equal to its rate divided by the sum of the rates of all the enabled activities.

Assuming to spend $x \in \mathbb{R}_{[0,t]}$ time units in state z, the first operand of the convolution is the exponentially distributed density function quantifying the sojourn time in z, i.e., the derivative of $1 - \mathrm{e}^{-\mathrm{E}(z)\times t}$ evaluated in x. For each state z' reachable from z by executing a, the first operand is then multiplied by the probability of performing within the remaining $t - x$ time units a computation from state z' that is labeled with the remaining trace α' and leads to a state in Z'. Formally, for $\alpha = a\,\alpha'$, $z \xrightarrow{a}_{\mathcal{Z}} \Delta$, and $t \in \mathbb{R}_{\geq 0}$:

$$\begin{aligned}
\mathcal{M}_{\mathcal{Z},\mathrm{ete}}(z, \alpha, Z')(t) &= \int_0^t (\mathrm{E}(z) \times \mathrm{e}^{-\mathrm{E}(z)\times x}) \times \sum_{z' \in supp(\Delta)} \Big(\tfrac{\Delta(z')}{\mathrm{E}(z)} \times \mathcal{M}_{\mathcal{Z},\mathrm{ete}}(z', \alpha', Z')(t - x) \Big)\, \mathrm{d}x \\
&= \int_0^t \mathrm{e}^{-\mathrm{E}(z)\times x} \times \sum_{z' \in supp(\Delta)} (\Delta(z') \times \mathcal{M}_{\mathcal{Z},\mathrm{ete}}(z', \alpha', Z')(t - x))\, \mathrm{d}x \\
&= f_{\mathcal{Z},\mathrm{ete}}\Big(\sum_{z' \in supp(\Delta)} (\Delta(z') \times \mathcal{M}_{\mathcal{Z},\mathrm{ete}}(z', \alpha', Z')(t - x)), z, a, \Delta\Big)(t)
\end{aligned}$$

where $f_{\mathcal{Z},\mathrm{ete}}(d, z, a, \Delta)(t) \;=\; \int_0^t \mathrm{e}^{-\mathrm{E}(z)\times x} \times d\, \mathrm{d}x \in \mathbb{R}_{\geq 0}$ for $d \in \mathbb{R}_{\geq 0}$.

The *step-by-step* option originates a measure schema $\mathcal{M}_{\mathrm{sbs}}$ such that $\mathcal{M}_{\mathcal{Z},\mathrm{sbs}}(z, \alpha, Z')(\theta)$ expresses the probability of performing within a sequence of time units $\theta \in (\mathbb{R}_{\geq 0})^*$ a computation from state z that is labeled with trace α and leads to a state in Z'. This value is computed as the product of two probability distributions. Assuming that $t \in \mathbb{R}_{\geq 0}$ is the first value in the sequence θ, the first operand of the product is the probability of leaving z within t time units, i.e., $1 - \mathrm{e}^{-\mathrm{E}(z)\times t}$. For each state z' reachable from z by executing a, the first operand is then multiplied by the probability of performing within the remaining sequence of time units $\theta' \in (\mathbb{R}_{\geq 0})^*$ a computation from state z' that is labeled with the remaining trace α' and leads to a state in Z'. Formally, for $\alpha = a\,\alpha'$, $z \xrightarrow{a}_{\mathcal{Z}} \Delta$, and $\theta = t\,\theta' \in (\mathbb{R}_{\geq 0})^*$ such that $|\theta| \geq |\alpha|$:

$$\begin{aligned}
\mathcal{M}_{\mathcal{Z},\mathrm{sbs}}(z, \alpha, Z')(\theta) &= (1 - \mathrm{e}^{-\mathrm{E}(z)\times t}) \times \sum_{z' \in supp(\Delta)} \Big(\tfrac{\Delta(z')}{\mathrm{E}(z)} \times \mathcal{M}_{\mathcal{Z},\mathrm{sbs}}(z', \alpha', Z')(\theta') \Big) \\
&= \tfrac{1 - \mathrm{e}^{-\mathrm{E}(z)\times t}}{\mathrm{E}(z)} \times \sum_{z' \in supp(\Delta)} (\Delta(z') \times \mathcal{M}_{\mathcal{Z},\mathrm{sbs}}(z', \alpha', Z')(\theta')) \\
&= f_{\mathcal{Z},\mathrm{sbs}}\Big(\sum_{z' \in supp(\Delta)} (\Delta(z') \times \mathcal{M}_{\mathcal{Z},\mathrm{sbs}}(z', \alpha', Z')(\theta')), z, a, \Delta\Big)(t)
\end{aligned}$$

where $f_{\mathcal{Z},\mathrm{sbs}}(d, z, a, \Delta)(t) \;=\; \frac{1 - \mathrm{e}^{-\mathrm{E}(z)\times t}}{\mathrm{E}(z)} \times d \in \mathbb{R}_{\geq 0}$ for $d \in \mathbb{R}_{\geq 0}$. ■

3.3. Definition and Comparison of Resolution-Based Behavioral Metaequivalences

We are now in a position of presenting the new formalization of behavioral metaequivalences on ULTRAS. For bisimulation semantics, unlike [4] we define two different metaequivalences $\sim_{\mathrm{B}}^{\mathrm{pre}}$ and $\sim_{\mathrm{B}}^{\mathrm{post}}$. Both of them are defined in the style of Larsen and Skou [29], which requires bisimulations to be equivalence relations. However, they deal with *sets* of equivalence classes, rather than only with *individual* equivalence classes. As shown in [7], in certain situations this avoids ending up with bisimulation equivalences that are too coarse. Moreover, this is consistent with approaches that have been adopted when dealing with continuous-state probabilistic processes to ensure transitivity of probabilistic bisimilarity [15, 10, 11].

The difference between the two metaequivalences lies in the position of the universal quantification over sets of equivalence classes, which is underlined in the definition below. In the first case, inspired by [40, 4, 38], the quantification occurs *before* the transition of the challenger and the transition of the defender, so that we speak of *pre*-bisimulation. In the second case, which is the widely accepted approach of [36], the quantification occurs *after* those two transitions, hence we speak of *post*-bisimulation. In the definition below, given an equivalence relation $\mathcal{B}$ over the state space S of an ULTRAS, and given a set of equivalence classes $\mathcal{G}$, which thus belongs to $2^{S/\mathcal{B}}$, the considered transitions are represented through 1-resolutions because $\bigcup \mathcal{G} \subseteq S$, with $\bigcup \mathcal{G}$ being the union of all the equivalence classes in $\mathcal{G}$.

Definition 3.3. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$:

- $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2$ iff there exists an $\mathcal{M}$-pre-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. An equivalence relation $\mathcal{B}$ over S is an *$\mathcal{M}$-pre-bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all $a \in A$ and $\underline{\mathcal{G} \in 2^{S/\mathcal{B}}}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s_2)$ such that:

$$\mathcal{M}(z_{s_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}(z_{s_2}, a, \bigcup \mathcal{G})$$

- $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$ iff there exists an $\mathcal{M}$-post-bisimulation $\mathcal{B}$ over S such that $(s_1, s_2) \in \mathcal{B}$. An equivalence relation $\mathcal{B}$ over S is an $\mathcal{M}$*-post-bisimulation* iff, whenever $(s_1, s_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s_2)$ such that for all $\underline{\mathcal{G} \in 2^{S/\mathcal{B}}}$:
$$\mathcal{M}(z_{s_1}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s_2}, a, \bigcup \mathcal{G})$$ ■

Also for trace semantics we define two distinct metaequivalences, $\sim_{\mathrm{T}}^{\mathrm{pre}}$ and $\sim_{\mathrm{T}}^{\mathrm{post}}$, with the difference being the position of the universal quantification over traces, which is underlined in the definition below. In the first case, inspired by [4], the quantification occurs *before* the computation of the challenger and the computation of the defender, so that we use superscript *pre*. In the second case, which is the widely accepted approach of [35], the quantification occurs *after* those two computations, hence we use superscript *post*. In the definition below, the considered computations are represented through resolutions.

Definition 3.4. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$:

- $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2$ iff for all $\underline{\alpha \in A^*}$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that:
$$\mathcal{M}(z_{s_1}, \alpha, Z_1) \; = \; \mathcal{M}(z_{s_2}, \alpha, Z_2)$$
and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied.

- $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}} s_2$ iff it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that for all $\underline{\alpha \in A^*}$:
$$\mathcal{M}(z_{s_1}, \alpha, Z_1) \; = \; \mathcal{M}(z_{s_2}, \alpha, Z_2)$$
and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied. ■

We continue by comparing the discriminating power of the four behavioral metaequivalences defined above for the revised ULTRAS framework. As expected, in general each post-metaequivalence turns out to be more discriminating than the corresponding pre-metaequivalence, with the two coinciding for bisimulation semantics in the absence of internal nondeterminism, i.e., of identically labeled transitions departing from the same state of the considered ULTRAS. It also holds that $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}}$ is more discriminating than $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$, while $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}}$ and $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$ are incomparable.

Proposition 3.5. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, and $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$. Then:

1. $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} \subseteq \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}}$, with $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} = \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}}$ if there is no internal nondeterminism in $\mathcal{U}$.
2. $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}} \subseteq \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}}$.
3. $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} \subseteq \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$.

Proof Given $s_1, s_2 \in S$, we proceed as follows:

1. If $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$ due to some $\mathcal{M}$-post-bisimulation $\mathcal{B}$, then $\mathcal{B}$ is also an $\mathcal{M}$-pre-bisimulation as can be seen by taking the same pairs of matching 1-resolutions considered in the $\mathcal{M}$-post-bisimulation game. Therefore, $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2$.
   In the case that there is no internal nondeterminism in $\mathcal{U}$, $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} \subseteq \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}}$ holds too. Indeed, if $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2$ due to some $\mathcal{M}$-pre-bisimulation $\mathcal{B}$, then $\mathcal{B}$ is also an $\mathcal{M}$-post-bisimulation as can be seen by taking the same pairs of matching 1-resolutions considered in the $\mathcal{M}$-pre-bisimulation game. The reason is that, given $a \in A$, due to the absence of internal nondeterminism in $\mathcal{U}$, either both s_1 and s_2 have a single 1-resolution starting with an a-transition – with the two 1-resolutions matching with respect to all sets of equivalence classes – or neither s_1 nor s_2 has a 1-resolution starting with an a-transition. Therefore, $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$.
2. If $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}} s_2$, then $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2$ as can be seen by taking the same pairs of matching resolutions considered for $\sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}}$.

3. Suppose that $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$ due to some $\mathcal{M}$-post-bisimulation $\mathcal{B}$. We show that $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}} s_2$ by proving that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ – resp. $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ – there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ – resp. $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ – such that for all $\alpha \in A^*$ it holds that:
$$\mathcal{M}(z_{s_1}, \alpha, Z_1) \ = \ \mathcal{M}(z_{s_2}, \alpha, Z_2)$$
Starting from s_1, we focus on an arbitrary $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$, which we assume not to consist of a single state without transitions so to avoid trivial cases. Indicating with $z_{s_1} \xrightarrow{a} \Delta_1$ the initial transition of $\mathcal{Z}_1$, since $(s_1, s_2) \in \mathcal{B}$ and $\mathcal{B}$ is an $\mathcal{M}$-post-bisimulation there must exist $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ with initial transition $z_{s_2} \xrightarrow{a} \Delta_2$ such that, in particular for each $C \subseteq Z_1 \cup Z_2$ corresponding to an equivalence class in $S/\mathcal{B}$, it holds that:
$$\mathcal{M}(z_{s_1}, a, C) \ = \ \bigoplus_{z' \in C} \Delta_1(z') \ = \ \Delta_1(C) \ = \ \Delta_2(C) \ = \ \bigoplus_{z' \in C} \Delta_2(z') \ = \ \mathcal{M}(z_{s_2}, a, C)$$
where finitely many D-values different from 0_D occur in both summations because Δ_1 and Δ_2 have finite support.
Among all the resolutions in $Res(s_2)$ satisfying the property above, we choose as $\mathcal{Z}_2$ one that can execute all the traces of $\mathcal{Z}_1$ (which must exist otherwise s_1 could execute a trace not executable by s_2 and hence $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$ would be contradicted) and only those traces (longer traces can be ruled out via pruning). Given an arbitrary $\alpha \in A^*$, we proceed by induction on $|\alpha| \in \mathbb{N}$:

   - If $|\alpha| = 0$, i.e., $\alpha = \varepsilon$, then it trivially holds that:
$$\mathcal{M}(z_{s_1}, \alpha, Z_1) \ = \ 1_D \ = \ \mathcal{M}(z_{s_2}, \alpha, Z_2)$$
   - Let $|\alpha| = n + 1$ for some $n \in \mathbb{N}$, with $\alpha = a' \, \alpha'$ and $|\alpha'| = n$, and suppose that the result holds for each trace of length n when considering two $\mathcal{M}$-post-bisimilar states. There are two cases:
     - If $a' \neq a$, since both $\mathcal{Z}_1$ and $\mathcal{Z}_2$ start with an a-transition it holds that:
$$\mathcal{M}(z_{s_1}, \alpha, Z_1) \ = \ 0_D \ = \ \mathcal{M}(z_{s_2}, \alpha, Z_2)$$
     - If $a' = a$, we observe that an arbitrary $C \subseteq Z_1 \cup Z_2$ corresponding to an equivalence class in $S/\mathcal{B}$ is either reachable via both a-transitions, or via neither; moreover, either α' is executable in all the states of C, or in none of them (this does not necessarily hold in the case of a set of classes). Let $\mathcal{G}$ be the set of subsets of $Z_1 \cup Z_2$, corresponding to equivalence classes in $S/\mathcal{B}$, that are reachable via both a-transitions (hence $\mathcal{G}$ is finite) and in which α' is executable; note that the other subsets do not contribute to $\mathcal{M}(z_{s_1}, \alpha, Z_1)$ and $\mathcal{M}(z_{s_2}, \alpha, Z_2)$. For each $C \in \mathcal{G}$, given an arbitrary $z_{C,1} \in C \cap supp(\Delta_1)$ and an arbitrary $z_{C,2} \in C \cap supp(\Delta_2)$ whose corresponding states in S are $s_{C,1}$ and $s_{C,2}$, since $s_{C,1} \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_{C,2}$ and $|\alpha'| = n$ by the induction hypothesis we have that:
$$\mathcal{M}(z_{C,1}, \alpha', Z_1) \ = \ \mathcal{M}(z_{C,2}, \alpha', Z_2)$$
As a consequence, by the distributivity of $\otimes$ with respect to $\oplus$ we have that:
$$\begin{aligned} \mathcal{M}(z_{s_1}, \alpha, Z_1) \ &= \ \bigoplus_{C \in \mathcal{G}} (\Delta_1(C) \otimes \mathcal{M}(z_{C,1}, \alpha', Z_1)) \ = \\ &= \ \bigoplus_{C \in \mathcal{G}} (\Delta_2(C) \otimes \mathcal{M}(z_{C,2}, \alpha', Z_2)) \ = \ \mathcal{M}(z_{s_2}, \alpha, Z_2) \end{aligned}$$
where finitely many D-values occur in both summations because $\mathcal{G}$ is finite. ■

Let us discuss the strictness of the inclusions in Prop. 3.5 and the role of internal nondeterminism by focusing on $(\mathbb{B}, \vee, \wedge)$, which is the simplest reachability-consistent semiring. Consider for instance the three $\mathbb{B}$-ULTRAS models in the upper part of Fig. 1, together with their maximal resolutions in the lower part of the same figure, where only the second model features internal nondeterminism ($\top$ is omitted in the case of target distributions with singleton support). It turns out that:

- $s_1 \sim_{\mathrm{B},\mathcal{M}_{\mathrm{nd}}}^{\mathrm{pre}} s_2$ but $s_1 \not\sim_{\mathrm{B},\mathcal{M}_{\mathrm{nd}}}^{\mathrm{post}} s_2$. The reason is that the only a-transition of s_1 cannot be matched, in the $\mathcal{M}_{\mathrm{nd}}$-post-bisimulation game, by any of the two a-transitions of s_2, because the transition of s_1 can reach two different equivalence classes, while the two transitions of s_2 can reach only one class each. Moreover $s_1 \not\sim_{\mathrm{B},\mathcal{M}_{\mathrm{nd}}}^{\mathrm{pre}} s_3$, and hence $s_1 \not\sim_{\mathrm{B},\mathcal{M}_{\mathrm{nd}}}^{\mathrm{post}} s_3$, because the state reached by the a-transition of s_3

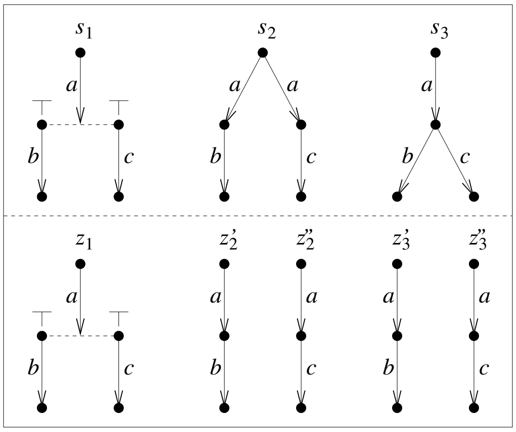

Figure 1: Counterexamples for Prop. 3.5: three $\mathbb{B}$-ULTRAS models and their maximal resolutions

enables two actions and, as a consequence, cannot be equivalent to any of the two states reached by the a-transition of s_1. Indeed, although s_2 and s_3 have the same resolutions, their maximal 1-resolutions are different; for s_2 they coincide with the two maximal resolutions, while for s_3 the only maximal 1-resolution coincides with the original model.

- $s_1 \sim^{\rm pre}_{\rm T,\mathcal{M}_{nd}} s_2$ but $s_1 \not\sim^{\rm post}_{\rm T,\mathcal{M}_{nd}} s_2$. This is due to the fact that the only maximal resolution of s_1 cannot be matched, in the case of $\sim^{\rm post}_{\rm T,\mathcal{M}_{nd}}$, by any of the two maximal resolutions of s_2, because the maximal resolution of s_1 has two different maximal traces, while the two maximal resolutions of s_2 have only one maximal trace each. Likewise, $s_1 \sim^{\rm pre}_{\rm T,\mathcal{M}_{nd}} s_3$ but $s_1 \not\sim^{\rm post}_{\rm T,\mathcal{M}_{nd}} s_3$ because s_3 has the same resolutions as s_2. This shows that, unlike bisimulation semantics, in general $\sim^{\rm pre}_{\rm T,\mathcal{M}_{nd}}$ and $\sim^{\rm post}_{\rm T,\mathcal{M}_{nd}}$ do not coincide even in the absence of internal nondeterminism, due to the existence of $\mathbb{B}$-ULTRAS models such as the first one that cannot be considered the canonical representation of any labeled transition system.

- $s_2 \sim^{\rm post}_{\rm T,\mathcal{M}_{nd}} s_3$ but $s_2 \not\sim^{\rm post}_{\rm B,\mathcal{M}_{nd}} s_3$.

We furthermore notice that $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ and $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ are generally incomparable. This can be derived from the spectrum of strong behavioral equivalences over nondeterministic and probabilistic processes studied in [6]. For instance, Fig. 13 in that paper shows two processes identified by $\sim^{\rm pre}_{\rm B,\mathcal{M}_{pb}}$ but distinguished by $\sim^{\rm pre}_{\rm T,\mathcal{M}_{pb}}$, while Fig. 12 shows other two processes that are identified by $\sim^{\rm pre}_{\rm T,\mathcal{M}_{pb}}$ but distinguished by $\sim^{\rm pre}_{\rm B,\mathcal{M}_{pb}}$. This also indicates that Thm. 3.8 of [4], i.e., the only metaresult contained in the original paper on ULTRAS, does not hold in general; it is fixed by Prop. 3.5 of the present paper.

We conclude by discussing the relationships between the two bisimulation metaequivalences introduced in Def. 3.3 and those defined over other metamodels appeared in the literature. In order for the comparison to be fair, function $f_{\mathcal{Z}}$ occurring in Def. 3.1 is assumed to return its first argument. Since the WLTS metamodel is subject to a weight determinacy condition that rules out internal nondeterminism, weighted bisimilarity [27, Def. 3] coincides with both $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ and $\sim^{\rm post}_{\rm B,\mathcal{M}}$ when the same commutative monoid is considered. This can be easily seen by recalling that, in Def. 3.3, 1-resolutions are simply a means to select a single transition from each of the related states and then consider the rest of the original state space. The same holds true for the FuTS metamodel, as long as the same semiring is considered, because its bisimulation equivalence [30, Def. 3.2] deals with deterministic state spaces and hence internal nondeterminism does not come into play. Different is the case of [31], where the ULTRAS metamodel is recast in terms of commutative monoids but, different from the WLTS metamodel, internal nondeterminism is allowed. The proposed bisimilarity [31, Def. 2.6], for which a general operational semantic rule format ensuring compositionality is developed together with a coalgebraic characterization, agrees only with $\sim^{\rm post}_{\rm B,\mathcal{M}}$ in the presence of internal nondeterminism, when considering the same commutative monoid.

3.4. Instantiation Results for Bisimulation Metaequivalences

The notion of measure schema in Def. 3.1 and the notion of measure function in [4, Def. 3.2] share the idea of measuring the reachability of a set of states when performing a certain trace from a given state, with both being inductively defined on the length of such a trace. However, the latter is more liberal. Therefore, although $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ in Def. 3.3 is akin to the bisimulation metaequivalence in [4, Def. 3.3], a general relationship between them cannot be established. However, it is easy to see that, when suitably selecting the measure schema $\mathcal{M}$, the relation $\sim^{\rm pre}_{\rm B,\mathcal{M}}$ captures all the bisimulation equivalences considered in [4]. On the other hand, among the same bisimulation equivalences, $\sim^{\rm post}_{\rm B,\mathcal{M}}$ captures only those defined over models not featuring the coexistence of internal nondeterminism and probability/stochasticity, plus the strong bisimulation equivalence of [36] that was not encompassed in [4].

Proposition 3.6. $\sim^{\rm pre}_{\rm B,\mathcal{M}_{\rm nd}}$ and $\sim^{\rm post}_{\rm B,\mathcal{M}_{\rm nd}}$ capture the strong bisimulation equivalence of [21] over labeled transition systems.

Proof Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be a labeled transition system and let $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding $\mathbb{B}$-ULTraS, which is defined by letting:[4]

- $s \xrightarrow{a}_{\mathcal{U}} \Delta_{s,a,s'}$ for all $s, s' \in S$ and $a \in A$ such that $s \xrightarrow{a}_{\mathcal{L}} s'$.
- $\Delta_{s,a,s'}(s'') = \begin{cases} \top & \text{if } s'' = s' \\ \bot & \text{if } s'' \neq s' \end{cases}$ for all $s'' \in S$.

Given $s_1, s_2 \in S$, let us assume that $(s_1, s_2) \in \mathcal{B}$ for some relation $\mathcal{B}$ over S that is a strong bisimulation on $\mathcal{L}$ according to [21]. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$:

- whenever $s'_1 \xrightarrow{a}_{\mathcal{L}} s''_1$, then $s'_2 \xrightarrow{a}_{\mathcal{L}} s''_2$ with $(s''_1, s''_2) \in \mathcal{B}$;
- whenever $s'_2 \xrightarrow{a}_{\mathcal{L}} s''_2$, then $s'_1 \xrightarrow{a}_{\mathcal{L}} s''_1$ with $(s''_1, s''_2) \in \mathcal{B}$.

Without loss of generality, we can suppose that $\mathcal{B}$ is an equivalence relation: should this not be the case, it suffices to take the reflexive and transitive closure of $\mathcal{B}$ as this is still a strong bisimulation according to [21]. As a consequence, the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$:

- whenever $s'_1 \xrightarrow{a}_{\mathcal{L}} s''_1$ with $s''_1 \in C$, then $s'_2 \xrightarrow{a}_{\mathcal{L}} s''_2$ with $s''_2 \in C$;
- whenever $s'_2 \xrightarrow{a}_{\mathcal{L}} s''_2$ with $s''_2 \in C$, then $s'_1 \xrightarrow{a}_{\mathcal{L}} s''_1$ with $s''_1 \in C$;

which boil down to the following:

- there exists $s''_1 \in C$ such that $s'_1 \xrightarrow{a}_{\mathcal{L}} s''_1$ iff there exists $s''_2 \in C$ such that $s'_2 \xrightarrow{a}_{\mathcal{L}} s''_2$.

Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that the existence of $s' \in \bigcup \mathcal{G}$ such that $s \xrightarrow{a}_{\mathcal{L}} s'$ corresponds to the existence of $s' \in C$ such that $s \xrightarrow{a}_{\mathcal{L}} s'$ for some $C \in \mathcal{G}$, we immediately derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$:

- there exists $s''_1 \in \bigcup \mathcal{G}$ such that $s'_1 \xrightarrow{a}_{\mathcal{L}} s''_1$ iff there exists $s''_2 \in \bigcup \mathcal{G}$ such that $s'_2 \xrightarrow{a}_{\mathcal{L}} s''_2$.

[4]This is what we call the canonical representation of a labeled transition system via a $\mathbb{B}$-ULTraS, which was mentioned on page 10. Unlike [4, Def. 2.4], it preserves the internal nondeterminism of the labeled transition system and generates transitions whose target has a finite support as required by Def. 2.1.

Since for all $s \in S$, $a \in A$, and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$(\exists s' \in \bigcup \mathcal{G}.\, s \xrightarrow{a}_{\mathcal{L}} s') \iff \bigvee_{s' \in \bigcup \mathcal{G}} \Delta_{s,a,s'}(s') = \top$$

where finitely many $\mathbb{B}$-values different from $\bot$ occur in the disjunction because $\Delta_{s,a,s'}$ has a singleton support, we further derive that the assumption is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ in $\mathcal{U}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that:

$$\mathcal{M}_{\mathrm{nd}}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\mathrm{nd}}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\mathrm{nd}}$-pre-bisimulation on $\mathcal{U}$ such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\mathrm{pre}}_{\mathrm{B},\mathcal{M}_{\mathrm{nd}}} s_2$.
Moreover, since the target of every transition in $\mathcal{U}$ has a singleton support, the assumption is also equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ in $\mathcal{U}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathcal{M}_{\mathrm{nd}}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\mathrm{nd}}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\mathrm{nd}}$-post-bisimulation on $\mathcal{U}$ too such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\mathrm{post}}_{\mathrm{B},\mathcal{M}_{\mathrm{nd}}} s_2$. ■

Proposition 3.7. $\sim^{\mathrm{pre}}_{\mathrm{B},\mathcal{M}_{\mathrm{pb}}}$ and $\sim^{\mathrm{post}}_{\mathrm{B},\mathcal{M}_{\mathrm{pb}}}$ capture the strong bisimulation equivalence of [18] over generative probabilistic labeled transition systems (a.k.a. action-labeled discrete-time Markov chains).

Proof A generative probabilistic labeled transition system can be represented as an $\mathbb{R}_{\geq 0}$-ULTraS $(S, A, \longrightarrow)$ in which for all $s \in S$:

- for all $a \in A$, $s \xrightarrow{a} \Delta_1 \wedge s \xrightarrow{a} \Delta_2 \Longrightarrow \Delta_1 = \Delta_2$ (absence of internal nondeterminism);
- $\sum_{s \xrightarrow{a} \Delta} \Delta(S) \in \{0, 1\}$ (the targets of all transitions from s collectively form a probability distribution).

Given $s_1, s_2 \in S$, let us assume that $(s_1, s_2) \in \mathcal{B}$ for some equivalence relation $\mathcal{B}$ over S that is a strong bisimulation according to [18]. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(C) = \Delta_2(C)$, which is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$. This is the same as requiring that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that:

$$\mathcal{M}_{\mathrm{pb}}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\mathrm{pb}}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\mathrm{pb}}$-pre-bisimulation such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\mathrm{pre}}_{\mathrm{B},\mathcal{M}_{\mathrm{pb}}} s_2$.
Moreover, due to the absence of internal nondeterminism, the assumption is also equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$. This is the same as requiring that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathcal{M}_{\mathrm{pb}}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\mathrm{pb}}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\mathrm{pb}}$-post-bisimulation too such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\mathrm{post}}_{\mathrm{B},\mathcal{M}_{\mathrm{pb}}} s_2$. ■

Proposition 3.8. $\sim^{\mathrm{pre}}_{\mathrm{B},\mathcal{M}_{\mathrm{pb}}}$ and $\sim^{\mathrm{post}}_{\mathrm{B},\mathcal{M}_{\mathrm{pb}}}$ capture the strong bisimulation equivalence of [29] over reactive probabilistic labeled transition systems (a.k.a. Markov decision processes).

Proof Observing that a reactive probabilistic labeled transition system can be represented as an $\mathbb{R}_{\geq 0}$-ULTraS $(S, A, \longrightarrow)$ in which for all $s \in S$ and $a \in A$:

- $s \xrightarrow{a} \Delta_1 \wedge s \xrightarrow{a} \Delta_2 \Longrightarrow \Delta_1 = \Delta_2$ (absence of internal nondeterminism);
- $\sum_{s \xrightarrow{a} \Delta} \Delta(S) \in \{0, 1\}$ (the target of each individual transition from s is a probability distribution);

the proof is identical to the one of Prop. 3.7. ■

Proposition 3.9. $\sim^{\rm pre}_{\rm B,\mathcal{M}_{pb}}$ captures the strong bisimulation equivalence of [7] over nondeterministic and probabilistic labeled transition systems (a.k.a. Segala simple probabilistic automata).

Proof A nondeterministic and probabilistic labeled transition system can be represented as an $\mathbb{R}_{\geq 0}$-ULTRAS $(S, A, \longrightarrow)$ in which for all $s \xrightarrow{a} \Delta$ it holds that $\Delta(S) = 1$, i.e., Δ is a probability distribution; notice that internal nondeterminism is allowed.
Given $s_1, s_2 \in S$, let us assume that $(s_1, s_2) \in \mathcal{B}$ for some equivalence relation $\mathcal{B}$ over S that is a strong bisimulation according to [7]. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(C) = \Delta_2(C)$, which is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$. This is the same as requiring that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that:

$$\mathcal{M}_{\rm pb}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\rm pb}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\rm pb}$-pre-bisimulation such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\rm pre}_{\rm B,\mathcal{M}_{pb}} s_2$. ■

Proposition 3.10. $\sim^{\rm post}_{\rm B,\mathcal{M}_{pb}}$ captures the strong bisimulation equivalence of [36] over nondeterministic and probabilistic labeled transition systems (a.k.a. Segala simple probabilistic automata).

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTRAS representing a nondeterministic and probabilistic labeled transition system as discussed at the beginning of the proof of Prop. 3.9.
Given $s_1, s_2 \in S$, let us assume that $(s_1, s_2) \in \mathcal{B}$ for some equivalence relation $\mathcal{B}$ over S that is a strong bisimulation according to [36]. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that for all $C \in S/\mathcal{B}$ it holds that $\Delta_1(C) = \Delta_2(C)$, which is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$. This is the same as requiring that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathcal{M}_{\rm pb}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\rm pb}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\rm pb}$-post-bisimulation such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\rm post}_{\rm B,\mathcal{M}_{pb}} s_2$. ■

Proposition 3.11. $\sim^{\rm pre}_{\rm B,\mathcal{M}_{ete}}$, $\sim^{\rm post}_{\rm B,\mathcal{M}_{ete}}$, $\sim^{\rm pre}_{\rm B,\mathcal{M}_{sbs}}$, $\sim^{\rm post}_{\rm B,\mathcal{M}_{sbs}}$ capture the strong bisimulation equivalence of [22] over generative stochastic labeled transition systems (a.k.a. action-labeled continuous-time Markov chains).

Proof A generative stochastic labeled transition system can be represented as an $\mathbb{R}_{\geq 0}$-ULTRAS $(S, A, \longrightarrow)$ in which for all $s \in S$:

- for all $a \in A$, $s \xrightarrow{a} \Delta_1 \wedge s \xrightarrow{a} \Delta_2 \Longrightarrow \Delta_1 = \Delta_2$ (absence of internal nondeterminism);
- for all $\mathcal{Z} \in 1\text{-}Res(s)$, $\mathrm{E}(z_s) = \sum_{s \xrightarrow{a} \Delta} \Delta(S)$ (collective application of the race policy).

Given $s_1, s_2 \in S$, let us assume that $(s_1, s_2) \in \mathcal{B}$ for some equivalence relation $\mathcal{B}$ over S that is a strong bisimulation according to [22]. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(C) = \Delta_2(C)$, which is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$. This is the same as requiring that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that:

$$\mathcal{M}_{\rm ete}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\rm ete}(z_{s'_2}, a, \bigcup \mathcal{G})$$
$$\mathcal{M}_{\rm sbs}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\rm sbs}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\rm ete}$-pre-bisimulation and an $\mathcal{M}_{\rm sbs}$-pre-bisimulation such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\rm pre}_{\rm B,\mathcal{M}_{ete}} s_2$ and $s_1 \sim^{\rm pre}_{\rm B,\mathcal{M}_{sbs}} s_2$.
Moreover, due to the absence of internal nondeterminism, the assumption is also equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that for

all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$. This is the same as requiring that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathcal{M}_{\text{ete}}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\text{ete}}(z_{s'_2}, a, \bigcup \mathcal{G})$$
$$\mathcal{M}_{\text{sbs}}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\text{sbs}}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\text{ete}}$-post-bisimulation and an $\mathcal{M}_{\text{sbs}}$-post-bisimulation too such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{ete}}} s_2$ and $s_1 \sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{sbs}}} s_2$.
Note that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ and $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ it holds that $\text{E}(z_{s'_1}) = \text{E}(z_{s'_2})$ both when starting with $\mathcal{B}$ as a strong bisimulation in the sense of [22] and when starting with $\mathcal{B}$ as an $\mathcal{M}_{\text{ete}}$-pre/post-bisimulation or $\mathcal{M}_{\text{sbs}}$-pre/post-bisimulation (just sum up over all actions a and classes C and then proceed as in the proofs of Lemmata 8.4, 8.5, 8.6 of [4]). ■

Proposition 3.12. $\sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{ete}}}, \sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{ete}}}, \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{sbs}}}, \sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{sbs}}}$ capture the strong bisimulation equivalence of [32] over reactive stochastic labeled transition systems (a.k.a. continuous-time Markov decision processes).

Proof Observing that a reactive stochastic labeled transition system can be represented as an $\mathbb{R}_{\geq 0}$-ULTraS $(S, A, \longrightarrow)$ in which for all $s \in S$ and $a \in A$:

- $s \xrightarrow{a} \Delta_1 \wedge s \xrightarrow{a} \Delta_2 \Longrightarrow \Delta_1 = \Delta_2$ (absence of internal nondeterminism);
- for all $\mathcal{Z} \in 1\text{-}Res(s)$ such that $z_s \xrightarrow{a}_{\mathcal{Z}} \Delta$, $\text{E}(z_s) = \Delta(S)$ (individual application of the race policy);

the proof is identical to the one of Prop. 3.11 (in the final note, action a is fixed and summations are taken over classes C only). ■

Proposition 3.13. $\sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{ete}}}$ and $\sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{sbs}}}$ capture the strong bisimulation equivalence of [4] over nondeterministic and stochastic labeled transition systems.

Proof A nondeterministic and stochastic labeled transition system can be represented as an $\mathbb{R}_{\geq 0}$-ULTraS $(S, A, \longrightarrow)$ in which, given $s \in S$ and $a \in A$, for all $\mathcal{Z} \in 1\text{-}Res(s)$ such that $z_s \xrightarrow{a}_{\mathcal{Z}} \Delta$ it holds that $\text{E}(z_s) = \Delta(S)$, i.e., the race policy is applied within the support of the target of individual transitions; notice that internal nondeterminism is allowed.
Given $s_1, s_2 \in S$, let us assume that $(s_1, s_2) \in \mathcal{B}$ for some equivalence relation $\mathcal{B}$ over S that is a strong bisimulation equivalence according to [4]. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $C \in S/\mathcal{B}$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(C) = \Delta_2(C)$ and $\Delta_1(S) = \Delta_2(S)$, which is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$ and $\Delta_1(S) = \Delta_2(S)$. This is the same as requiring that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ and $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that:

$$\mathcal{M}_{\text{ete}}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\text{ete}}(z_{s'_2}, a, \bigcup \mathcal{G})$$
$$\mathcal{M}_{\text{sbs}}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \mathcal{M}_{\text{sbs}}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\text{ete}}$-pre-bisimulation and an $\mathcal{M}_{\text{sbs}}$-pre-bisimulation such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{ete}}} s_2$ and $s_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{sbs}}} s_2$.
Note that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ and $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ it holds that $\text{E}(z_{s'_1}) = \text{E}(z_{s'_2})$ both when starting with $\mathcal{B}$ as a strong bisimulation in the sense of [4] (by definition) and when starting with $\mathcal{B}$ as an $\mathcal{M}_{\text{ete}}$-pre-bisimulation or $\mathcal{M}_{\text{sbs}}$-pre-bisimulation (fixing action a, just sum up over all classes C and then proceed similar to the proofs of Lemmata 8.5 and 8.6 of [4]). ■

Proposition 3.14. $\sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{ete}}}$ and $\sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{sbs}}}$ capture the strong bisimulation equivalence in the style of [36] over nondeterministic and stochastic labeled transition systems.

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTraS representing a nondeterministic and stochastic labeled transition system as discussed at the beginning of the proof of Prop. 3.13.
Given $s_1, s_2 \in S$, let us assume that $(s_1, s_2) \in \mathcal{B}$ for some equivalence relation $\mathcal{B}$ over S that is a strong

bisimulation in the style of [36]. This means that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that for all $C \in S/\mathcal{B}$ it holds that $\Delta_1(C) = \Delta_2(C)$, which is equivalent to having that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $s'_1 \xrightarrow{a} \Delta_1$ there exists $s'_2 \xrightarrow{a} \Delta_2$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that $\Delta_1(\bigcup \mathcal{G}) = \Delta_2(\bigcup \mathcal{G})$. This is the same as requiring that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $a \in A$ it holds that for each $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ there exists $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathcal{M}_{\text{ete}}(z_{s'_1}, a, \bigcup \mathcal{G}) \ = \ \mathcal{M}_{\text{ete}}(z_{s'_2}, a, \bigcup \mathcal{G})$$
$$\mathcal{M}_{\text{sbs}}(z_{s'_1}, a, \bigcup \mathcal{G}) \ = \ \mathcal{M}_{\text{sbs}}(z_{s'_2}, a, \bigcup \mathcal{G})$$

This means that $\mathcal{B}$ is an $\mathcal{M}_{\text{ete}}$-post-bisimulation and an $\mathcal{M}_{\text{sbs}}$-post-bisimulation such that $(s_1, s_2) \in \mathcal{B}$, hence $s_1 \sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{ete}}} s_2$ and $s_1 \sim^{\text{post}}_{\text{B},\mathcal{M}_{\text{sbs}}} s_2$.
Note that, whenever $(s'_1, s'_2) \in \mathcal{B}$, then for all $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ and $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ it holds that $\text{E}(z_{s'_1}) = \text{E}(z_{s'_2})$ both when starting with $\mathcal{B}$ as a strong bisimulation in the style of [36] and when starting with $\mathcal{B}$ as an $\mathcal{M}_{\text{ete}}$-post-bisimulation or $\mathcal{M}_{\text{sbs}}$-post-bisimulation (fixing action a, just sum up over all classes C and then proceed as in the proofs of Lemmata 8.4, 8.5, 8.6 of [4]). ■

3.5. Instantiation Results for Trace Metaequivalences

Similar to the case of bisimulation semantics, although $\sim^{\text{pre}}_{\text{T},\mathcal{M}}$ in Def. 3.4 is akin to the trace metaequivalence in [4, Def. 3.4], a general relationship between them cannot be established. However, it is easy to see that, when suitably selecting the measure schema $\mathcal{M}$, the relation $\sim^{\text{pre}}_{\text{T},\mathcal{M}}$ captures all the trace equivalences considered in [4]. On the other hand, among the same trace equivalences, $\sim^{\text{post}}_{\text{T},\mathcal{M}}$ captures only those defined over models not featuring the coexistence of internal nondeterminism and probability/stochasticity, plus the trace-distribution equivalence of [35] that was not encompassed in [4]. Unlike bisimulation semantics, different specific trace equivalences are captured in the stochastic case depending on whether the end-to-end option or the step-by-step option is adopted when considering time. As observed in [4], the former option results in a higher discriminating power because it is able to keep track of the time instants at which the various actions of a trace start/complete their execution.

Proposition 3.15. $\sim^{\text{pre}}_{\text{T},\mathcal{M}_{\text{nd}}}$ and $\sim^{\text{post}}_{\text{T},\mathcal{M}_{\text{nd}}}$ capture the strong trace equivalence of [9] over labeled transition systems.

Proof Let $\mathcal{L} = (S, A, \longrightarrow_{\mathcal{L}})$ be a labeled transition system and let $\mathcal{U} = (S, A, \longrightarrow_{\mathcal{U}})$ be its corresponding $\mathbb{B}$-ULTraS as defined at the beginning of the proof of Prop. 3.6.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent according to [9]. This means that for all $\alpha \in A^*$ it holds that s_1 can perform in $\mathcal{L}$ a computation labeled with α iff so can s_2. Since for all $s \in S$ and $\alpha \in A^*$ it holds that the existence of a computation in $\mathcal{L}$ labeled with α is equivalent to the existence of a computation in some resolution $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ of $\mathcal{U}$ labeled with α and hence to $\mathcal{M}_{\text{nd}}(z_s, \alpha, Z)$ being equal to $\top$, we derive that the assumption is equivalent to having that for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that:

$$\mathcal{M}_{\text{nd}}(z_{s_1}, \alpha, Z_1) \ = \ \mathcal{M}_{\text{nd}}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\text{pre}}_{\text{T},\mathcal{M}_{\text{nd}}} s_2$.
Moreover, since the target of every transition in $\mathcal{U}$ has a singleton support and hence every resolution in $\mathcal{U}$ boils down to a computation, the assumption is also equivalent to having that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that for all $\alpha \in A^*$:

$$\mathcal{M}_{\text{nd}}(z_{s_1}, \alpha, Z_1) \ = \ \mathcal{M}_{\text{nd}}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\text{post}}_{\text{T},\mathcal{M}_{\text{nd}}} s_2$. ■

Proposition 3.16. $\sim^{\text{pre}}_{\text{T},\mathcal{M}_{\text{pb}}}$ and $\sim^{\text{post}}_{\text{T},\mathcal{M}_{\text{pb}}}$ capture the strong trace equivalence of [25] over generative probabilistic labeled transition systems (a.k.a. action-labeled discrete-time Markov chains).

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTraS representing a generative probabilistic labeled transition system as discussed at the beginning of the proof of Prop. 3.7.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent according to [25]. This means that for

all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in \mathit{Res}(s_2)$ such that:

$$\mathcal{M}_{\mathrm{pb}}(z_{s_1}, \alpha, Z_1) \;=\; \mathcal{M}_{\mathrm{pb}}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}_{\mathrm{pb}}} s_2$.
Moreover, due to the absence of internal nondeterminism, the assumption is also equivalent to having that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in \mathit{Res}(s_2)$ such that for all $\alpha \in A^*$:

$$\mathcal{M}_{\mathrm{pb}}(z_{s_1}, \alpha, Z_1) \;=\; \mathcal{M}_{\mathrm{pb}}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{pb}}} s_2$. ■

Proposition 3.17. $\sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}_{\mathrm{pb}}}$ and $\sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{pb}}}$ capture the strong trace equivalence of [37] over reactive probabilistic labeled transition systems (a.k.a. Markov decision processes).

Proof Recalling that a reactive probabilistic labeled transition system can be represented as discussed at the beginning of the proof of Prop. 3.8, the proof is identical to the one of Prop. 3.16. ■

Proposition 3.18. $\sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}_{\mathrm{pb}}}$ captures the strong trace equivalence of [5] over nondeterministic and probabilistic labeled transition systems (a.k.a. Segala simple probabilistic automata).

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTraS representing a nondeterministic and probabilistic labeled transition system as discussed at the beginning of the proof of Prop. 3.9.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent according to [5]. This means that for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in \mathit{Res}(s_2)$ such that:

$$\mathcal{M}_{\mathrm{pb}}(z_{s_1}, \alpha, Z_1) \;=\; \mathcal{M}_{\mathrm{pb}}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}_{\mathrm{pb}}} s_2$. ■

Proposition 3.19. $\sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{pb}}}$ captures the strong trace equivalence of [35] over nondeterministic and probabilistic labeled transition systems (a.k.a. Segala simple probabilistic automata).

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTraS representing a nondeterministic and probabilistic labeled transition system as discussed at the beginning of the proof of Prop. 3.9.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent according to [35]. This means that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in \mathit{Res}(s_2)$ such that for all $\alpha \in A^*$:

$$\mathcal{M}_{\mathrm{pb}}(z_{s_1}, \alpha, Z_1) \;=\; \mathcal{M}_{\mathrm{pb}}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{pb}}} s_2$. ■

Proposition 3.20. $\sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}_{\mathrm{ete}}}$ and $\sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{ete}}}$ capture the strong trace equivalence of [41] over generative stochastic labeled transition systems (a.k.a. action-labeled continuous-time Markov chains).

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTraS representing a generative stochastic labeled transition system as discussed at the beginning of the proof of Prop. 3.11.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent according to [41]. This means that for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in \mathit{Res}(s_2)$ such that:

$$\mathcal{M}_{\mathrm{ete}}(z_{s_1}, \alpha, Z_1) \;=\; \mathcal{M}_{\mathrm{ete}}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}_{\mathrm{ete}}} s_2$.
Moreover, due to the absence of internal nondeterminism, the assumption is also equivalent to having that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in \mathit{Res}(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in \mathit{Res}(s_2)$ such that for all $\alpha \in A^*$:

$$\mathcal{M}_{\mathrm{ete}}(z_{s_1}, \alpha, Z_1) \;=\; \mathcal{M}_{\mathrm{ete}}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{ete}}} s_2$. ■

Proposition 3.21. $\sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}_{\mathrm{sbs}}}$ and $\sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}_{\mathrm{sbs}}}$ capture the strong trace equivalence of [3] over generative stochastic labeled transition systems (a.k.a. action-labeled continuous-time Markov chains).

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTRAS representing a generative stochastic labeled transition system as discussed at the beginning of the proof of Prop. 3.11.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent according to [3]. This means that for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that:

$$\mathcal{M}_{\rm sbs}(z_{s_1}, \alpha, Z_1) \ = \ \mathcal{M}_{\rm sbs}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\rm pre}_{{\rm T},\mathcal{M}_{\rm sbs}} s_2$.
Moreover, due to the absence of internal nondeterminism, the assumption is also equivalent to having that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that for all $\alpha \in A^*$:

$$\mathcal{M}_{\rm sbs}(z_{s_1}, \alpha, Z_1) \ = \ \mathcal{M}_{\rm sbs}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\rm post}_{{\rm T},\mathcal{M}_{\rm sbs}} s_2$. ■

Proposition 3.22. $\sim^{\rm pre}_{{\rm T},\mathcal{M}_{\rm ete}}$ and $\sim^{\rm post}_{{\rm T},\mathcal{M}_{\rm ete}}$ capture the strong trace equivalence in the style of [41] over reactive stochastic labeled transition systems (a.k.a. continuous-time Markov decision processes).

Proof Recalling that a reactive stochastic labeled transition system can be represented as discussed at the beginning of the proof of Prop. 3.12, the proof is identical to the one of 3.20. ■

Proposition 3.23. $\sim^{\rm pre}_{{\rm T},\mathcal{M}_{\rm sbs}}$ and $\sim^{\rm post}_{{\rm T},\mathcal{M}_{\rm sbs}}$ capture the strong trace equivalence in the style of [3] over reactive stochastic labeled transition systems (a.k.a. continuous-time Markov decision processes).

Proof Recalling that a reactive stochastic labeled transition system can be represented as discussed at the beginning of the proof of Prop. 3.12, the proof is identical to the one of 3.21. ■

Proposition 3.24. $\sim^{\rm pre}_{{\rm T},\mathcal{M}_{\rm ete}}$ captures the ete-variant of the strong trace equivalence of [4] over nondeterministic and stochastic labeled transition systems.

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTRAS representing a nondeterministic and stochastic labeled transition system as discussed at the beginning of the proof of Prop. 3.13.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent in the ete-sense according to [4]. This means that for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that:

$$\mathcal{M}_{\rm ete}(z_{s_1}, \alpha, Z_1) \ = \ \mathcal{M}_{\rm ete}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\rm pre}_{{\rm T},\mathcal{M}_{\rm ete}} s_2$. ■

Proposition 3.25. $\sim^{\rm pre}_{{\rm T},\mathcal{M}_{\rm sbs}}$ captures the sbs-variant of the strong trace equivalence of [4] over nondeterministic and stochastic labeled transition systems.

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTRAS representing a nondeterministic and stochastic labeled transition system as discussed at the beginning of the proof of Prop. 3.13.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent in the sbs-sense according to [4]. This means that for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that:

$$\mathcal{M}_{\rm sbs}(z_{s_1}, \alpha, Z_1) \ = \ \mathcal{M}_{\rm sbs}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\rm pre}_{{\rm T},\mathcal{M}_{\rm sbs}} s_2$. ■

Proposition 3.26. $\sim^{\rm post}_{{\rm T},\mathcal{M}_{\rm ete}}$ captures the ete-variant of the strong trace equivalence in the style of [35] over nondeterministic and stochastic labeled transition systems.

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTRAS representing a nondeterministic and stochastic labeled transition system as discussed at the beginning of the proof of Prop. 3.13.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent in the ete-sense according to the style of [35]. This means that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that for all $\alpha \in A^*$:

$$\mathcal{M}_{\rm ete}(z_{s_1}, \alpha, Z_1) \ = \ \mathcal{M}_{\rm ete}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\rm post}_{{\rm T},\mathcal{M}_{\rm ete}} s_2$. ■

Proposition 3.27. $\sim^{\rm post}_{\rm T,\mathcal{M}_{sbs}}$ captures the sbs-variant of the strong trace equivalence in the style of [35] over nondeterministic and stochastic labeled transition systems.

Proof Let $(S, A, \longrightarrow)$ be an $\mathbb{R}_{\geq 0}$-ULTRaS representing a nondeterministic and stochastic labeled transition system as discussed at the beginning of the proof of Prop. 3.13.
Given $s_1, s_2 \in S$, let us assume that they are strong trace equivalent in the sbs-sense according to the style of [35]. This means that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ such that for all $\alpha \in A^*$:

$$\mathcal{M}_{\rm sbs}(z_{s_1}, \alpha, Z_1) \;=\; \mathcal{M}_{\rm sbs}(z_{s_2}, \alpha, Z_2)$$

and also the analogous condition obtained by exchanging $\mathcal{Z}_1$ with $\mathcal{Z}_2$ is satisfied, i.e., $s_1 \sim^{\rm post}_{\rm T,\mathcal{M}_{sbs}} s_2$. ■

4. Compositionality Metaresults

In the specific setting of nondeterministic and probabilistic processes, the two equivalences $\sim^{\rm pre}_{\rm B,\mathcal{M}_{pb}}$ and $\sim^{\rm pre}_{\rm T,\mathcal{M}_{pb}}$ originated in [4] have surprising compositionality properties. In [7], it has been proved that $\sim^{\rm pre}_{\rm B,\mathcal{M}_{pb}}$ is *not* a congruence with respect to parallel composition, although bisimilarity is considered a robust notion of equivalence. In [5], it has been proved that $\sim^{\rm pre}_{\rm T,\mathcal{M}_{pb}}$ *is* a congruence with respect to parallel composition, although the standard trace semantics is known not to be compositional for probabilistic processes [25, 35].

In this section, we provide some compositionality results for bisimulation and trace semantics in the revised ULTRaS framework enriched with resolutions and based on semiring structures and measure schemata. We examine a selection of generalizations of typical operators such as action prefix, guarded choice, nondeterministic choice, and parallel composition, which are at the basis of any process calculus. While all the four behavioral metaequivalences defined in the previous section turn out to be congruences with respect to action prefix and the two choice operators, in the case of parallel composition our results confirm the existence of a foundational difference in the compositionality of bisimulation and trace semantics, which shows up in the presence of internal nondeterminism.

4.1. Action Prefix

Given an ULTRaS $\mathcal{U} = (S, A, \longrightarrow)$ on a reachability-consistent semiring $(D, \oplus, \otimes, 0_D, 1_D)$, we represent a generalized action prefix operator in the following way:

$$a \,.\, (d_j \triangleright s_j)_{j \in J}$$

where $a \in A$, $J \neq \emptyset$ is a finite index set, and $d_j \in D$ and $s_j \in S$ for all $j \in J$.

The operational semantics of action prefix is formalized as follows:

$$a \,.\, (d_j \triangleright s_j)_{j \in J} \;\xrightarrow{a}\; \bigoplus_{j \in J} (d_j \circledast \delta_{s_j})$$

where:

- $\oplus$ is naturally lifted from D-values to D-distributions by letting $(\Delta_1 \oplus \Delta_2)(s) = \Delta_1(s) \oplus \Delta_2(s)$.
- $\circledast$ denotes the multiplication of a D-distribution by a D-value as defined by $(d \circledast \Delta)(s) = d \otimes \Delta(s)$.
- δ_s is the singleton-support D-distribution such that $\delta_s(s) = 1_D$ and $\delta_s(s') = 0_D$ for all $s' \in S \setminus \{s\}$.

Notice that, in the presence of a replicated state in the sequence $(d_j \triangleright s_j)_{j \in J}$, the operational semantic rule sums up according to $\oplus$ all the d_j values associated with the various occurrences of that state. This is important to support a correct encoding of probabilistic/stochastic process calculi [12].

Each of the four metaequivalences of Sect. 3 is a congruence with respect to action prefix. More precisely, the two bisimulation metaequivalences are full congruences, while the two trace metaequivalences are conditional congruences in the case $|J| > 1$ and full congruences in the case $|J| = 1$.

Theorem 4.1. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRaS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, $J \neq \emptyset$ be a finite index set, and $s_{1,j}, s_{2,j} \in S$ for all $j \in J$. Let $\sim_{\rm B,\mathcal{M}} \in \{\sim^{\rm pre}_{\rm B,\mathcal{M}}, \sim^{\rm post}_{\rm B,\mathcal{M}}\}$. If $s_{1,j} \sim_{\rm B,\mathcal{M}} s_{2,j}$ for all $j \in J$, then $a \,.\, (d_j \triangleright s_{1,j})_{j \in J} \;\sim_{\rm B,\mathcal{M}}\; a \,.\, (d_j \triangleright s_{2,j})_{j \in J}$ for all $a \in A$ and $\{d_j \mid j \in J\} \subseteq D$.

Proof Let $\mathcal{B}$ be an $\mathcal{M}$-bisimulation witnessing $s_{1,j} \sim_{\mathrm{B},\mathcal{M}} s_{2,j}$ for all $j \in J$ and consider the relation:

$\mathcal{B}' = \mathcal{B} \cup \{(a\,.\,(d_j \triangleright s'_{1,j})_{j\in J},\, a\,.\,(d_j \triangleright s'_{2,j})_{j\in J}) \mid \{(s'_{1,j}, s'_{2,j}) \mid j \in J\} \subseteq \mathcal{B},\, a \in A,\, \{d_j \mid j \in J\} \subseteq D\}$

which is an equivalence relation because so is $\mathcal{B}$. We prove that $\mathcal{B}'$ is an $\mathcal{M}$-bisimulation too.

The only interesting case is the one in which the pair of states $(s'_1, s'_2) \in \mathcal{B}'$ is of the form $(a\,.\,(d_j \triangleright s'_{1,j})_{j\in J}$, $a\,.\,(d_j \triangleright s'_{2,j})_{j\in J})$ and we consider their initial a-transitions, respectively leading to $\bigoplus_{j\in J}(d_j \circledast \delta_{s'_{1,j}})$ and $\bigoplus_{j\in J}(d_j \circledast \delta_{s'_{2,j}})$. Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be the two corresponding 1-resolutions starting with an a-transition. Given $\mathcal{G} \in 2^{S/\mathcal{B}'}$ and observing that for all $j \in J$ it holds that $s'_{1,j} \in \bigcup\mathcal{G}$ iff $s'_{2,j} \in \bigcup\mathcal{G}$ because $(s'_{1,j}, s'_{2,j}) \in \mathcal{B} \subseteq \mathcal{B}'$, we have that:

$$\mathcal{M}(z_{s'_1}, a, \textstyle\bigcup\mathcal{G}) \;=\; \displaystyle\bigoplus_{j\in J}^{s'_{1,j}\in\bigcup\mathcal{G}} d_j \;=\; \bigoplus_{j\in J}^{s'_{2,j}\in\bigcup\mathcal{G}} d_j \;=\; \mathcal{M}(z_{s'_2}, a, \textstyle\bigcup\mathcal{G})$$

■

Theorem 4.2. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, $J \neq \emptyset$ be a finite index set, and $s_{1,j}, s_{2,j} \in S$ for all $j \in J$. Let $\sim_{\mathrm{T},\mathcal{M}} \in \{\sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}}, \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}}\}$. If $s_{1,j} \sim_{\mathrm{T},\mathcal{M}} s_{2,j}$ for all $j \in J$, then $a\,.\,(d_j \triangleright s_{1,j})_{j\in J} \sim_{\mathrm{T},\mathcal{M}} a\,.\,(d_j \triangleright s_{2,j})_{j\in J}$ for all $a \in A$ and $\{d_j \mid j \in J\} \subseteq D$ provided that $s_{1,j} \neq s_{1,j'}$ and $s_{2,j} \neq s_{2,j'}$ for all $j, j' \in J$ such that $j \neq j'$.

Proof Given $s_1 = a\,.\,(d_j \triangleright s_{1,j})_{j\in J}$ and $s_2 = a\,.\,(d_j \triangleright s_{2,j})_{j\in J}$, we observe that their initial a-transitions respectively lead to $\bigoplus_{j\in J}(d_j \circledast \delta_{s_{1,j}})$ and $\bigoplus_{j\in J}(d_j \circledast \delta_{s_{2,j}})$. Therefore, for $h \in \{1, 2\}$, each resolution of s_h that is not trivial (i.e., not consisting of a single state without transitions) starts with an a-transition that, for all $j \in J$, reaches with degree d_j state $s_{h,j}$ and hence all of its resolutions (the degree is precisely d_j because all $s_{h,j}$ states are distinct from each other).

The only interesting case is the one in which we consider $\alpha = a\,\alpha' \in A^*$ and $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in \mathit{Res}(s_1)$ that after performing a continues as $\mathcal{Z}_{1,j} = (Z_{1,j}, A, \longrightarrow_{\mathcal{Z}_{1,j}}) \in \mathit{Res}(s_{1,j})$ for all $j \in J$:

- If $s_{1,j} \sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}} s_{2,j}$ for all $j \in J$, then:
  $$\mathcal{M}(z_{s_1}, \alpha, Z_1) \;=\; \bigoplus_{j\in J}(d_j \otimes \mathcal{M}(z_{s_{1,j}}, \alpha', Z_{1,j})) \;=\; \bigoplus_{j\in J}(d_j \otimes \mathcal{M}(z_{s_{2,j}}, \alpha', Z_{2,j,\alpha'})) \;=\; \mathcal{M}(z_{s_2}, \alpha, Z_{2,\alpha})$$
  where, for all $j \in J$, $\mathcal{Z}_{2,j,\alpha'} = (Z_{2,j,\alpha'}, A, \longrightarrow_{\mathcal{Z}_{2,j,\alpha'}}) \in \mathit{Res}(s_{2,j})$ matches $\mathcal{Z}_{1,j}$ with respect to α' by virtue of $s_{1,j} \sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}} s_{2,j}$. Therefore $\mathcal{Z}_{2,\alpha} = (Z_{2,\alpha}, A, \longrightarrow_{\mathcal{Z}_{2,\alpha}}) \in \mathit{Res}(s_2)$, which after performing a continues as $\mathcal{Z}_{2,j,\alpha'}$ for all $j \in J$, matches $\mathcal{Z}_1$ with respect to α.

- If $s_{1,j} \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}} s_{2,j}$ for all $j \in J$, then:
  $$\mathcal{M}(z_{s_1}, \alpha, Z_1) \;=\; \bigoplus_{j\in J}(d_j \otimes \mathcal{M}(z_{s_{1,j}}, \alpha', Z_{1,j})) \;=\; \bigoplus_{j\in J}(d_j \otimes \mathcal{M}(z_{s_{2,j}}, \alpha', Z_{2,j})) \;=\; \mathcal{M}(z_{s_2}, \alpha, Z_2)$$
  where, for all $j \in J$, $\mathcal{Z}_{2,j} = (Z_{2,j}, A, \longrightarrow_{\mathcal{Z}_{2,j}}) \in \mathit{Res}(s_{2,j})$ matches $\mathcal{Z}_{1,j}$ with respect to all traces by virtue of $s_{1,j} \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}} s_{2,j}$. Therefore $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in \mathit{Res}(s_2)$, which after performing a continues as $\mathcal{Z}_{2,j}$ for all $j \in J$, matches $\mathcal{Z}_1$ with respect to all traces. ■

The condition $s_{1,j} \neq s_{1,j'}$ and $s_{2,j} \neq s_{2,j'}$ for all $j, j' \in J$ such that $j \neq j'$ is necessary in the proof above for the following reason. Suppose that $s_1 = a\,.\,(d_1 \triangleright s_{1,1}, d_2 \triangleright s_{1,2})$ and $s_2 = a\,.\,(d_1 \triangleright s'_2, d_2 \triangleright s'_2)$ where $s_{1,1} \neq s_{1,2}$, $s_{1,1} \sim_{\mathrm{T},\mathcal{M}} s'_2$, and $s_{1,2} \sim_{\mathrm{T},\mathcal{M}} s'_2$. When considering $\alpha = a\,\alpha' \in A^*$, if $\mathcal{Z}_{1,1} = (Z_{1,1}, A, \longrightarrow_{\mathcal{Z}_{1,1}}) \in \mathit{Res}(s_{1,1})$ and $\mathcal{Z}_{1,2} = (Z_{1,2}, A, \longrightarrow_{\mathcal{Z}_{1,2}}) \in \mathit{Res}(s_{1,2})$ are selected such that $\mathcal{M}(z_{s_{1,1}}, \alpha', Z_{1,1}) \neq \mathcal{M}(z_{s_{1,2}}, \alpha', Z_{1,2})$, then it is not possible to match both $\mathcal{Z}_{1,1}$ and $\mathcal{Z}_{1,2}$ with a single resolution of s'_2.

4.2. Guarded Choice

Guarded choice is an extension of action prefix that describes a choice among several action-prefixed alternative behaviors, whose generalization we represent in the following way:

$$\sum_{i\in I} a_i\,.\,(d_{i,j} \triangleright s_{i,j})_{j\in J_i}$$

where $I \neq \emptyset$ is a finite index set and $a_i\,.\,(d_{i,j} \triangleright s_{i,j})_{j\in J_i}$ is a generalized action prefix expression for all $i \in I$. This construct is useful in probabilistic and stochastic settings to express quantified choices and impose specific constraints:

- For generative probabilistic processes, an action can occur several times and it is necessary to ensure that the summation over all $i \in I$ and $j \in J_i$ of the $d_{i,j}$ values is 1.
- For reactive probabilistic processes, an action can occur at most once and it is necessary to ensure that for each $i \in I$ the summation over all $j \in J_i$ of the $d_{i,j}$ values is 1.
- In the case of stochastic processes, it allows the race policy to be implemented.

The operational semantics of guarded choice is formalized as follows:

$$\sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s_{i,j})_{j \in J_i} \;\xrightarrow{a_i}\; \bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \circledast \delta_{s_{k,j}}), \quad i \in I$$

As can be noted, a single transition is generated for each distinct a_i-action as the target distributions of prefixes starting with the same action are summed up according to $\oplus$, which enables a correct encoding of probabilistic and stochastic process calculi [12].

For the four metaequivalences of Sect. 3, the congruence results with respect to guarded choice are analogous to the ones with respect to action prefix in Sect. 4.1.

Theorem 4.3. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, $I \neq \emptyset$ be a finite index set, $J_i \neq \emptyset$ be a finite index set for all $i \in I$, and $s_{1,i,j}, s_{2,i,j} \in S$ for all $i \in I$ and $j \in J_i$. Let $\sim_{\mathrm{B},\mathcal{M}} \in \{\sim^{\mathrm{pre}}_{\mathrm{B},\mathcal{M}}, \sim^{\mathrm{post}}_{\mathrm{B},\mathcal{M}}\}$. If $s_{1,i,j} \sim_{\mathrm{B},\mathcal{M}} s_{2,i,j}$ for all $i \in I$ and $j \in J_i$, then $\sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s_{1,i,j})_{j \in J_i} \;\sim_{\mathrm{B},\mathcal{M}}\; \sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s_{2,i,j})_{j \in J_i}$ for all $\{a_i \mid i \in I\} \subseteq A$ and $\{d_{i,j} \mid i \in I, j \in J_i\} \subseteq D$.

Proof Let $\mathcal{B}$ be an $\mathcal{M}$-bisimulation witnessing $s_{1,i,j} \sim_{\mathrm{B},\mathcal{M}} s_{2,i,j}$ for all $i \in I$ and $j \in J_i$ and consider the relation:

$$\begin{array}{rcl} \mathcal{B}' & = & \mathcal{B} \cup \{(\sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s'_{1,i,j})_{j \in J_i}, \sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s'_{2,i,j})_{j \in J_i}) \mid \\ & & \quad \{(s'_{1,i,j}, s'_{2,i,j}) \mid i \in I, j \in J_i\} \subseteq \mathcal{B}, \{a_i \mid i \in I\} \subseteq A, \{d_{i,j} \mid i \in I, j \in J_i\} \subseteq D\} \end{array}$$

which is an equivalence relation because so is $\mathcal{B}$. We prove that $\mathcal{B}'$ is an $\mathcal{M}$-bisimulation too. The only interesting case is the one in which the pair of states $(s'_1, s'_2) \in \mathcal{B}'$ is $(\sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s'_{1,i,j})_{j \in J_i}, \sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s'_{2,i,j})_{j \in J_i})$ and we consider one of their initial a_i-transitions, respectively leading to $\bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \circledast \delta_{s'_{1,k,j}})$ and $\bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \circledast \delta_{s'_{2,k,j}})$. Let $\mathcal{Z}_{1,a_i}$ and $\mathcal{Z}_{2,a_i}$ be the two corresponding 1-resolutions starting with an a_i-transition. Given $\mathcal{G} \in 2^{S/\mathcal{B}'}$ and observing that for all $i \in I$ and $j \in J_i$ it holds that $s'_{1,i,j} \in \bigcup \mathcal{G}$ iff $s'_{2,i,j} \in \bigcup \mathcal{G}$ because $(s'_{1,i,j}, s'_{2,i,j}) \in \mathcal{B} \subseteq \mathcal{B}'$, we have that:

$$\mathcal{M}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k}^{s'_{1,k,j} \in \bigcup \mathcal{G}} d_{k,j} \;=\; \bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k}^{s'_{2,k,j} \in \bigcup \mathcal{G}} d_{k,j} \;=\; \mathcal{M}(z_{s'_2}, a, \bigcup \mathcal{G})$$

■

Theorem 4.4. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, $I \neq \emptyset$ be a finite index set, $J_i \neq \emptyset$ be a finite index set for all $i \in I$, and $s_{1,i,j}, s_{2,i,j} \in S$ for all $i \in I$ and $j \in J_i$. Let $\sim_{\mathrm{T},\mathcal{M}} \in \{\sim^{\mathrm{pre}}_{\mathrm{T},\mathcal{M}}, \sim^{\mathrm{post}}_{\mathrm{T},\mathcal{M}}\}$. If $s_{1,i,j} \sim_{\mathrm{T},\mathcal{M}} s_{2,i,j}$ for all $i \in I$ and $j \in J_i$, then $\sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s_{1,i,j})_{j \in J_i} \;\sim_{\mathrm{T},\mathcal{M}}\; \sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s_{2,i,j})_{j \in J_i}$ for all $\{a_i \mid i \in I\} \subseteq A$ and $\{d_{i,j} \mid i \in I, j \in J_i\} \subseteq D$ provided that $s_{1,i,j} \neq s_{1,k,j'}$ and $s_{2,i,j} \neq s_{2,k,j'}$ for all $i, k \in I$, $j \in J_i$, and $j' \in J_k$ such that $j \neq j'$ when $i = k$.

Proof Given $s_1 = \sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s_{1,i,j})_{j \in J_i}$ and $s_2 = \sum_{i \in I} a_i \,.\, (d_{i,j} \triangleright s_{2,i,j})_{j \in J_i}$, we observe that their initial a_i-transitions respectively lead to $\bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \circledast \delta_{s_{1,k,j}})$ and $\bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \circledast \delta_{s_{2,k,j}})$. Therefore, for $h \in \{1, 2\}$, each resolution of s_h that is not trivial (i.e., not consisting of a single state without transitions) starts with an a_i-transition that, for all $j \in J_i$, reaches with degree $d_{i,j}$ state $s_{h,i,j}$ and hence all of its resolutions (the degree is precisely $d_{i,j}$ because all $s_{h,i,j}$ states are distinct from each other and from all $s_{h,k,j}$ states for $a_k = a_i$).
The only interesting case is the one in which we consider $\alpha = a_i \, \alpha' \in A^*$ and $\mathcal{Z}_{1,i} = (Z_{1,i}, A, \longrightarrow_{\mathcal{Z}_{1,i}}) \in \mathit{Res}(s_1)$ that after performing a_i continues as $\mathcal{Z}_{1,k,j} = (Z_{1,k,j}, A, \longrightarrow_{\mathcal{Z}_{1,k,j}}) \in \mathit{Res}(s_{1,k,j})$ for all $k \in I$ such that $a_k = a_i$ and $j \in J_k$:

- If $s_{1,i,j} \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_{2,i,j}$ for all $i \in I$ and $j \in J_i$, then:

$$\begin{aligned} \mathcal{M}(z_{s_1}, \alpha, Z_{1,i}) &= \bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \otimes \mathcal{M}(z_{s_{1,k,j}}, \alpha', Z_{1,k,j})) = \\ &= \bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \otimes \mathcal{M}(z_{s_{2,k,j}}, \alpha', Z_{2,k,j,\alpha'})) = \mathcal{M}(z_{s_2}, \alpha, Z_{2,i,\alpha}) \end{aligned}$$

where, for all $k \in I$ such that $a_k = a_i$ and $j \in J_k$, $\mathcal{Z}_{2,k,j,\alpha'} = (Z_{2,k,j,\alpha'}, A, \longrightarrow_{\mathcal{Z}_{2,k,j,\alpha'}}) \in \mathit{Res}(s_{2,k,j})$ matches $\mathcal{Z}_{1,k,j}$ with respect to α' by virtue of $s_{1,k,j} \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_{2,k,j}$. Therefore $\mathcal{Z}_{2,i,\alpha} = (Z_{2,i,\alpha}, A, \longrightarrow_{\mathcal{Z}_{2,i,\alpha}})$ $\in \mathit{Res}(s_2)$, which after performing a_i continues as $\mathcal{Z}_{2,k,j,\alpha'}$ for all $k \in I$ such that $a_k = a_i$ and $j \in J_k$, matches $\mathcal{Z}_{1,i}$ with respect to α.

- If $s_{1,i,j} \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}} s_{2,i,j}$ for all $i \in I$ and $j \in J_i$, then:

$$\begin{aligned} \mathcal{M}(z_{s_1}, \alpha, Z_{1,i}) &= \bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \otimes \mathcal{M}(z_{s_{1,k,j}}, \alpha', Z_{1,k,j})) = \\ &= \bigoplus_{k \in I}^{a_k = a_i} \bigoplus_{j \in J_k} (d_{k,j} \otimes \mathcal{M}(z_{s_{2,k,j}}, \alpha', Z_{2,k,j})) = \mathcal{M}(z_{s_2}, \alpha, Z_{2,i}) \end{aligned}$$

where, for all $k \in I$ such that $a_k = a_i$ and $j \in J_k$, $\mathcal{Z}_{2,k,j} = (Z_{2,k,j}, A, \longrightarrow_{\mathcal{Z}_{2,k,j}}) \in \mathit{Res}(s_{2,k,j})$ matches $\mathcal{Z}_{1,k,j}$ with respect to all traces by virtue of $s_{1,k,j} \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{post}} s_{2,k,j}$. Therefore $\mathcal{Z}_{2,i} = (Z_{2,i}, A, \longrightarrow_{\mathcal{Z}_{2,i}}) \in \mathit{Res}(s_2)$, which after performing a_i continues as $\mathcal{Z}_{2,k,j}$ for all $k \in I$ such that $a_k = a_i$ and $j \in J_k$, matches $\mathcal{Z}_{1,i}$ with respect to all traces. ■

4.3. Nondeterministic Choice

Nondeterministic choice, denoted by $+$, is governed by the following two operational semantic rules:

$$\frac{s_1 \xrightarrow{a} \Delta}{s_1 + s_2 \xrightarrow{a} \Delta} \qquad \frac{s_2 \xrightarrow{a} \Delta}{s_1 + s_2 \xrightarrow{a} \Delta}$$

This construct, which is typical of process calculi admitting internal nondeterminism, does not support any constraint and does not combine target distributions. According to the two rules above, $s + s$ is isomorphic to s itself, i.e., nondeterministic choice is idempotent. Observing that $+$ is also associative and commutative, we denote by $\sum$ the extension of binary nondeterministic choice to arbitrarily many operands.

All the four metaequivalences of Sect. 3 are congruences with respect to nondeterministic choice.

Theorem 4.5. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$. Let $\sim_{\mathrm{B},\mathcal{M}} \in \{\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}}, \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}}\}$. If $s_1 \sim_{\mathrm{B},\mathcal{M}} s_2$, then $s_1 + s \sim_{\mathrm{B},\mathcal{M}} s_2 + s$ for all $s \in S$.

Proof Let $\mathcal{B}$ be an $\mathcal{M}$-bisimulation witnessing $s_1 \sim_{\mathrm{B},\mathcal{M}} s_2$ and consider the relation:

$$\mathcal{B}' = \mathcal{B} \cup \{(s_1' + s, \, s_2' + s) \mid (s_1', s_2') \in \mathcal{B}, \, s \in S\}$$

which is an equivalence relation – because so is $\mathcal{B}$ – whose equivalence classes are the classes of $\mathcal{B}$ along with the new classes of the form $C + s = \{s' + s \mid s' \in C\}$ for $C \in S/\mathcal{B}$ and $s \in S$. We prove that $\mathcal{B}'$ is an $\mathcal{M}$-bisimulation too.

The only interesting case is the one in which the considered pair of states is $(s_1' + s, s_2' + s) \in \mathcal{B}'$. Suppose that $s_1' + s \xrightarrow{a} \Delta_1'$ for some $a \in A$ and D-distribution Δ_1' and let $\mathcal{Z}_1' \in 1\text{-}\mathit{Res}(s_1' + s)$ be the 1-resolution selecting that transition. There are two possibilities:

- If $s_1' + s \xrightarrow{a} \Delta_1'$ is originated from $s_1' \xrightarrow{a} \Delta_1'$, with $\mathcal{Z}_1 \in 1\text{-}\mathit{Res}(s_1')$ being the 1-resolution selecting the latter transition, then from $(s_1', s_2') \in \mathcal{B}$ it follows that:

  - In the case of $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}}$, for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ there exists $s_2' \xrightarrow{a} \Delta_{2,\mathcal{G}}'$, with $\mathcal{Z}_{2,\mathcal{G}} \in 1\text{-}\mathit{Res}(s_2')$ being the 1-resolution selecting that transition, such that:

    $$\mathcal{M}(z_{s_1'}, a, \bigcup \mathcal{G}) = \bigoplus_{s' \in \cup \mathcal{G}} \Delta_1'(s') = \bigoplus_{s' \in \cup \mathcal{G}} \Delta_{2,\mathcal{G}}'(s') = \mathcal{M}(z_{s_2'}, a, \bigcup \mathcal{G})$$

    where finitely many D-values different from 0_D occur in both summations because Δ_1' and $\Delta_{2,\mathcal{G}}'$

have finite support. As a consequence, for all $\mathcal{G}' \in 2^{S/\mathcal{B}'}$ containing $\mathcal{G}$ as the larger set of classes of $\mathcal{B}$ there exists $s'_2 + s \xrightarrow{a} \Delta'_{2,\mathcal{G}}$, with $\mathcal{Z}'_{2,\mathcal{G}} \in 1\text{-}Res(s'_2 + s)$ being the 1-resolution selecting that transition, such that:

$$\mathcal{M}(z_{s'_1+s}, a, \bigcup \mathcal{G}') \; = \; \mathcal{M}(z_{s'_1}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s'_2}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s'_2+s}, a, \bigcup \mathcal{G}')$$

  - In the case of $\sim^{\text{post}}_{\text{B},\mathcal{M}}$, there exists $s'_2 \xrightarrow{a} \Delta'_2$, with $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ being the 1-resolution selecting that transition, such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:

$$\mathcal{M}(z_{s'_1}, a, \bigcup \mathcal{G}) \; = \; \bigoplus_{s' \in \cup \mathcal{G}} \Delta'_1(s') \; = \; \bigoplus_{s' \in \cup \mathcal{G}} \Delta'_2(s') \; = \; \mathcal{M}(z_{s'_2}, a, \bigcup \mathcal{G})$$

    where finitely many D-values different from 0_D occur in both summations because Δ'_1 and Δ'_2 have finite support. As a consequence $s'_2 + s \xrightarrow{a} \Delta'_2$, with $\mathcal{Z}'_2 \in 1\text{-}Res(s'_2 + s)$ being the 1-resolution selecting that transition, and for all $\mathcal{G}' \in 2^{S/\mathcal{B}'}$, indicating with $\mathcal{G}$ the larger set of classes of $\mathcal{B}$ contained in $\mathcal{G}'$, it holds that:

$$\mathcal{M}(z_{s'_1+s}, a, \bigcup \mathcal{G}') \; = \; \mathcal{M}(z_{s'_1}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s'_2}, a, \bigcup \mathcal{G}) \; = \; \mathcal{M}(z_{s'_2+s}, a, \bigcup \mathcal{G}')$$

- If $s'_1 + s \xrightarrow{a} \Delta'_1$ is originated from $s \xrightarrow{a} \Delta'_1$, then $s'_2 + s \xrightarrow{a} \Delta'_1$. Since the two a-transitions from the $\_ + s$ states have the same target distribution, they match one another under $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ and $\sim^{\text{post}}_{\text{B},\mathcal{M}}$. ■

Theorem 4.6. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$. Let $\sim_{\text{T},\mathcal{M}} \in \{\sim^{\text{pre}}_{\text{T},\mathcal{M}}, \sim^{\text{post}}_{\text{T},\mathcal{M}}\}$. If $s_1 \sim_{\text{T},\mathcal{M}} s_2$, then $s_1 + s \sim_{\text{T},\mathcal{M}} s_2 + s$ for all $s \in S$.

Proof Given $s_h + s$ for $h \in \{1, 2\}$, we observe that the set of traces executable by $s_h + s$ is the union of the set of traces executable by s_h and the set of traces executable by s, and that each resolution of s_h that is not trivial (i.e., not consisting of a single state without transitions) starts with a transition of s_h or s. To avoid trivial cases, we consider a nontrivial $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1 + s)$. There are two possibilities:

- Suppose that $\mathcal{Z}_1$ starts with a transition of s_1:
  - If $s_1 \sim^{\text{pre}}_{\text{T},\mathcal{M}} s_2$, then for $\mathcal{Z}'_1 = (Z'_1, A, \longrightarrow_{\mathcal{Z}'_1}) \in Res(s_1)$ isomorphic to $\mathcal{Z}_1$ it holds that for all $\alpha \in A^*$ there exists $\mathcal{Z}'_{2,\alpha} = (Z'_{2,\alpha}, A, \longrightarrow_{\mathcal{Z}'_{2,\alpha}}) \in Res(s_2)$ such that:

$$\mathcal{M}(z'_{s_1}, \alpha, Z'_1) \; = \; \mathcal{M}(z'_{s_2}, \alpha, Z'_{2,\alpha})$$

    Since for each such $\mathcal{Z}'_{2,\alpha}$ there exists $\mathcal{Z}_{2,\alpha} = (Z_{2,\alpha}, A, \longrightarrow_{\mathcal{Z}_{2,\alpha}}) \in Res(s_2 + s)$ isomorphic to $\mathcal{Z}'_{2,\alpha}$, for all $\alpha \in A^*$ it holds that:

$$\mathcal{M}(z_{s_1+s}, \alpha, Z_1) \; = \; \mathcal{M}(z'_{s_1}, \alpha, Z'_1) \; = \; \mathcal{M}(z'_{s_2}, \alpha, Z'_{2,\alpha}) \; = \; \mathcal{M}(z_{s_2+s}, \alpha, Z_{2,\alpha})$$

  - If $s_1 \sim^{\text{post}}_{\text{T},\mathcal{M}} s_2$, then for $\mathcal{Z}'_1 = (Z'_1, A, \longrightarrow_{\mathcal{Z}'_1}) \in Res(s_1)$ isomorphic to $\mathcal{Z}_1$ there exists $\mathcal{Z}'_2 = (Z'_2, A, \longrightarrow_{\mathcal{Z}'_2}) \in Res(s_2)$ such that for all $\alpha \in A^*$ it holds that:

$$\mathcal{M}(z'_{s_1}, \alpha, Z'_1) \; = \; \mathcal{M}(z'_{s_2}, \alpha, Z'_2)$$

    Since there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2 + s)$ isomorphic to $\mathcal{Z}'_2$, for all $\alpha \in A^*$ it holds that:

$$\mathcal{M}(z_{s_1+s}, \alpha, Z_1) \; = \; \mathcal{M}(z'_{s_1}, \alpha, Z'_1) \; = \; \mathcal{M}(z'_{s_2}, \alpha, Z'_2) \; = \; \mathcal{M}(z_{s_2+s}, \alpha, Z_2)$$

- Suppose that $\mathcal{Z}_1$ starts with a transition of s. Then $\mathcal{Z}_1$ is isomorphic to some $\mathcal{Z}'_1 = (Z'_1, A, \longrightarrow_{\mathcal{Z}'_1}) \in Res(s)$ and it is sufficient to take $\mathcal{Z}'_2 = \mathcal{Z}'_1$ and $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2 + s)$ isomorphic to $\mathcal{Z}'_2$ to ensure that $\mathcal{Z}_1$ and $\mathcal{Z}_2$ match one another under $\sim^{\text{pre}}_{\text{T},\mathcal{M}}$ and $\sim^{\text{post}}_{\text{T},\mathcal{M}}$. ■

4.4. Parallel Composition

Among the numerous parallel composition operators proposed in the literature, the one in the CSP style [9] that we consider here, denoted by $\|$, is governed by the following three operational semantic rules:

$$\frac{s_1 \xrightarrow{a} \Delta_1 \quad a \notin L}{s_1 \,\|_L\, s_2 \xrightarrow{a} \Delta_1 \otimes \delta_{s_2}} \qquad \frac{s_2 \xrightarrow{a} \Delta_2 \quad a \notin L}{s_1 \,\|_L\, s_2 \xrightarrow{a} \delta_{s_1} \otimes \Delta_2} \qquad \frac{s_1 \xrightarrow{a} \Delta_1 \quad s_2 \xrightarrow{a} \Delta_2 \quad a \in L}{s_1 \,\|_L\, s_2 \xrightarrow{a} \Delta_1 \otimes \Delta_2}$$

where $L \subseteq A$ and $\otimes$ is naturally lifted from D-values to D-distributions over the parallel composition of states by letting $(\Delta_1 \otimes \Delta_2)(s_1 \,\|_L\, s_2) = \Delta_1(s_1) \otimes \Delta_2(s_2)$.

Based on the operational semantic rules introduced so far, parallel composition is related to nondeterministic choice and guarded choice according to the following generalized expansion law. For $K_1 \neq \emptyset \neq K_2$ finite index sets, given two expressions such as:

$$s_1 \; = \sum_{k \in K_1} \sum_{i \in I_k} a_{1,k,i} \,.\, (d_{1,k,i,j} \triangleright s_{1,k,i,j})_{j \in J_i} \qquad s_2 \; = \sum_{k' \in K_2} \sum_{i' \in I_{k'}} a_{2,k',i'} \,.\, (d_{2,k',i',j'} \triangleright s_{2,k',i',j'})_{j' \in J_{i'}}$$

it holds that $s_1 \parallel_L s_2$ is isomorphic to:

$$\sum_{k \in K_1} \sum_{i \in I_k}^{a_{1,k,i} \notin L} a_{1,k,i} \,.\, (d_{1,k,i,j} \triangleright (s_{1,k,i,j} \parallel_L s_2))_{j \in J_i} + \sum_{k' \in K_2} \sum_{i' \in I_{k'}}^{a_{2,k',i'} \notin L} a_{2,k',i'} \,.\, (d_{2,k',i',j'} \triangleright (s_1 \parallel_L s_{2,k',i',j'}))_{j' \in J_{i'}}$$
$$+ \sum_{k \in K_1} \sum_{k' \in K_2} \sum_{i \in I_k}^{a_{1,k,i} \in L} \sum_{i' \in I_{k'}}^{a_{2,k',i'} = a_{1,k,i}} a_{1,k,i} \,.\, ((d_{1,k,i,j} \otimes d_{2,k',i',j'}) \triangleright (s_{1,k,i,j} \parallel_L s_{2,k',i',j'}))_{j \in J_i, j' \in J_{i'}}$$

The compositionality of the four metaequivalences of Sect. 3 with respect to $\parallel$ is discussed in the following two subsections: the former focuses on bisimulation semantics, while the latter on trace semantics.

4.4.1. Compositionality of Bisimulation Metaequivalences

Our first metaresult establishes that $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}}$ is always – i.e., for every measure schema $\mathcal{M}$ defined over an arbitrary ULTRAS – a congruence with respect to parallel composition. As a consequence of Prop. 3.5, this is the case also for $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}}$ in the absence of internal nondeterminism.

Theorem 4.7. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$. If $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$, then $s_1 \parallel_L s \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2 \parallel_L s$ for all $L \subseteq A$ and $s \in S$.

Proof Let $\mathcal{B}$ be an $\mathcal{M}$-post-bisimulation witnessing $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$ and for $L \subseteq A$ consider the relation:

$$\mathcal{B}' \; = \; \{(s'_1 \parallel_L s, \; s'_2 \parallel_L s) \mid (s'_1, s'_2) \in \mathcal{B}, \; s \in S\}$$

which is an equivalence relation over $S \parallel_L S$ – because $\mathcal{B}$ is an equivalence relation over S – whose equivalence classes are of the form $C \parallel_L s = \{s' \parallel_L s \mid s' \in C\}$ for $C \in S/\mathcal{B}$ and $s \in S$. We prove that $\mathcal{B}'$ is an $\mathcal{M}$-post-bisimulation too by examining the transitions departing from a pair of states $(s'_1 \parallel_L s, s'_2 \parallel_L s) \in \mathcal{B}'$. Suppose that $s'_1 \parallel_L s \xrightarrow{a} \Delta'_1$ for some $a \in A$ and D-distribution Δ'_1 and let $\mathcal{Z}'_1 \in 1\text{-}Res(s'_1 \parallel_L s)$ be the 1-resolution selecting that transition. There are two cases:

- Assuming that $a \notin L$, there are two subcases:
  - If $s'_1 \parallel_L s \xrightarrow{a} \Delta'_1$ is originated from $s'_1 \xrightarrow{a} \Delta_1$, with $\Delta'_1 = \Delta_1 \otimes \delta_s$ and $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ being the 1-resolution selecting the latter transition, then from $(s'_1, s'_2) \in \mathcal{B}$ it follows that there exists $s'_2 \xrightarrow{a} \Delta_2$, with $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ being the 1-resolution selecting that transition, such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:
  $$\mathcal{M}(z_{s'_1}, a, \bigcup \mathcal{G}) \; = \bigoplus_{s' \in \cup \mathcal{G}} \Delta_1(s') \; = \bigoplus_{s' \in \cup \mathcal{G}} \Delta_2(s') \; = \; \mathcal{M}(z_{s'_2}, a, \bigcup \mathcal{G})$$
  where finitely many D-values different from 0_D occur in both summations because Δ_1 and Δ_2 have finite support. As a consequence $s'_2 \parallel_L s \xrightarrow{a} \Delta'_2$, with $\Delta'_2 = \Delta_2 \otimes \delta_s$ and $\mathcal{Z}'_2 \in 1\text{-}Res(s'_2 \parallel_L s)$ being the 1-resolution selecting that transition. Given an arbitrary $\mathcal{G}' \in 2^{(S \parallel_L S)/\mathcal{B}'}$, which is of the form $\{C_i \parallel_L r_i \mid C_i \in S/\mathcal{B}, r_i \in S, i \in I\}$, it holds that:
  $$\mathcal{M}(z_{s'_1 \parallel_L s}, a, \bigcup \mathcal{G}') \; = \bigoplus_{i \in I}^{r_i = s} \mathcal{M}(z_{s'_1}, a, \bigcup \{C_i\}) \; = \bigoplus_{i \in I}^{r_i = s} \mathcal{M}(z_{s'_2}, a, \bigcup \{C_i\}) \; = \; \mathcal{M}(z_{s'_2 \parallel_L s}, a, \bigcup \mathcal{G}')$$
  where finitely many D-values different from 0_D occur in both summations because finitely many C_i classes yield $\mathcal{M}(z_{s'_1}, a, \bigcup\{C_i\}) \neq 0_D \neq \mathcal{M}(z_{s'_2}, a, \bigcup\{C_i\})$ – as Δ_1 and Δ_2 have finite support.
  - If $s'_1 \parallel_L s \xrightarrow{a} \Delta'_1$ is originated from $s \xrightarrow{a} \Delta$, with $\Delta'_1 = \delta_{s'_1} \otimes \Delta$ and $\mathcal{Z} \in 1\text{-}Res(s)$ being the 1-resolution selecting the latter transition, then $s'_2 \parallel_L s \xrightarrow{a} \Delta'_2$, with $\Delta'_2 = \delta_{s'_2} \otimes \Delta$ and $\mathcal{Z}'_2 \in 1\text{-}Res(s'_2 \parallel_L s)$ being the 1-resolution selecting that transition. Given an arbitrary $\mathcal{G}' \in 2^{(S \parallel_L S)/\mathcal{B}'}$, which is of the form $\{C_i \parallel_L r_i \mid C_i \in S/\mathcal{B}, r_i \in S, i \in I\}$, it holds that:

$$\begin{array}{rcl}\mathcal{M}(z_{s'_1 \,\|_L\, s}, a, \bigcup \mathcal{G}') & = & \bigoplus\limits_{r \in S}^{\exists i \in I. s'_1 \in C_i \wedge r_i = r} \mathcal{M}(z_s, a, \{r\}) \; = \\ & = & \bigoplus\limits_{r \in S}^{\exists i \in I. s'_1 \in C_i \wedge r_i = r} \Delta(r) \; = \\ & = & \bigoplus\limits_{r \in S}^{\exists i \in I. s'_2 \in C_i \wedge r_i = r} \Delta(r) \; = \\ & = & \bigoplus\limits_{r \in S}^{\exists i \in I. s'_2 \in C_i \wedge r_i = r} \mathcal{M}(z_s, a, \{r\}) \; = \; \mathcal{M}(z_{s'_2 \,\|_L\, s}, a, \bigcup \mathcal{G}')\end{array}$$

where $s'_1 \in C_i$ iff $s'_2 \in C_i$ because $(s'_1, s'_2) \in \mathcal{B}$ and finitely many D-values different from 0_D occur in all summations because Δ has finite support.

- Assume that $a \in L$. Then $s'_1 \,\|_L\, s \xrightarrow{a} \Delta'_1$ is originated from $s'_1 \xrightarrow{a} \Delta_1$, with $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ being the 1-resolution selecting that transition, and $s \xrightarrow{a} \Delta$, with $\mathcal{Z} \in 1\text{-}Res(s)$ being the 1-resolution selecting that transition, where $\Delta'_1 = \Delta_1 \otimes \Delta$. From $(s'_1, s'_2) \in \mathcal{B}$, it follows that there exists $s'_2 \xrightarrow{a} \Delta_2$, with $\mathcal{Z}_2 \in 1\text{-}Res(s'_2)$ being the 1-resolution selecting that transition, such that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ it holds that:
$$\mathcal{M}(z_{s'_1}, a, \bigcup \mathcal{G}) \; = \bigoplus_{s' \in \cup \mathcal{G}} \Delta_1(s') \; = \bigoplus_{s' \in \cup \mathcal{G}} \Delta_2(s') \; = \; \mathcal{M}(z_{s'_2}, a, \bigcup \mathcal{G})$$
where finitely many D-values different from 0_D occur in both summations because Δ_1 and Δ_2 have finite support. As a consequence $s'_2 \,\|_L\, s \xrightarrow{a} \Delta'_2$, with $\Delta'_2 = \Delta_2 \otimes \Delta$ and $\mathcal{Z}'_2 \in 1\text{-}Res(s'_2 \,\|_L\, s)$ being the 1-resolution selecting that transition. Given an arbitrary $\mathcal{G}' \in 2^{(S \,\|_L\, S)/\mathcal{B}'}$, which is of the form $\{C_i \,\|_L\, r_i \mid C_i \in S/\mathcal{B}, r_i \in S, i \in I\}$, it holds that:
$$\begin{array}{rcl}\mathcal{M}(z_{s'_1 \,\|_L\, s}, a, \bigcup \mathcal{G}') & = & \bigoplus\limits_{i \in I} (\mathcal{M}(z_{s'_1}, a, \bigcup \{C_i\}) \otimes \mathcal{M}(z_s, a, \{r_i\})) \; = \\ & = & \bigoplus\limits_{i \in I} (\mathcal{M}(z_{s'_2}, a, \bigcup \{C_i\}) \otimes \mathcal{M}(z_s, a, \{r_i\})) \; = \; \mathcal{M}(z_{s'_2 \,\|_L\, s}, a, \bigcup \mathcal{G}')\end{array}$$
where finitely many D-values different from 0_D occur in both summations because finitely many C_i classes yield $\mathcal{M}(z_{s'_1}, a, \bigcup \{C_i\}) \neq 0_D \neq \mathcal{M}(z_{s'_2}, a, \bigcup \{C_i\})$ – as Δ_1 and Δ_2 have finite support – and finitely many r_i states yield $\mathcal{M}(z_s, a, \{r_i\}) \neq \bot_D$ – as Δ and has finite support. ■

Corollary 4.8. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS with no internal nondeterminism, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$. If $s_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}} s_2$, then $s_1 \,\|_L\, s \sim^{\text{pre}}_{\text{B},\mathcal{M}} s_2 \,\|_L\, s$ for all $L \subseteq A$ and $s \in S$.

Proof A straightforward consequence of the fact that, if there is no internal nondeterminism in $\mathcal{U}$, then $\sim^{\text{pre}}_{\text{B},\mathcal{M}} = \sim^{\text{post}}_{\text{B},\mathcal{M}}$ as established by Prop. 3.5(1). ■

It remains to investigate the compositionality of $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ in the presence of internal nondeterminism. We start with the case $|D| = 2$, i.e., with the simplest reachability-consistent semiring, which is $(\mathbb{B}, \vee, \wedge)$, and the corresponding measure schema, which is $\mathcal{M}_{\text{nd}}$. In this specific case, $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ turns out to be a congruence with respect to parallel composition. Intuitively, in addition to the coinductive nature of bisimulation semantics, the reason is that, starting from transitions whose target distributions can only contain $\top$ and $\bot$ as values, their parallel composition cannot generate, for the target distributions of the resulting transitions, any value different from $\top$ and $\bot$ themselves.

Theorem 4.9. Let $\mathcal{U} = (S, A, \longrightarrow)$ be a $\mathbb{B}$-ULTraS and $s_1, s_2 \in S$. If $s_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} s_2$, then $s_1 \,\|_L\, s \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} s_2 \,\|_L\, s$ for all $L \subseteq A$ and $s \in S$.

Proof Let $\mathcal{B}$ be an $\mathcal{M}_{\text{nd}}$-pre-bisimulation witnessing $s_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}_{\text{nd}}} s_2$ and for $L \subseteq A$ consider the relation:
$$\mathcal{B}' \; = \; \{(s'_1 \,\|_L\, s, \; s'_2 \,\|_L\, s) \mid (s'_1, s'_2) \in \mathcal{B}, \; s \in S\}$$
which is an equivalence relation over $S \,\|_L\, S$ – because $\mathcal{B}$ is an equivalence relation over S – whose equivalence classes are of the form $C \,\|_L\, s = \{s' \,\|_L\, s \mid s' \in C\}$ for $C \in S/\mathcal{B}$ and $s \in S$. We prove that $\mathcal{B}'$ is an $\mathcal{M}_{\text{nd}}$-pre-bisimulation too by examining the transitions departing from a pair of states $(s'_1 \,\|_L\, s, s'_2 \,\|_L\, s) \in \mathcal{B}'$.
Suppose that $s'_1 \,\|_L\, s \xrightarrow{a} \Delta'_1$ for some $a \in A$ and $\mathbb{B}$-distribution Δ'_1 and let $\mathcal{Z}'_1 \in 1\text{-}Res(s'_1 \,\|_L\, s)$ be the 1-resolution selecting that transition. There are two cases:

- Assuming that $a \notin L$, there are two subcases:

  - If $s'_1 \,\|_L\, s \xrightarrow{a} \Delta'_1$ is originated from $s'_1 \xrightarrow{a} \Delta_1$, with $\Delta'_1 = \Delta_1 \otimes \delta_s$ and $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ being the 1-resolution selecting the latter transition, then from $(s'_1, s'_2) \in \mathcal{B}$ it follows that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ there exists $s'_2 \xrightarrow{a} \Delta_{2,\mathcal{G}}$, with $\mathcal{Z}_{2,\mathcal{G}} \in 1\text{-}Res(s'_2)$ being the 1-resolution selecting that transition, such that:
  $$\mathcal{M}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \bigvee_{s' \in \cup \mathcal{G}} \Delta_1(s') \;=\; \bigvee_{s' \in \cup \mathcal{G}} \Delta_{2,\mathcal{G}}(s') \;=\; \mathcal{M}(z_{s'_2}, a, \bigcup \mathcal{G})$$
  where finitely many $\mathbb{B}$-values different from $\bot$ occur in both disjunctions because Δ_1 and $\Delta_{2,\mathcal{G}}$ have finite support. As a consequence there exist transitions of the form $s'_2 \,\|_L\, s \xrightarrow{a} \Delta'_{2,\mathcal{G}}$, with $\Delta'_{2,\mathcal{G}} = \Delta_{2,\mathcal{G}} \otimes \delta_s$ and $\mathcal{Z}'_{2,\mathcal{G}} \in 1\text{-}Res(s'_2 \,\|_L\, s)$ being the 1-resolution selecting a transition of that form. Given an arbitrary $\mathcal{G}' \in 2^{(S \,\|_L\, S)/\mathcal{B}'}$, which is of the form $\{C_i \,\|_L\, r_i \mid C_i \in S/\mathcal{B}, r_i \in S, i \in I\}$, since:
  $$\mathcal{M}(z_{s'_1 \,\|_L\, s}, a, \bigcup \mathcal{G}') \;=\; \begin{cases} \top & \text{if } \mathcal{M}(z_{s'_1}, a, \bigcup\{C_j\}) = \top \wedge r_j = s \text{ for some } j \in I \\ \bot & \text{if } \mathcal{M}(z_{s'_1}, a, \bigcup\{C_i\}) = \bot \vee r_i \neq s \text{ for all } i \in I \end{cases}$$
  we obtain:
  $$\mathcal{M}(z_{s'_1 \,\|_L\, s}, a, \bigcup \mathcal{G}') \;=\; \mathcal{M}(z_{s'_2 \,\|_L\, s}, a, \bigcup \mathcal{G}')$$
  by taking $\mathcal{Z}'_{2,\mathcal{G}_j}$ in the first case or any $\mathcal{Z}'_{2,\mathcal{G}_i}$ in the second case.

  - If $s'_1 \,\|_L\, s \xrightarrow{a} \Delta'_1$ is originated from $s \xrightarrow{a} \Delta$, with $\Delta'_1 = \delta_{s'_1} \otimes \Delta$ and $\mathcal{Z} \in 1\text{-}Res(s)$ being the 1-resolution selecting the latter transition, then $s'_2 \,\|_L\, s \xrightarrow{a} \Delta'_2$, with $\Delta'_2 = \delta_{s'_2} \otimes \Delta$ and $\mathcal{Z}'_2 \in 1\text{-}Res(s'_2 \,\|_L\, s)$ being the 1-resolution selecting that transition. Given an arbitrary $\mathcal{G}' \in 2^{(S \,\|_L\, S)/\mathcal{B}'}$, which is of the form $\{C_i \,\|_L\, r_i \mid C_i \in S/\mathcal{B}, r_i \in S, i \in I\}$, it holds that:
  $$\begin{aligned} \mathcal{M}(z_{s'_1 \,\|_L\, s}, a, \bigcup \mathcal{G}') \;&=\; \bigvee_{r \in S}^{\exists i \in I. s'_1 \in C_i \wedge r_i = r} \mathcal{M}(z_s, a, \{r\}) \;= \\ &=\; \bigvee_{r \in S}^{\exists i \in I. s'_1 \in C_i \wedge r_i = r} \Delta(r) \;= \\ &=\; \bigvee_{r \in S}^{\exists i \in I. s'_2 \in C_i \wedge r_i = r} \Delta(r) \;= \\ &=\; \bigvee_{r \in S}^{\exists i \in I. s'_2 \in C_i \wedge r_i = r} \mathcal{M}(z_s, a, \{r\}) \;=\; \mathcal{M}(z_{s'_2 \,\|_L\, s}, a, \bigcup \mathcal{G}') \end{aligned}$$
  where $s'_1 \in C_i$ iff $s'_2 \in C_i$ because $(s'_1, s'_2) \in \mathcal{B}$ and finitely many $\mathbb{B}$-values different from $\bot$ occur in all disjunctions because Δ has finite support.

- Assume that $a \in L$. Then $s'_1 \,\|_L\, s \xrightarrow{a} \Delta'_1$ is originated from $s'_1 \xrightarrow{a} \Delta_1$, with $\mathcal{Z}_1 \in 1\text{-}Res(s'_1)$ being the 1-resolution selecting that transition, and $s \xrightarrow{a} \Delta$, with $\mathcal{Z} \in 1\text{-}Res(s)$ being the 1-resolution selecting that transition, where $\Delta'_1 = \Delta_1 \otimes \Delta$. From $(s'_1, s'_2) \in \mathcal{B}$, it follows that for all $\mathcal{G} \in 2^{S/\mathcal{B}}$ there exists $s'_2 \xrightarrow{a} \Delta_{2,\mathcal{G}}$, with $\mathcal{Z}_{2,\mathcal{G}} \in 1\text{-}Res(s'_2)$ being the 1-resolution selecting that transition, such that:
$$\mathcal{M}(z_{s'_1}, a, \bigcup \mathcal{G}) \;=\; \bigvee_{s' \in \cup \mathcal{G}} \Delta_1(s') \;=\; \bigvee_{s' \in \cup \mathcal{G}} \Delta_{2,\mathcal{G}}(s') \;=\; \mathcal{M}(z_{s'_2}, a, \bigcup \mathcal{G})$$
where finitely many $\mathbb{B}$-values different from $\bot$ occur in both disjunctions because Δ_1 and $\Delta_{2,\mathcal{G}}$ have finite support. As a consequence there exist transitions of the form $s'_2 \,\|_L\, s \xrightarrow{a} \Delta'_{2,\mathcal{G}}$, with $\Delta'_{2,\mathcal{G}} = \Delta_{2,\mathcal{G}} \otimes \Delta$ and $\mathcal{Z}'_{2,\mathcal{G}} \in 1\text{-}Res(s'_2 \,\|_L\, s)$ being the 1-resolution selecting a transition of that form. Given an arbitrary $\mathcal{G}' \in 2^{(S \,\|_L\, S)/\mathcal{B}'}$, which is of the form $\{C_i \,\|_L\, r_i \mid C_i \in S/\mathcal{B}, r_i \in S, i \in I\}$, since:
$$\mathcal{M}(z_{s'_1 \,\|_L\, s}, a, \bigcup \mathcal{G}') = \begin{cases} \top & \text{if } \mathcal{M}(z_{s'_1}, a, \bigcup\{C_j\}) \wedge \mathcal{M}(z_s, a, \{r_j\}) = \top \text{ for some } j \in I \\ \bot & \text{if } \mathcal{M}(z_{s'_1}, a, \bigcup\{C_i\}) \wedge \mathcal{M}(z_s, a, \{r_i\}) = \bot \text{ for all } i \in I \end{cases}$$
we obtain:
$$\mathcal{M}(z_{s'_1 \,\|_L\, s}, a, \bigcup \mathcal{G}') \;=\; \mathcal{M}(z_{s'_2 \,\|_L\, s}, a, \bigcup \mathcal{G}')$$
by taking $\mathcal{Z}'_{2,\mathcal{G}_j}$ in the first case or any $\mathcal{Z}'_{2,\mathcal{G}_i}$ in the second case. ■

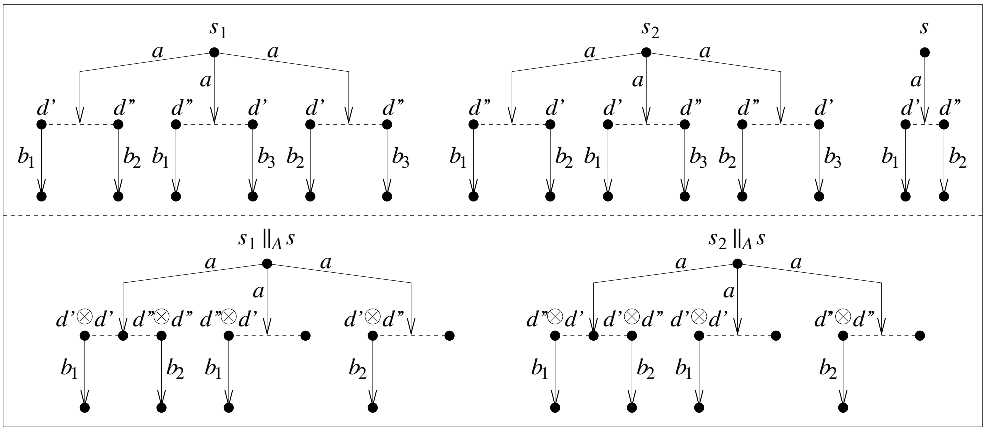

Figure 2: $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ is not compositional when $|D| > 2$ and there is internal nondeterminism

In all the other cases, i.e., when $|D| > 2$ and we are in the presence of internal nondeterminism, the relation $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ is no longer guaranteed to be a congruence with respect to parallel composition. Consider for instance the first two D-ULTraS models in the upper part of Fig. 2, where $d', d'' \in D$ are such that $d' \neq d''$ and $d' \neq 0_D \neq d''$ (these values exist because $|D| > 2$). Given a D-measure schema $\mathcal{M}$, it holds that $s_1 \sim^{\text{pre}}_{\text{B},\mathcal{M}} s_2$ (while $s_1 \not\sim^{\text{post}}_{\text{B},\mathcal{M}} s_2$). However, if we take into account the last D-ULTraS in the upper part, we obtain the two D-ULTraS models in the lower part of Fig. 2, which show that $s_1 \,\|_A\, s \not\sim^{\text{pre}}_{\text{B},\mathcal{M}} s_2 \,\|_A\, s$. The reason is that, when examining the set of equivalence classes whose states can perform b_1 or b_2, the leftmost a-transition of $s_1 \,\|_A\, s$ is not matched by any a-transition of $s_2 \,\|_A\, s$ whenever we have that $(d' \otimes d') \oplus (d'' \otimes d'') \notin \{(d'' \otimes d') \oplus (d' \otimes d''),\, d' \otimes d',\, d'' \otimes d''\}$.

We conclude by observing that a *coarsest congruence metaresult* relating $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ and $\sim^{\text{pre}}_{\text{B},\mathcal{M}}$ can be established whenever the reachability-consistent semiring $(D, \oplus, \otimes, 0_D, 1_D)$ is a *field*, which means that the inverse operations with respect to $\oplus$ and $\otimes$ exist:

- $d \ominus d = d \oplus inv_{\oplus}(d) = inv_{\oplus}(d) \oplus d = 0_D$ for all $d \in D$.
- $d \oslash d = d \otimes inv_{\otimes}(d) = inv_{\otimes}(d) \otimes d = 1_D$ for all $d \in D \setminus \{0_D\}$.

Examples of fields are $(\mathbb{Q}, +, \times)$, $(\mathbb{R}, +, \times)$, and $(\mathbb{C}, +, \times)$. The coarsest congruence result holds for ULTraS models that are *image finite* – i.e., in which the number of identically labeled transitions departing from any state is finite – and relies on the fact that transitions have target distributions with *finite support*.

The idea behind the proof is to exploit the algebraic and topological properties of one of the infinitely many *vector spaces* $(\vec{D}, \boxplus, \circledast, \vec{0}_D)$ that can be built on top of the field structure, where:

- $\vec{D}$ is a set of tuples of D-values having the same finite length.
- $\vec{0}_D$ is the tuple whose D-values are all equal to 0_D.
- $\boxplus$ is the natural lifting of $\oplus$ to $\vec{D}$, thus it is associative and commutative and admits neutral element $\vec{0}_D$.
- $\circledast : D \times \vec{D} \to \vec{D}$ satisfies:
  - $d_1 \circledast (d_2 \circledast \vec{d}) = (d_1 \otimes d_2) \circledast \vec{d}$.
  - $1_D \circledast \vec{d} = \vec{d}$.
  - $d \circledast (\vec{d}_1 \boxplus \vec{d}_2) = (d \circledast \vec{d}_1) \boxplus (d \circledast \vec{d}_2)$.
  - $(d_1 \oplus d_2) \circledast \vec{d} = (d_1 \circledast \vec{d}) \boxplus (d_2 \circledast \vec{d})$.

It is worth pointing out that the proof of the metaresult below differs from the one of an analogous result specific to the field $(\mathbb{R}, +, \times)$ contained in [7]. In both proofs, at a certain point a system of inequalities is built, such that each associated equation describes a hyperplane passing through the origin of the considered vector space. A hyperplane in a vector space on $\mathbb{R}$ always separates the vector space into two half spaces, and from this fact it is possible to derive the existence of a nonempty subspace whose points satisfy all the inequalities. However, it is not guaranteed that a hyperplane in a vector space on a generic reachability-consistent field separates the vector space into two half spaces. In this case, the key observation is that the field is infinite because it has *characteristic zero*, hence also the vector space is infinite and, as a consequence, it strictly includes any union of proper subspaces like the previously mentioned hyperplanes.

Theorem 4.10. Let $(D, \oplus, \ominus, \otimes, \oslash, 0_D, 1_D)$ be a reachability-consistent field, $\mathcal{U} = (S, A, \longrightarrow)$ be an image-finite D-ULTRAS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$. Then $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$ iff $s_1 \parallel_L s \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2 \parallel_L s$ for all $L \subseteq A$ and $s \in S$.

Proof If $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$, then for all $L \subseteq A$ and $s \in S$ it holds that $s_1 \parallel_L s \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2 \parallel_L s$ by virtue of Thm. 4.7, and hence $s_1 \parallel_L s \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2 \parallel_L s$ by virtue of Prop. 3.5(1).
As far as the reverse implication is concerned, we reason on the contrapositive. Thus, we show that, if $s_1 \not\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$, then there exist $L \subseteq A$ and $s \in S$ such that $s_1 \parallel_L s \not\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2 \parallel_L s$. We assume that $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2$, otherwise trivially $s_1 \parallel_\emptyset \underline{0} \not\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2 \parallel_\emptyset \underline{0}$ where $\underline{0}$ is a terminal state, i.e., a state without any outgoing transition. From $s_1 \not\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$ and $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2$, it follows that internal nondeterminism is present in $\mathcal{U}$.
Let us observe that the discriminating power of $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}}$ does not change if equivalence classes are considered instead of sets of equivalence classes. From $s_1 \not\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} s_2$, it follows that there exists a transition $s_1 \xrightarrow{a} \Delta_1$ – with $\mathcal{Z}_1 \in 1\text{-}\mathit{Res}(s_1)$ being the 1-resolution selecting that transition – such that for each transition $s_2 \xrightarrow{a} \Delta_{2,j}$ – with $\mathcal{Z}_{2,j} \in 1\text{-}\mathit{Res}(s_2)$ being the 1-resolution selecting that transition – there exists an equivalence class $C \in S/\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}}$ for which it holds that:

$$\mathcal{M}(z_{s_1}, a, C) \; = \; \bigoplus_{s' \in C} \Delta_1(s') \; = \; \Delta_1(C) \; \neq \; \Delta_{2,j}(C) \; = \; \bigoplus_{s' \in C} \Delta_{2,j}(s') \; = \; \mathcal{M}(z_{s_2,j}, a, C)$$

where finitely many D-values different from 0_D occur in both summations because Δ_1 and $\Delta_{2,j}$ have finite support. Since $s_1 \xrightarrow{a} \Delta_1$ and $s_1 \sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} s_2$, state s_2 must have several outgoing a-transitions – which is consistent with the fact that internal nondeterminism is present in $\mathcal{U}$ – otherwise the only a-transition of s_2 should match each a-transition of s_1 with respect to all the equivalence classes, thus contradicting the reasoning above. From the image finiteness of $\mathcal{U}$, it follows that the number of such transitions must be finite, say $m \in \mathbb{N}_{\geq 2}$, so that $1 \leq j \leq m$.
Let us consider the following set of equivalence classes with respect to $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}}$:

$$\mathcal{C} \; = \; \{C \in S/\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}} \mid \Delta_1(C) \neq 0_D\} \; = \; \{C_1, C_2, \ldots, C_n\}$$

which is not empty, because Δ_1 is the target of a transition and hence $\mathit{supp}(\Delta_1) \neq \emptyset$, and finite, because $\mathit{supp}(\Delta_1)$ is finite; thus $n \in \mathbb{N}_{\geq 1}$. Moreover, let s be a state having a single outgoing transition $s \xrightarrow{a} \Delta$, where Δ assigns reachability degree $d_i \in D \setminus \{0_D\}$ to the representative state s_{C_i} of class C_i for each $i = 1, \ldots, n$.
Recalling that the equivalence classes with respect to $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}}$ are unions of equivalence classes with respect to $\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{post}}$ due to Prop. 3.5(1), we focus on:

$$\mathcal{G} \; = \; \{C' \in (S \parallel_A S)/\sim_{\mathrm{B},\mathcal{M}}^{\mathrm{pre}} \mid \exists C \in \mathcal{C}.\, (C \parallel_A C) \subseteq C'\}$$

and prove that the transition $s_1 \parallel_A s \xrightarrow{a} \Delta'_1$ – with $\mathcal{Z}'_1 \in 1\text{-}\mathit{Res}(s_1 \parallel_A s)$ being the 1-resolution selecting that transition – deriving from the synchronization of the transition $s_1 \xrightarrow{a} \Delta_1$ with the transition $s \xrightarrow{a} \Delta$, is such that for each transition $s_2 \parallel_A s \xrightarrow{a} \Delta'_{2,j}$ – with $\mathcal{Z}'_{2,j} \in 1\text{-}\mathit{Res}(s_2 \parallel_A s)$ being the 1-resolution selecting that transition – deriving from the synchronization of one of the m transitions $s_2 \xrightarrow{a} \Delta_{2,j}$ with the transition $s \xrightarrow{a} \Delta$, it holds that:

$$\mathcal{M}(z_{s_1 \parallel_A s}, a, \bigcup \mathcal{G}) \; = \; \bigoplus_{s' \in \cup \mathcal{G}} \Delta'_1(s') \; = \Delta'_1(\bigcup \mathcal{G}) \; \neq \; \Delta'_{2,j}(\bigcup \mathcal{G}) \; = \; \bigoplus_{s' \in \cup \mathcal{G}} \Delta'_{2,j}(s') \; = \; \mathcal{M}(z_{s_2 \parallel_A s, j}, a, \bigcup \mathcal{G})$$

where finitely many D-values different from 0_D occur in both summations because Δ'_1 and $\Delta'_{2,j}$ have finite support. There are two cases:

- If the set of actions labeling the outgoing transitions of s_{C_i} is disjoint from the set of actions labeling the outgoing transitions of $s_{C_{i'}}$ for all $i \neq i'$, then $s_{C_i} \parallel_A s_{C_{i'}}$ is a terminal state not belonging to $\bigcup \mathcal{G}$. In this case, for each of the m transitions $s_2 \parallel_A s \xrightarrow{a} \Delta'_{2,j}$ it holds that $\Delta'_1(\bigcup \mathcal{G}) \neq \Delta'_{2,j}(\bigcup \mathcal{G})$, i.e., $\Delta'_1(\bigcup \mathcal{G}) \ominus \Delta'_{2,j}(\bigcup \mathcal{G}) \neq 0_D$ thanks to the field structure, iff:
$$\left(\bigoplus_{i=1}^{n} (d_i \otimes \Delta_1(C_i))\right) \ominus \left(\bigoplus_{i=1}^{n} (d_i \otimes \Delta_{2,j}(C_i))\right) \;=\; \bigoplus_{i=1}^{n} (d_i \otimes (\Delta_1(C_i) \ominus \Delta_{2,j}(C_i))) \;\neq\; 0_D$$
which yields a system of m inequalities with n unknowns $d_1, d_2, \dots, d_n \in D \setminus \{0_D\}$.
Each of the m associated equations:
$$\bigoplus_{i=1}^{n} (d_i \otimes (\Delta_1(C_i) \ominus \Delta_{2,j}(C_i))) \;=\; 0_D$$
where at least one of the coefficients $\Delta_1(C_i) \ominus \Delta_{2,j}(C_i)$ is different from 0_D as $s_1 \not\sim^{\text{post}}_{\text{B},\mathcal{M}} s_2$ (see the initial part of the proof), describes a hyperplane passing through the origin of the n-dimensional vector space on D, thus it is a proper subspace of the considered vector space. Since the field is reachability consistent, it has characteristic zero and hence it is infinite. It is not difficult to prove that the infinite, n-dimensional vector space on D strictly includes any union of $k \in \mathbb{N}_{\geq 1}$ proper subspaces.
Indeed, this is trivial for $k = 1$, so assume $k > 1$ and suppose that the result holds for all $1 \leq k' \leq k - 1$. By the induction hypothesis, we can choose a vector $\vec{d}$ not contained in the proper subspaces $\vec{D}_1, \vec{D}_2, \dots, \vec{D}_{k-1}$. There are two cases. If $\vec{d}$ is not contained in the proper subspace $\vec{D}_k$ either, then we are done. If $\vec{d} \in \vec{D}_k$, we choose $\vec{d'} \notin \vec{D}_k$, so that $\vec{d'} \neq \vec{d}$. Let V be the span of $\vec{d'} \boxminus \vec{d} \neq \vec{0}_D$, so the translated line $\vec{d} \boxplus V$ passes through both $\vec{d'}$ and $\vec{d}$. Note that $V \cap \vec{D}_h = \{\vec{0}_D\}$ for all $1 \leq h \leq k$, because this intersection is a *proper* subspace of the 1-dimensional V (as V contains both $\vec{d'}$ and $\vec{d}$, at least one of which is not in $\vec{D}_h$). As a consequence, the intersection $(\vec{d} \boxplus V) \cap \vec{D}_h = (\vec{d'} \boxplus V) \cap \vec{D}_h$ is either empty or a single point; it cannot contain two points as the difference would be a nonzero element in $V \cap \vec{D}_h = \{\vec{0}_D\}$. Thus $(\vec{d} \boxplus V) \cap \bigcup_{1 \leq h \leq k} \vec{D}_h = \bigcup_{1 \leq h \leq k} ((\vec{d} \boxplus V) \cap \vec{D}_h)$ is a possibly empty, *finite* union of points. Since V is 1-dimensional over an infinite field and hence $\vec{d} \boxplus V$ is *infinite*, we can find $\vec{d''} \in \vec{d} \boxplus V$ not contained in any $\vec{D}_h$. Therefore $\bigcup_{1 \leq h \leq k} \vec{D}_h$ is strictly included in the original n-dimensional vector space.
Now, a point with coordinates all different from 0_D of the n-dimensional vector space on D satisfying the m inequalities:
$$\bigoplus_{i=1}^{n} (d_i \otimes (\Delta_1(C_i) \ominus \Delta_{2,j}(C_i))) \;\neq\; 0_D$$
should lay in the complement of the union of the m proper subspaces constituted by the hyperplanes described by the associated equations and the n proper subspaces corresponding to the axes $d_i = 0_D$. Since the complement of the union of these finitely many proper subspaces is not empty as a consequence of what we have proved above, the considered point exists.

- If the set of actions labeling the outgoing transitions of s_{C_i} is not disjoint from the set of actions labeling the outgoing transitions of $s_{C_{i'}}$ for some $i \neq i'$, then it is sufficient to decorate each "shared" action labeling a transition from one of those states with the name of the equivalence class with respect to $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ to which the source state of the transition belongs. Observing that two states are related by $\sim^{\text{post}}_{\text{B},\mathcal{M}}$ iff so are their "decorated" versions, we can then proceed as in the previous case. ■

Note that in the theorem above $|D| > 2$ because there is no reachability-consistent field with $|D| = 2$. Indeed, $(\mathbb{B}, \vee, \wedge)$ is reachability consistent but not a field, while $(\mathbb{Z}_2, +_2, \times_2)$ is neither a field – it is only a ring – nor reachability consistent. Moreover, the metaresult trivially holds in the absence of internal nondeterminism because in that case $\sim^{\text{post}}_{\text{B},\mathcal{M}} = \sim^{\text{pre}}_{\text{B},\mathcal{M}}$.

4.4.2. Compositionality of Trace Metaequivalences

While in the case of bisimulation semantics we have established the full compositionality of $\sim^{\rm post}_{\rm B,\mathcal{M}}$, for trace semantics it is $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ that is always – i.e., for every measure schema $\mathcal{M}$ defined over an arbitrary ULTRAS – a congruence with respect to parallel composition. Furthermore, no compositionality metaresult can be established for $\sim^{\rm post}_{\rm T,\mathcal{M}}$, only specific results when it coincides with $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ (see Props. 3.15 to 3.27).

To prove the compositionality of $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ for an ULTRAS $\mathcal{U} = (S, A, \longrightarrow)$ on a reachability-consistent semiring $(D, \oplus, \otimes, 0_D, 1_D)$, we provide an alternative characterization of this metaequivalence based on associating with each state the set of D-traces that it can perform. The additional information attached to every D-trace is the corresponding *degree of executability* in a certain resolution, i.e., the degree of multi-step reachability of the states of the resolution in which one ends up after executing that trace. Notice that a trace may have different degrees of executability in different resolutions.

In the following, for $a \in A$, $d \in D \setminus \{0_D\}$, and $T, T_1, T_2 \subseteq A^* \times (D \setminus \{0_D\})$ we let:

- $a \,.\, T \;=\; \{(a\,\alpha, d') \mid (\alpha, d') \in T\}$.
- $d \otimes T \;=\; \{(\alpha, d \otimes d') \mid (\alpha, d') \in T\}$.
- $T_1 \oplus T_2 \;=\; \{(\alpha, d_1 \oplus d_2) \mid \alpha \text{ occurring in } T_1 \text{ and } T_2, (\alpha, d_1) \in T_1, (\alpha, d_2) \in T_2\}$
  $\cup\; \{(\alpha, d_1) \in T_1 \mid \alpha \text{ occurring only in } T_1\}$
  $\cup\; \{(\alpha, d_2) \in T_2 \mid \alpha \text{ occurring only in } T_2\}$.

Using the notation above, we define the set of D-traces of state $s \in S$ as follows:

$$T_D(s) \;=\; \bigcup_{k \in \mathbb{N}} T_{D,k}(s)$$

where $T_{D,k}(s)$ is the set of D-traces of s having length at most k as recursively defined below:

$$\begin{array}{rcl} T_{D,0}(s) & = & \{(\varepsilon, 1_D)\} \\ T_{D,k+1}(s) & = & \{(\varepsilon, 1_D)\} \cup \displaystyle\bigcup_{s \xrightarrow{a} \Delta} a \,.\, \left(\bigoplus_{s' \in supp(\Delta)} (\Delta(s') \otimes T_{D,k}(s')) \right) \end{array}$$

As we will see shortly, the presence of $(\varepsilon, 1_D)$ in every set makes the construction monotonic.

The alternative characterization, which is a generalization of a result in [5], relies on the observation that, since $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ treats traces individually regardless of the resolutions in which they can be executed, two states turn out to be equivalent according to $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ iff they have the same set of D-traces.

Lemma 4.11. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTRAS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, $(\alpha, d) \in A^* \times (D \setminus \{0_D\})$, and $s, s_1, s_2 \in S$. Then:

1. $T_{D,k}(s) \subseteq T_{D,k+1}(s)$ for all $k \in \mathbb{N}$.
2. $(\alpha, d) \in T_D(s)$ iff there exists $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ such that $\mathcal{M}(z_s, \alpha, Z) = d$.
3. $s_1 \sim^{\rm pre}_{\rm T,\mathcal{M}} s_2$ iff $T_D(s_1) = T_D(s_2)$.

Proof We proceed as follows:

1. To prove that $T_{D,k}(s) \subseteq T_{D,k+1}(s)$ for all $k \in \mathbb{N}$, given $(\alpha, d) \in A^* \times (D \setminus \{0_D\})$ we show that $(\alpha, d) \in T_{D,k}(s)$ implies $(\alpha, d) \in T_{D,k+1}(s)$ for all $k \in \mathbb{N}_{\geq |\alpha|}$ by proceeding by induction on $|\alpha| \in \mathbb{N}$:
   - If $|\alpha| = 0$, i.e., $\alpha = \varepsilon$, then for all $k \in \mathbb{N}$ we have that $(\varepsilon, d) \in T_{D,k}(s)$ iff $d = 1_D$ by definition, from which the result trivially follows.
   - Let $|\alpha| = n + 1$ for some $n \in \mathbb{N}$, with $\alpha = a\,\alpha'$ and $|\alpha'| = n$, and suppose that the result holds for each trace of length n. If $(\alpha, d) \in T_{D,k}(s)$ for some $k \geq n + 1$, then there exists a transition $s \xrightarrow{a} \Delta$ such that:

$$(\alpha', d) \in \bigoplus_{s' \in supp(\Delta)} (\Delta(s') \otimes T_{D,k-1}(s'))$$

For each $s' \in supp(\Delta)$, either α' does not occur in $T_{D,k-1}(s')$, or α' occurs in $T_{D,k-1}(s')$ with some degree of executability $d_{s'} \in D \setminus \{0_D\}$. Denoting with S' the set of states $s' \in supp(\Delta)$ such that α' occurs in $T_{D,k-1}(s')$, we have that:

$$d = \bigoplus_{s' \in S'} (\Delta(s') \otimes d_{s'})$$

Since for all $s' \in S'$ it holds that $(\alpha', d_{s'}) \in T_{D,k-1}(s')$, from the induction hypothesis it follows that $(\alpha', d_{s'}) \in T_{D,k}(s')$. As a consequence:

$$(\alpha', d) \in \bigoplus_{s' \in supp(\Delta)} (\Delta(s') \otimes T_{D,k}(s'))$$

and hence $(\alpha, d) \in T_{D,k+1}(s)$.

2. We prove that $(\alpha, d) \in T_D(s)$ iff there exists $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ such that $\mathcal{M}(z_s, \alpha, Z) = d$ by proceeding by induction on $|\alpha| \in \mathbb{N}$:

   - If $|\alpha| = 0$, i.e., $\alpha = \varepsilon$, then $(\varepsilon, d) \in T_D(s)$ iff $d = 1_D$ by definition. On the other hand, for each $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ it holds that $\mathcal{M}(z_s, \varepsilon, Z) = 1_D$. Therefore, the result holds.
   - Let $|\alpha| = n + 1$ for some $n \in \mathbb{N}$, with $\alpha = a\,\alpha'$ and $|\alpha'| = n$, and suppose that the result holds for each trace of length n. The proof is divided into two parts:
     - If $(\alpha, d) \in T_D(s)$, then $(\alpha, d) \in T_{D,k}(s)$ for some $k \geq n+1$ and hence there exists a transition $s \xrightarrow{a} \Delta$ such that:

       $$(\alpha', d) \in \bigoplus_{s' \in supp(\Delta)} (\Delta(s') \otimes T_{D,k-1}(s'))$$

       For each $s' \in supp(\Delta)$, either α' does not occur in $T_{D,k-1}(s')$, or α' occurs in $T_{D,k-1}(s')$ with some degree of executability $d_{s'} \in D \setminus \{0_D\}$. Denoting with S' the set of states $s' \in supp(\Delta)$ such that α' occurs in $T_{D,k-1}(s')$, we have that:

       $$d = \bigoplus_{s' \in S'} (\Delta(s') \otimes d_{s'})$$

       Since for all $s' \in S'$ it holds that $(\alpha', d_{s'}) \in T_{D,k-1}(s') \subseteq T_D(s')$, from the induction hypothesis it follows that there exists $\mathcal{Z}_{s'} = (Z_{s'}, A, \longrightarrow_{\mathcal{Z}_{s'}}) \in Res(s')$ such that $\mathcal{M}(z_{s'}, \alpha', Z_{s'}) = d_{s'}$. Therefore, if we consider the resolution $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ such that its initial transition $z_s \xrightarrow{a}_{\mathcal{Z}} \Delta'$ corresponds to $s \xrightarrow{a} \Delta$ and, for each $z_{s'} \in supp(\Delta')$, it behaves as $\mathcal{Z}_{s'}$ when $s' \in S'$ whereas it halts when $s' \in supp(\Delta) \setminus S'$, it is easy to see that $\mathcal{M}(z_s, \alpha, Z) = d$.
     - If there exists $\mathcal{Z} = (Z, A, \longrightarrow_{\mathcal{Z}}) \in Res(s)$ such that $\mathcal{M}(z_s, \alpha, Z) = d$, then there exists a transition $z_s \xrightarrow{a}_{\mathcal{Z}} \Delta$ such that:

       $$d = \bigoplus_{z' \in supp(\Delta)} (\Delta(z') \otimes \mathcal{M}(z', \alpha', Z))$$

       For each $z' \in supp(\Delta)$, either α' is not executable from z', or there exists $d_{z'} \in D \setminus \{0_D\}$ such that $d_{z'} = \mathcal{M}(z', \alpha', Z)$. Denoting with Z' the set of states $z' \in supp(\Delta)$ for which there exists $d_{z'} \in D \setminus \{0_D\}$ such that $d_{z'} = \mathcal{M}(z', \alpha', Z)$, we have that:

       $$d = \bigoplus_{z' \in Z'} (\Delta(z') \otimes d_{z'})$$

       Indicating with $corr_{\mathcal{Z}}$ the correspondence function for $\mathcal{Z}$, from the induction hypothesis it follows that $(\alpha', d_{z'}) \in T_D(corr_{\mathcal{Z}}(z'))$ for all $z' \in Z'$, hence $(\alpha, d) \in T_D(corr_{\mathcal{Z}}(z_s))$. The result follows from $corr_{\mathcal{Z}}(z_s) = s$.

3. By virtue of Def. 3.4, we have that $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2$ iff for all $\alpha \in A^*$ it holds that for each $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ – resp. $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ – there exists $\mathcal{Z}_2 = (Z_2, A, \longrightarrow_{\mathcal{Z}_2}) \in Res(s_2)$ – resp. $\mathcal{Z}_1 = (Z_1, A, \longrightarrow_{\mathcal{Z}_1}) \in Res(s_1)$ – such that:

   $$\mathcal{M}(z_{s_1}, \alpha, Z_1) = d = \mathcal{M}(z_{s_2}, \alpha, Z_2)$$

   Suppose that $d \neq 0_D$, because for $d = 0_D$, which implies $\alpha \neq \varepsilon$, the resolution of the challenger can trivially be matched by the resolution of the defender containing only the initial state without any outgoing transition. By virtue of Lemma 4.11(2), we thus have that $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2$ iff for all $(\alpha, d) \in A^* \times (D \setminus \{0_D\})$

it holds that:

$$(\alpha, d) \in T_D(s_1) \iff (\alpha, d) \in T_D(s_2)$$

which means that $T_D(s_1) = T_D(s_2)$. ■

Theorem 4.12. Let $(D, \oplus, \otimes, 0_D, 1_D)$ be a reachability-consistent semiring, $\mathcal{U} = (S, A, \longrightarrow)$ be a D-ULTraS, $\mathcal{M}$ be a D-measure schema for $\mathcal{U}$, and $s_1, s_2 \in S$. If $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2$, then $s_1 \,\|_L\, s \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2 \,\|_L\, s$ for all $L \subseteq A$ and $s \in S$.

Proof As a preliminary step, we extend the notion of parallel composition to the set A^* of traces by introducing $\vdash$ as the smallest relation satisfying the following rules:

$$\frac{}{\varepsilon \,\|_L\, \varepsilon \vdash \varepsilon} \qquad \frac{\alpha_1 \,\|_L\, \alpha_2 \vdash \alpha}{a\,\alpha_1 \,\|_L\, \alpha_2 \vdash a\,\alpha}\; a \notin L \qquad \frac{\alpha_1 \,\|_L\, \alpha_2 \vdash \alpha}{\alpha_1 \,\|_L\, a\,\alpha_2 \vdash a\,\alpha}\; a \notin L \qquad \frac{\alpha_1 \,\|_L\, \alpha_2 \vdash \alpha}{a\,\alpha_1 \,\|_L\, a\,\alpha_2 \vdash a\,\alpha}\; a \in L$$

then we lift the same notion to sets of D-traces by letting:

$$T_1 \,\|_L\, T_2 \;=\; \{(\alpha, d_1 \otimes d_2) \mid (\alpha_1, d_1) \in T_1, (\alpha_2, d_2) \in T_2, \alpha_1 \,\|_L\, \alpha_2 \vdash \alpha\}$$

If $s_1 \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2$, then from Lemma 4.11(3) it follows that $T_D(s_1) = T_D(s_2)$. Given $L \subseteq A$ and $s \in S$, assume that $T_D(s_i \,\|_L\, s) = T_D(s_i) \,\|_L\, T_D(s)$ for $i = 1, 2$. Then we would have:

$$T_D(s_1 \,\|_L\, s) \;=\; T_D(s_1) \,\|_L\, T_D(s) \;=\; T_D(s_2) \,\|_L\, T_D(s) \;=\; T_D(s_2 \,\|_L\, s)$$

and hence $s_1 \,\|_L\, s \sim_{\mathrm{T},\mathcal{M}}^{\mathrm{pre}} s_2 \,\|_L\, s$ by virtue of Lemma 4.11(3).

It remains to prove the assumption above. To this purpose, we now show that for all $(\alpha, d) \in A^* \times (D \setminus \{0_D\})$ and $r \in S$ it holds that $(\alpha, d) \in T_{D,k}(r \,\|_L\, s)$ for some $k \in \mathbb{N}$ iff $(\alpha, d) \in T_{D,k'}(r) \,\|_L\, T_{D,k''}(s)$ for some $k', k'' \in \mathbb{N}$ such that $k', k'' \leq k$ by proceeding by induction on $|\alpha| \in \mathbb{N}$:

- If $|\alpha| = 0$, i.e., $\alpha = \varepsilon$, then $d = 1_D$ by definition and the result immediately follows from:

  $$T_{D,0}(r \,\|_L\, s) \;=\; \{(\varepsilon, 1_D)\} \;=\; \{(\varepsilon, 1_D)\} \,\|_L \{(\varepsilon, 1_D)\} \;=\; T_{D,0}(r) \,\|_L\, T_{D,0}(s)$$

  and from the fact that the D-trace $(\varepsilon, 1_D)$ occurs in every $T_{D,k}$ set.

- Let $|\alpha| = n + 1$ for some $n \in \mathbb{N}$, with $\alpha = a\,\alpha'$ and $|\alpha'| = n$, and suppose that the result holds for each trace of length n. The fact that $(\alpha, d) \in T_{D,n+1}(r \,\|_L\, s)$ means that there exists a transition $r \,\|_L\, s \xrightarrow{a} \Delta$ such that:

  $$(\alpha', d) \;\in \bigoplus_{r' \,\|_L\, s' \in supp(\Delta)} (\Delta(r' \,\|_L\, s') \otimes T_{D,n}(r' \,\|_L\, s'))$$

  There are two cases:

  - If $a \notin L$, then $r \,\|_L\, s \xrightarrow{a} \Delta$ is originated from either $r \xrightarrow{a} \Delta'$ with $\Delta = \Delta' \otimes \delta_s$, or $s \xrightarrow{a} \Delta''$ with $\Delta = \delta_r \otimes \Delta''$, so that:

    $$(\alpha', d) \;\in\; \left(\bigoplus_{r' \in supp(\Delta')} (\Delta'(r') \otimes T_{D,n}(r' \,\|_L\, s))\right) \bigcup \left(\bigoplus_{s' \in supp(\Delta'')} (\Delta''(s') \otimes T_{D,n}(r \,\|_L\, s'))\right)$$

    For each $r' \in supp(\Delta')$, either α' does not occur in $T_{D,n}(r' \,\|_L\, s)$, or α' occurs in $T_{D,n}(r' \,\|_L\, s)$ with some degree of executability $d_{r' \,\|_L\, s} \in D \setminus \{0_D\}$, with the various $\Delta'(r') \otimes d_{r' \,\|_L\, s}$ summing up to d. Likewise, for each $s' \in supp(\Delta'')$, either α' does not occur in $T_{D,n}(r \,\|_L\, s')$, or α' occurs in $T_{D,n}(r \,\|_L\, s')$ with some degree of executability $d_{r \,\|_L\, s'} \in D \setminus \{0_D\}$, with the various $\Delta''(s') \otimes d_{r \,\|_L\, s'}$ summing up to d. By applying the induction hypothesis to all $r' \in supp(\Delta')$ for which $(\alpha', d_{r' \,\|_L\, s}) \in T_{D,n}(r' \,\|_L\, s)$ and to all $s' \in supp(\Delta'')$ for which $(\alpha', d_{r \,\|_L\, s'}) \in T_{D,n}(r \,\|_L\, s')$, and exploiting Lemma 4.11(1) so to obtain a single pair $n', n'' \leq n$ from the various pairs $n'_{r'}, n''_s \leq n$ and $n'_r, n''_{s'} \leq n$, it follows that $(\alpha, d) \in T_{D,n+1}(r \,\|_L\, s)$ means that there exist $n' + 1, n'' + 1 \leq n + 1$ such that:

$$
\begin{aligned}
(\alpha, d) \ \in \ & \left(a \, . \left(\bigoplus_{r' \in supp(\Delta')} \left(\Delta'(r') \otimes \left(T_{D,n'}(r') \parallel_L T_{D,n''}(s)\right)\right)\right)\right) \cup \\
& \left(a \, . \left(\bigoplus_{s' \in supp(\Delta'')} \left(\Delta''(s') \otimes \left(T_{D,n'}(r) \parallel_L T_{D,n''}(s')\right)\right)\right)\right) \\
= \ & \left(a \, . \left(\left(\bigoplus_{r' \in supp(\Delta')} (\Delta'(r') \otimes T_{D,n'}(r'))\right) \parallel_L T_{D,n''}(s)\right)\right) \cup \\
& \left(a \, . \left(T_{D,n'}(r) \parallel_L \left(\bigoplus_{s' \in supp(\Delta'')} (\Delta''(s') \otimes T_{D,n''}(s'))\right)\right)\right) \\
= \ & \left(\left(a \, . \left(\bigoplus_{r' \in supp(\Delta')} (\Delta'(r') \otimes T_{D,n'}(r'))\right)\right) \parallel_L T_{D,n''}(s)\right) \cup \\
& \left(T_{D,n'}(r) \parallel_L \left(a \, . \left(\bigoplus_{s' \in supp(\Delta'')} (\Delta''(s') \otimes T_{D,n''}(s'))\right)\right)\right) \\
\subseteq \ & \left(T_{D,n'+1}(r) \parallel_L T_{D,n''}(s)\right) \cup \left(T_{D,n'}(r) \parallel_L T_{D,n''+1}(s)\right) \\
\subseteq \ & \left(T_{D,n'+1}(r) \parallel_L T_{D,n''+1}(s)\right) \cup \left(T_{D,n'+1}(r) \parallel_L T_{D,n''+1}(s)\right) \\
= \ & T_{D,n'+1}(r) \parallel_L T_{D,n''+1}(s)
\end{aligned}
$$

where at the end we have exploited again Lemma 4.11(1) so to uniformize to $n' + 1$ and $n'' + 1$.

– If $a \in L$, then $r \parallel_L s \xrightarrow{a} \Delta$ is originated from $r \xrightarrow{a} \Delta'$ and $s \xrightarrow{a} \Delta''$, where $\Delta = \Delta' \otimes \Delta''$, so that:

$$
(\alpha', d) \ \in \ \bigoplus_{r' \in supp(\Delta')} \ \bigoplus_{s' \in supp(\Delta'')} (\Delta'(r') \otimes \Delta''(s') \otimes T_{D,n}(r' \parallel_L s'))
$$

For each $r' \in supp(\Delta')$ and $s' \in supp(\Delta'')$, either α' does not occur in $T_{D,n}(r' \parallel_L s')$, or α' occurs in $T_{D,n}(r' \parallel_L s')$ with some degree of executability $d_{r' \parallel_L s'} \in D \setminus \{0_D\}$, with the various $\Delta'(r') \otimes \Delta''(s') \otimes d_{r' \parallel_L s'}$ summing up to d. By applying the induction hypothesis to all $r' \in supp(\Delta')$ and $s' \in supp(\Delta'')$ for which $(\alpha, d_{r' \parallel_L s'}) \in T_{D,n}(r' \parallel_L s')$, and exploiting Lemma 4.11(1) so to obtain a single pair $n', n'' \leq n$ from the various pairs $n'_{r'}, n''_{s'} \leq n$, it follows that $(\alpha, d) \in T_{D,n+1}(r \parallel_L s)$ means that there exist $n' + 1, n'' + 1 \leq n + 1$ such that:

$$
\begin{aligned}
(\alpha, d) \ \in \ & a \, . \left(\bigoplus_{r' \in supp(\Delta')} \ \bigoplus_{s' \in supp(\Delta'')} \left(\Delta'(r') \otimes \Delta''(s') \otimes \left(T_{D,n'}(r') \parallel_L T_{D,n''}(s')\right)\right)\right) \\
= \ & a \, . \left(\bigoplus_{r' \in supp(\Delta')} \ \bigoplus_{s' \in supp(\Delta'')} \left((\Delta'(r') \otimes T_{D,n'}(r')) \parallel_L (\Delta''(s') \otimes T_{D,n''}(s'))\right)\right) \\
= \ & a \, . \left(\left(\bigoplus_{r' \in supp(\Delta')} (\Delta'(r') \otimes T_{D,n'}(r'))\right) \parallel_L \left(\bigoplus_{s' \in supp(\Delta'')} (\Delta''(s') \otimes T_{D,n''}(s'))\right)\right) \\
= \ & \left(a \, . \left(\bigoplus_{r' \in supp(\Delta')} (\Delta'(r') \otimes T_{D,n'}(r'))\right)\right) \parallel_L \left(a \, . \left(\bigoplus_{s' \in supp(\Delta'')} (\Delta''(s') \otimes T_{D,n''}(s'))\right)\right) \\
\subseteq \ & T_{D,n'+1}(r) \parallel_L T_{D,n''+1}(s)
\end{aligned}
$$

■

The metaresult above is not in disagreement with [25], where trace equivalence over fully probabilistic processes was shown not to be a congruence with respect to restriction, an operator which we could recast in terms of parallel composition with a terminal state by enforcing synchronization on all the actions that have to be restricted. The violation of compositionality in [25] is a consequence of the fact that probability subdistributions are not admitted in that model, hence in the case of restriction it is necessary to normalize

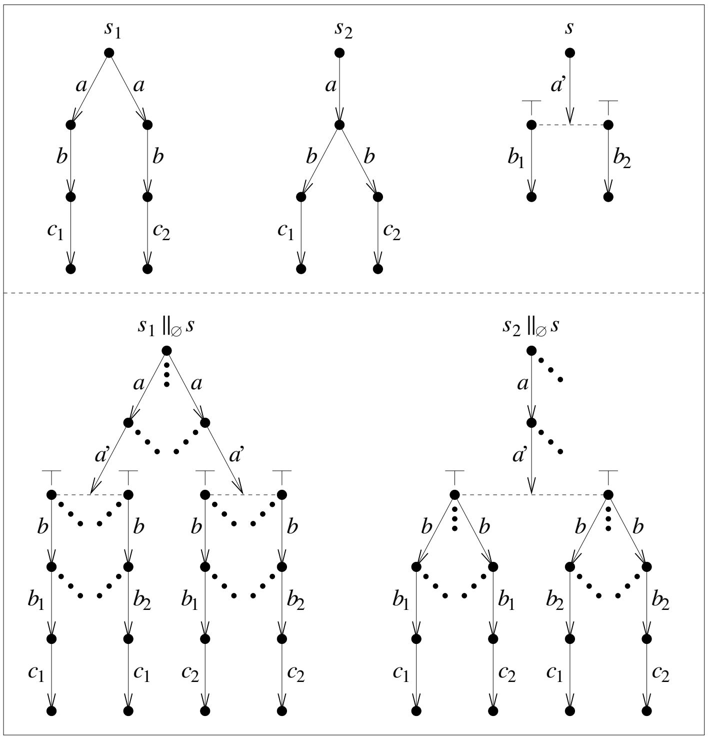

Figure 3: $\sim^{\rm post}_{\rm T,\mathcal{M}}$ is not compositional

residual probabilities.

It remains to investigate the compositionality of $\sim^{\rm post}_{\rm T,\mathcal{M}}$. Even with the simplest reachability-consistent semiring, i.e., $(\mathbb{B}, \vee, \wedge)$, it turns out that $\sim^{\rm post}_{\rm T,\mathcal{M}}$ is not a congruence with respect to parallel composition. Consider for instance the first two $\mathbb{B}$-ULTRAS models in the upper part of Fig. 3, where $s_1 \sim^{\rm post}_{\rm T,\mathcal{M}_{\rm nd}} s_2$. If we take into account the last $\mathbb{B}$-ULTRAS in the upper part, we obtain $s_1 \parallel_\emptyset s \not\sim^{\rm post}_{\rm T,\mathcal{M}_{\rm nd}} s_2 \parallel_\emptyset s$. This is witnessed by the maximal resolutions of $s_1 \parallel_\emptyset s$ and $s_2 \parallel_\emptyset s$ that start with trace $a\,a'$ and then continue with one of the traces in $\{b\,b_1\,c_1, b\,b_1\,c_2, b\,b_2\,c_1, b\,b_2\,c_2\}$, where $s_1 \parallel_\emptyset s$ and $s_2 \parallel_\emptyset s$ are in the lower part of Fig. 3 (dots stands for transitions that are not shown). As an example, the maximal resolution of $s_2 \parallel_\emptyset s$ whose associated set of maximal traces is $\{a\,a'\,b\,b_1\,c_1, a\,a'\,b\,b_2\,c_2\}$ is not matched under $\sim^{\rm post}_{\rm T,\mathcal{M}_{\rm nd}}$ by any maximal resolution of $s_1 \parallel_\emptyset s$. We observe that this counterexample is not in contrast with Prop. 3.15, because the last $\mathbb{B}$-ULTRAS in the upper part of Fig. 3 cannot be the canonical representation of any labeled transition system.

It is worth noting that no coarsest congruence metaresult relating $\sim^{\rm post}_{\rm T,\mathcal{M}}$ and $\sim^{\rm pre}_{\rm T,\mathcal{M}}$ can be established. The reason is that, unlike bisimulation semantics, for trace semantics it is the pre-metaequivalence that is compositional, and the post-metaequivalence is contained in it.

5. Conclusion

In this paper, we have extended the ULTRAS metamodel of [4] with the notion of resolution borrowed from testing theories for nondeterministic and probabilistic processes [42, 24, 23, 13]. On the one hand, introducing resolutions as a first-class concept has reconciled the ULTRAS metamodel with the FUTS metamodel of [12], thanks to the elicitation of a reachability-consistent semiring structure made possible by the deterministic nature of resolutions. These can be viewed as specializations of the WLTS metamodel of [27] having a tree-like structure and admitting at most one transition from any state. On the other hand, it has allowed us to revise the definition of bisimulation and trace metaequivalences in such a way that, for nondeterministic and probabilistic processes, the new equivalences arosen in [4] are kept and, in addition, the widely accepted equivalences of [36] and [35] are captured.

It is worth observing that the ULTraS metamodel is still defined in terms of a preordered set equipped with minimum, as this is enough to express reachability degrees – i.e., how reachable every state is through a given transition – as well as the notion of unreachability. This also opens the way to the development of behavioral metapreorders, an issue that is outside the scope of this paper. As a matter of fact, the semiring structure is only instrumental to the definition of behavioral metaequivalences, with the additive operation being necessary for bisimulation semantics and both operations being necessary for trace semantics. Based on the comparison at the end of Sect. 3.3 with bisimulation metaequivalences defined over other metamodels appeared in the literature, we can also affirm that our behavioral metaequivalences enables the impact of internal nondeterminism to be fully appreciated.

We have then investigated the congruence property of the four newly defined metaequivalences with respect to generalizations of typical process algebraic operators such as action prefix, guarded choice, nondeterministic choice, and parallel composition. This last operator has emphasized a foundational difference in the compositionality of bisimulation and trace semantics, which had recently emerged in the specific setting of probabilistic and nondeterministic processes [7, 5]. For bisimulation semantics, only the post-metaequivalence is always compositional, while the compositionality of the pre-metaequivalence may depend on the presence of internal nondeterminism in the model; a coarsest congruence metaresult relates the two metaequivalences under certain conditions. In contrast, for trace semantics, it is only the pre-metaequivalence that is always compositional, while no compositionality metaresult can be established for the post-metaequivalence.

As far as future work is concerned, we would like to further extend compositionality metaresults for bisimulation and trace pre-/post-metaequivalences along two directions. Firstly, we intend to address different forms of parallel composition – which may be binary as in CCS [21] rather than multiway as in CSP [9] or, in the case $a \notin L$, may be governed by rules with premises $s_1 \xrightarrow{a} \Delta_1$ and $s_2 \xrightarrow{a} \Delta_2$ and conclusion $s_1 \parallel_L s_2 \xrightarrow{a} (\Delta_1 \otimes \delta_{s_2}) \oplus (\delta_{s_1} \otimes \Delta_2)$ as in [12] – as well as more general forms of parallel composition – which, following [20, 12], allow normalizing factors or operations different from the semiring ones in the calculation of the target of a synchronization. Secondly, by exploiting more recent results in the field of operational semantic rule formats such as those in [31], we hope to prove compositionality metaresults for the four considered metaequivalences in a way that abstracts from specific process algebraic operators. Finally, we plan to keep putting ULTraS at work in search for other general results, such as equational and logical characterizations for bisimulation and trace semantics.

Acknowledgment: We are grateful to the anonymous referees for their constructive comments that contributed to improve the quality of the paper. Special thanks go to Devis Abriani for his help with vector spaces in the proof of the coarsest congruence metaresult for bisimulation metaequivalences. This work has been funded by MIUR-PRIN project CINA.

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