A Theory of Testing for Markovian Processes

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Abstract. We present a testing theory for Markovian processes in order to formalize a notion of efficiency which may be useful for the analysis of soft real time systems. Our Markovian testing theory is shown to enjoy close connections with the classical testing theory of De Nicola-Hennessy and the probabilistic testing theory of Cleaveland-Smolka et al. The Markovian testing equivalence is also shown to be coarser than the Markovian bisimulation equivalence. A fully abstract characterization is developed to ease the task of establishing testing related relationships between Markovian processes. It is then demonstrated that our Markovian testing equivalence, which is based on the (easier to work with) probability of executing a successful computation whose average duration is not greater than a given amount of time, coincides with the Markovian testing equivalence based on the (more intuitive) probability of reaching success within a given amount of time. Finally, it is shown that it is not possible to define a Markovian preorder which is consistent with reward based performance measures, thus justifying why a generic notion of efficiency has been considered.

1 Introduction

Markovian process algebras (see, e.g., [8, 7, 1, 6]) provide a linguistic means to formally model and analyze performance characteristics of concurrent systems in the early stages of their design. Since timing aspects are described by means of exponentially distributed durations, the memoryless property of exponential distributions is exploited to define the semantics for these calculi in the classical interleaving style. Moreover, performance measures can be effectively computed on Markov chains derived by applying semantic rules to process terms.

Markovian process algebras are equipped with a notion of equivalence which relates terms possessing the same functional and performance properties. Markovian bisimulation equivalence, originally proposed in [8, 7] and then further elaborated on in [1, 6], constitutes a useful semantic theory as it has been proved to be the coarsest congruence contained in the intersection of a purely functional bisimulation equivalence and a purely performance bisimulation equivalence, to have a clear relationship with the aggregation technique for Markov chains known as ordinary lumping, and to have a sound and complete axiomatization.

From the performance standpoint, Markovian bisimulation equivalence relates terms having the same transient and steady state behavior. However, it would be useful to develop a notion of Markovian preorder which orders functionally equivalent terms according to their performance. As recognized in [9], an application of such a Markovian preorder would be the approximation of a term with another term whose underlying Markov chain is amenable to efficient solution, in the case in which the original term cannot be replaced with a suitable equivalent term. This would allow bounds on the performance of the original term to be efficiently derived.

In this paper, we extend the probabilistic testing framework of [4] by presenting a testing theory for Markovian processes based on a generic notion of efficiency (Sect. 2). Both Markovian processes and tests will be formally described in the Markovian process algebra EMPA [1]. Since EMPA allows for both exponentially timed actions and prioritized weighted immediate actions, Markovian testing preorders can be developed for continuous time processes, discrete time processes, and mixed processes where the duration of a transition can be either exponentially distributed or zero (Sect. 3).

The Markovian testing theory is shown to be a refinement of the classical testing theory of [5]: whenever a process passes a test with probability 1 (greater than 0) within some amount of time, then the process must (may) pass the test in the classical theory. The Markovian testing theory is also shown to be a refinement of the probabilistic testing theory of [4]. Moreover, the Markovian testing equivalence is proved to be coarser than the Markovian bisimulation equivalence. Since verifying that one process is related to another one requires a consideration of the behavior of both processes in the context of all possible tests, following [13] we also present a fully abstract characterization of the Markovian preorder based on extended traces. From such an alternative characterization, we derive a proof technique that simplifies the task of establishing preorder relationships between Markovian processes (Sect. 4).

The quantification of the probability that tests are passed within a given amount of time on which our Markovian testing theory relies is the probability of executing a successful computation whose average duration is not greater than a given amount of time. Such a quantification is relatively easy to work with. A more intuitive approach would quantify over the probability of reaching success within a given period of time. Such an approach is more difficult to treat formally because it requires handling nonexponential distributions. However, we show that the Markovian testing equivalences induced by the two different quantifications coincide, thus justifying why the testing theory has been developed for the less intuitive, but easier to work with, quantification (Sect. 5).

The paper concludes with some counterexamples which show that it is not possible to define a Markovian preorder that can be used to order processes according to reward based performance measures. This justifies the fact that a generic notion of efficiency has been considered (Sect. 6).

Due to space limitations, we present our testing theory only for continuous time Markovian processes. The interested reader is referred to [2] for discrete time and mixed Markovian processes, proofs of results, and comparisons with other efficiency preorders.

2 Efficiency Preorders for Concurrent Processes

The concept of efficiency for concurrent processes can be characterized both in terms of probability and in terms of time. A process is more efficient than another one if it is able to perform certain computations with a greater probability or by taking a lesser time. In our Markovian framework, both probabilistic and timing aspects are present, so we should set up a definition of efficiency which takes both into account.

Three types of Markovian processes can be identified: continuous time, discrete time, and mixed. They can be viewed as state transition graphs with initial state probabilities labeling states and execution rates or probabilities labeling transitions. Due to lack of space, in the following we consider only continuous time Markovian processes. In such a case, the duration of a transition is described by an exponentially distributed random variable X whose probability distribution function is $F_X(t) = 1 - e^{-\lambda \cdot t}$ with $\lambda \in \mathbb{R}_+$ called the rate and used as label of the transition. The execution probability of a transition turns out to be the ratio of its rate to the sum of the rates of the transitions having the same source state. According to the race policy, the sojourn time in a state is exponentially distributed as well with expected value equal to the reciprocal of the sum of the rates of the outgoing transitions. The average duration of a computation for a continuous time Markovian process is the sum of the average sojourn times of the visited states (except the last one), while the execution probability of a computation is the product of the execution probabilities of the involved transitions.

The notion of efficiency for Markovian processes we shall develop in this paper is based on a quantification of the probability with which processes pass tests within a given amount of time. More precisely, given a notion of success related to test passing, the quantification we shall consider is concerned with the probability of executing a successful computation whose average duration is not greater than a given amount of time. Based on such a notion of efficiency and the probabilistic testing theory of [4], in this paper we define a Markovian preorder in such a way that a Markovian process is less than another one if the probability that it executes a successful computation whose average duration is not greater than a given amount of time is less than the probability that the other process executes a successful computation whose average duration is not greater than the same amount of time. For notational conciseness, in the sequel Markovian processes will be formalized as terms in EMPA instead of as state transition graphs.

3 Extended Markovian Process Algebra

In this section we present actions, operators, and semantics for $\mathrm{EMPA}_{\mathrm{ct}}$, the sublanguage of EMPA [1] for continuous time Markovian processes. Each action $<a, \tilde{\lambda}>$ is characterized by its type, a, and its rate, $\tilde{\lambda}$. Its second component indicates the speed at which the action occurs and is used as a concise way to denote the random variable specifying the duration of the action. Based on rates, an action is classified as exponentially timed, if its rate is a positive real number, or passive, if its rate is left unspecified (denoted "$*$"). The set of actions is given by $\mathit{Act}_{\mathrm{ct}} = \mathit{AType} \times \mathit{ARate}_{\mathrm{ct}}$ where AType is the set of types, including τ for invisible actions, and $\mathit{ARate}_{\mathrm{ct}} = \mathbb{R}_+ \cup \{*\}$ is the set of rates.

Let Const be a set of constants (ranged over by A) and let $\mathit{ATRFun} = \{\varphi : \mathit{AType} \longrightarrow \mathit{AType} \mid \varphi^{-1}(\tau) = \{\tau\}\}$ be a set of action type relabeling functions.

Definition 1. *The set $\mathcal{L}_{\mathrm{ct}}$ of process terms of $\mathrm{EMPA}_{\mathrm{ct}}$ is generated by the following syntax*

$$E ::= \underline{0} \mid <a, \tilde{\lambda}>.E \mid E/L \mid E[\varphi] \mid E + E \mid E \parallel_S E \mid A$$

where $L, S \subseteq \mathit{AType} - \{\tau\}$. Constant defining equations are written $A \stackrel{\Delta}{=} E$. We denote by $\mathcal{G}_{\mathrm{ct}}$ the set of closed and guarded terms. ■

Term $\underline{0}$ cannot execute any action. Term $<a, \tilde{\lambda}>.E$ can execute action $<a, \tilde{\lambda}>$ and then behaves as E. Term E/L behaves as term E except that the type of each executed action is turned into τ whenever it belongs to L. Term $E[\varphi]$ behaves as term E except that the type a of each executed action becomes $\varphi(a)$. Term $E_1 + E_2$ behaves as either term E_1 or term E_2 depending on whether an action of E_1 or an action of E_2 is executed first. If the involved actions are exponentially timed, then the choice is resolved according to the race policy: the action sampling the least duration succeeds. If only passive actions are involved, then the choice is purely nondeterministic. Term $E_1 \parallel_S E_2$ asynchronously executes actions of E_1 or E_2 whose type does not belong to S and synchronously executes actions of E_1 and E_2 whose type belongs to S if at least one of the two actions is passive (the other action determines the rate of the resulting action).

The integrated interleaving semantics for $\mathrm{EMPA}_{\mathrm{ct}}$ is represented by a labeled transition system (LTS for short) whose labels are actions. The two layer definition of the semantics given in Table 1 is needed to correctly handle transition rates without burdening transitions with auxiliary labels. Transition relation $\longrightarrow$, which is the least subset of $\mathcal{G}_{\mathrm{ct}} \times \mathit{Act}_{\mathrm{ct}} \times \mathcal{G}_{\mathrm{ct}}$ satisfying the inference rule in the first part of Table 1, merges together the potential moves having the same action type, the same priority level, and the same derivative term by computing the resulting rate according to the race policy:

$$\begin{aligned}\mathit{Melt}(\mathit{PM}) = \{(<a, \tilde{\lambda}>, E) \mid \exists \tilde{\mu} \in \mathit{ARate}_{\mathrm{ct}}.\, (<a, \tilde{\mu}>, E) \in \mathit{PM} \,\wedge \\ \tilde{\lambda} = \mathit{Aggr}\{\!|\, \tilde{\gamma} \mid (<a, \tilde{\gamma}>, E) \in \mathit{PM} \wedge \mathit{PL}(<a, \tilde{\gamma}>) = \mathit{PL}(<a, \tilde{\mu}>) \,|\!\}\}\end{aligned}$$

where $\mathit{PL}(<a, *>) = -1$, $\mathit{PL}(<a, \lambda>) = 0$, $* \,\mathit{Aggr}\, * = *$, and $\lambda_1 \,\mathit{Aggr}\, \lambda_2 = \lambda_1 + \lambda_2$. The multiset [1] $\mathit{PM}(E) \in \mathcal{M}u_{\mathit{fin}}(\mathit{Act}_{\mathrm{ct}} \times \mathcal{G}_{\mathrm{ct}})$ of potential moves of

[1] We use "$\{\!|$" and "$|\!\}$" as brackets for multisets, "$\_ \oplus \_$" to denote multiset union, $\mathcal{M}u_{\mathit{fin}}(S)$ ($\mathcal{P}_{\mathit{fin}}(S)$) to denote the collection of finite multisets (sets) over set S, and $M(s)$ to denote the multiplicity of element s in multiset M.

$$\frac{(<a, \tilde{\lambda}>, E') \in \mathit{Melt}(\mathit{PM}(E))}{E \xrightarrow{a, \tilde{\lambda}} E'}$$

$$\begin{aligned}
\mathit{PM}(\underline{0}) &= \emptyset \\
\mathit{PM}(<a, \tilde{\lambda}>.E) &= \{\!|\, (<a, \tilde{\lambda}>, E) \,|\!\} \\
\mathit{PM}(E/L) &= \{\!|\, (<a, \tilde{\lambda}>, E'/L) \mid (<a, \tilde{\lambda}>, E') \in \mathit{PM}(E) \wedge a \notin L \,|\!\} \oplus \\
&\quad\ \{\!|\, (<\tau, \tilde{\lambda}>, E'/L) \mid \exists a \in L.\, (<a, \tilde{\lambda}>, E') \in \mathit{PM}(E) \,|\!\} \\
\mathit{PM}(E[\varphi]) &= \{\!|\, (<\varphi(a), \tilde{\lambda}>, E'[\varphi]) \mid (<a, \tilde{\lambda}>, E') \in \mathit{PM}(E) \,|\!\} \\
\mathit{PM}(E_1 + E_2) &= \mathit{PM}(E_1) \oplus \mathit{PM}(E_2) \\
\mathit{PM}(E_1 \parallel_S E_2) &= \{\!|\, (<a, \tilde{\lambda}>, E'_1 \parallel_S E_2) \mid a \notin S \wedge (<a, \tilde{\lambda}>, E'_1) \in \mathit{PM}(E_1) \,|\!\} \oplus \\
&\quad\ \{\!|\, (<a, \tilde{\lambda}>, E_1 \parallel_S E'_2) \mid a \notin S \wedge (<a, \tilde{\lambda}>, E'_2) \in \mathit{PM}(E_2) \,|\!\} \oplus \\
&\quad\ \{\!|\, (<a, \tilde{\gamma}>, E'_1 \parallel_S E'_2) \mid a \in S \wedge \exists \tilde{\lambda}_1, \tilde{\lambda}_2 \in \mathit{ARate}. \\
&\qquad\qquad (<a, \tilde{\lambda}_1>, E'_1) \in \mathit{PM}(E_1) \wedge \\
&\qquad\qquad (<a, \tilde{\lambda}_2>, E'_2) \in \mathit{PM}(E_2) \wedge \\
&\qquad\qquad \tilde{\gamma} = \mathit{Norm}(a, \tilde{\lambda}_1, \tilde{\lambda}_2, \mathit{PM}(E_1), \mathit{PM}(E_2)) \,|\!\} \\
\mathit{PM}(A) &= \mathit{PM}(E) \qquad \text{if } A \stackrel{\Delta}{=} E
\end{aligned}$$

Table 1. $\mathrm{EMPA_{ct}}$ integrated interleaving semantics

$E \in \mathcal{G}_{\mathrm{ct}}$ is defined by structural induction in the second part of Table 1 according to the intuitive meaning of the operators. In the rule for the parallel composition operator a normalization is carried out when computing the rates of the potential moves resulting from the synchronization of the same nonpassive action with several independent or alternative passive actions:

$$\mathit{Norm}(a, \tilde{\lambda}_1, \tilde{\lambda}_2, \mathit{PM}_1, \mathit{PM}_2) = \begin{cases} \mathit{Split}(\tilde{\lambda}_1, 1/(\pi_1(\mathit{PM}_2))(<a, *>)) & \text{if } \tilde{\lambda}_2 = * \\ \mathit{Split}(\tilde{\lambda}_2, 1/(\pi_1(\mathit{PM}_1))(<a, *>)) & \text{if } \tilde{\lambda}_1 = * \end{cases}$$

where $\mathit{Split}(*, p) = *$ and $\mathit{Split}(\lambda, p) = \lambda \cdot p$, while $\pi_1(\mathit{PM})$ denotes the action multiset obtained by projecting the potential moves in PM on their first component. Applying Aggr to the rates of the synchronizations involving the same nonpassive action gives as a result the rate of the nonpassive action itself, thus complying with the bounded capacity assumption.

Definition 2. *The integrated interleaving semantics of* $E \in \mathcal{G}_{\mathrm{ct}}$ *is the LTS* $\mathcal{I}[\![E]\!] = (S_{E,\mathcal{I}}, \mathit{Act}_{\mathrm{ct}}, \longrightarrow_{E,\mathcal{I}}, E)$ *where* $S_{E,\mathcal{I}}$ *is the least subset of* $\mathcal{G}_{\mathrm{ct}}$ *reachable from E via* $\longrightarrow$ *and* $\longrightarrow_{E,\mathcal{I}}$ *is the restriction of* $\longrightarrow$ *to* $S_{E,\mathcal{I}} \times \mathit{Act}_{\mathrm{ct}} \times S_{E,\mathcal{I}}$. *Term* $E \in \mathcal{G}_{\mathrm{ct}}$ *is performance closed iff* $\mathcal{I}[\![E]\!]$ *does not contain passive transitions. We denote by* $\mathcal{E}_{\mathrm{ct}}$ *the set of performance closed terms of* $\mathcal{G}_{\mathrm{ct}}$. ■

From $\mathcal{I}[\![E]\!]$ we can derive two projected semantic models by simply dropping action rates and action types, respectively. We define below the functional one as it will be used in the following.

Definition 3. *The functional semantics of* $E \in \mathcal{G}_{\rm ct}$ *is the LTS* $\mathcal{F}[\![E]\!] = (S_{E,\mathcal{F}}, \mathit{AType}, \longrightarrow_{E,\mathcal{F}}, E)$ *where* $S_{E,\mathcal{F}} = S_{E,\mathcal{I}}$ *and* $\longrightarrow_{E,\mathcal{F}}$ *is the restriction of* $\longrightarrow_{E,\mathcal{I}}$ *to* $S_{E,\mathcal{F}} \times \mathit{AType} \times S_{E,\mathcal{F}}$. ■

The Markovian bisimulation equivalence for $\mathrm{EMPA}_{\rm ct}$ is defined as follows.

Definition 4. *Let function* $\mathit{Rate} : (\mathcal{G}_{\rm ct} \times \mathit{AType} \times \{-1, 0\} \times \mathcal{P}(\mathcal{G}_{\rm ct})) \multimap \mathit{ARate}_{\rm ct}$ *be defined by*

$$\mathit{Rate}(E, a, l, C) = \mathit{Aggr}\{\!|\, \tilde{\lambda} \mid \exists E' \in C.\, E \xrightarrow{a,\tilde{\lambda}} E' \wedge \mathit{PL}(<a, \tilde{\lambda}>) = l \,|\!\}$$

An equivalence relation $\mathcal{B} \subseteq \mathcal{G}_{\rm ct} \times \mathcal{G}_{\rm ct}$ *is a strong extended Markovian bisimulation (strong EMB) iff, whenever* $(E_1, E_2) \in \mathcal{B}$, *then for all* $a \in \mathit{AType}$, $l \in \{-1, 0\}$, *and equivalence classes* $C \in \mathcal{G}_{\rm ct}/\mathcal{B}$

$$\mathit{Rate}(E_1, a, l, C) = \mathit{Rate}(E_2, a, l, C)$$

The union $\sim_{\rm EMB}$ *of all the strong EMBs is called the strong extended Markovian bisimulation equivalence.* ■

4 A Preorder for Continuous Time Markovian Processes

In this section we develop a testing preorder for continuous time Markovian processes by extending the probabilistic framework of [4]. Our objective is to relate continuous time Markovian processes whose behavior is completely specified from the performance point of view. Such processes never engage in passive actions, although they may have subprocesses that perform passive actions in the context of a synchronization with exponentially timed actions. Formally, continuous time Markovian processes are represented through the set $\mathcal{E}_{\rm ct,ndiv}$ of performance closed terms which are not divergent, i.e. not capable of performing a sequence of infinitely many internal actions. In the testing approach, processes are composed in parallel with tests forcing the synchronization over every observable action. Because of the synchronization discipline on action rates adopted in EMPA, tests are naturally expressed as terms composed of observable passive actions, which must synchronize with observable nonpassive actions of the process, and internal nonpassive actions, which introduce arbitrary delays and probabilistic branches. Formally, tests are represented via $\mathcal{T}_{\rm ct}$, the set of closed and guarded terms defined over the syntax of Def. 1 extended with the deadlocked term *success* but restricted to the action set $\mathit{TAct}_{\rm ct} = \{<a, \tilde{\lambda}> \in \mathit{Act}_{\rm ct} \mid (a \in \mathit{AType} - \{\tau\} \wedge \tilde{\lambda} = *) \vee (a = \tau \wedge \tilde{\lambda} \in \mathbb{R}_+)\}$, which are finite state and acyclic (hence divergence free).

Definition 5. *Let* $E \in \mathcal{E}_{\rm ct,ndiv}$ *and* $T \in \mathcal{T}_{\rm ct}$.

- *The interaction system of* E *and* T *is term* $E \,\|_{\mathit{AType}-\{\tau\}}\, T$.
- *A configuration is a state of* $\mathcal{I}[\![E \,\|_{\mathit{AType}-\{\tau\}}\, T]\!]$.
- *A configuration is successful iff its test component is success.*
- *A computation is a maximal sequence*
  $$E \,\|_{\mathit{AType}-\{\tau\}}\, T \equiv s_0 \xrightarrow{a_1,\lambda_1} s_1 \xrightarrow{a_2,\lambda_2} \ldots \xrightarrow{a_n,\lambda_n} s_n$$
  where configuration s_i *is not successful for any* $0 \leq i \leq n-1$.

- *A computation is successful iff so is its last configuration.*
- *We denote by* $\mathcal{C}(E, T)$ *and* $\mathcal{S}(E, T)$ *the set of computations and successful computations, respectively.* ■

Note that the interaction system is a performance closed term. Moreover, since E is divergence free and T is finite state and acyclic, every computation of the interaction system has finite length. In the following, we use c to range over computations and s to range over configurations: $c = s$ denotes a computation composed of a single configuration. Furthermore, $Rate(E, AType, l, C)$ will stand for $Aggr\{\!| \, Rate(E, a, l, C) \mid a \in AType \, |\!\}$.

Definition 6. *Let* $E \in \mathcal{E}_{\mathrm{ct,ndiv}}$, $T \in \mathcal{T}_{\mathrm{ct}}$, $c \in \mathcal{C}(E, T)$, *and* $C \subseteq \mathcal{C}(E, T)$.

- $prob(c) = \begin{cases} 1 & \text{if } c = s \\ \frac{\lambda}{\Lambda} \cdot prob(c') & \text{if } c = s \xrightarrow{a,\lambda} c' \text{ with } \Lambda = Rate(s, AType, 0, \mathcal{E}_{\mathrm{ct}}) \end{cases}$
- $time(c) = \begin{cases} 0 & \text{if } c = s \\ \frac{1}{\Lambda} + time(c') & \text{if } c = s \xrightarrow{a,\lambda} c' \text{ with } \Lambda = Rate(s, AType, 0, \mathcal{E}_{\mathrm{ct}}) \end{cases}$
- $prob(C) = \sum_{c \in C} prob(c)$.
- $C_{\leq t} = \{c \in C \mid time(c) \leq t\}$. ■

Definition 7. *Let* $E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$.

- $E_1 \sqsubseteq_{\mathrm{MT}} E_2$ *iff* $\forall T \in \mathcal{T}_{\mathrm{ct}}. \forall t \in \mathbb{R}_+. \, prob(\mathcal{S}_{\leq t}(E_1, T)) \leq prob(\mathcal{S}_{\leq t}(E_2, T))$.
- $E_1 \sim_{\mathrm{MT}} E_2$ *iff* $E_1 \sqsubseteq_{\mathrm{MT}} E_2 \wedge E_2 \sqsubseteq_{\mathrm{MT}} E_1$. ■

We shall say that $E_1 \sqsubseteq_{\mathrm{MT}} E_2$ ($E_1 \sim_{\mathrm{MT}} E_2$) w.r.t. test T if the related condition in the definition above holds true when $\mathcal{T}_{\mathrm{ct}}$ is restricted to $\{T\}$.

Some remarks are now in order. First of all, note that in the definition of $time(c)$ we consider for each transition of c the average sojourn time of the source state instead of the rate of the transition. In fact, if we consider $E_1 \equiv <a, \lambda>.\underline{0} + <a, \lambda>.\underline{0}$ and $E_2 \equiv <a, 2 \cdot \lambda>.\underline{0}$, then $E_1 \sim_{\mathrm{MT}} E_2$ w.r.t. $T \equiv <a, *>.success$. If we considered rates instead, then we would obtain $prob(\mathcal{S}_{\leq t}(E_1, T)) = 0 < 1 = prob(\mathcal{S}_{\leq t}(E_2, T))$ for $\frac{1}{2 \cdot \lambda} \leq t < \frac{1}{\lambda}$. The reason why the average sojourn times of the traversed states are used instead of the rates of the executed transitions is that durations are described in a stochastic way, not in a deterministic way.

$\sqsubseteq_{\mathrm{MT}}$ and $\sim_{\mathrm{MT}}$ retain the link between the functional part and the performance part of every action, which is a necessary condition to achieve compositionality [1]. If we consider $E_1 \equiv <a, \lambda>.\underline{0} + <b, \mu>.\underline{0}$ and $E_2 \equiv <a, \mu>.\underline{0} + <b, \lambda>.\underline{0}$ where $\lambda < \mu$, then $E_1 \sqsubseteq_{\mathrm{MT}} E_2$ w.r.t. $T \equiv <a, *>.success + <b, *>.\underline{0}$, but $E_2 \not\sqsubseteq_{\mathrm{MT}} E_1$ w.r.t. T. As a consequence, $E_1 \not\sim_{\mathrm{MT}} E_2$ as expected.

Finally, we observe that allowing for internal, exponentially timed actions in tests increases the discriminating power. If we consider $E_1 \equiv <a, \lambda/2>.<b, \mu>.\underline{0} + <a, \lambda/2>.\underline{0}$ and $E_2 \equiv <a, \lambda>.<b, \mu>.\underline{0}$, then $E_1 \sqsubseteq_{\mathrm{MT}} E_2$ w.r.t. $T_1 \equiv <a, *>.<b, *>.success$ while $E_1 \not\sqsubseteq_{\mathrm{MT}} E_2$ w.r.t. $T_2 \equiv <a, *>.(<\tau, \gamma>.success + <b, *>.\underline{0})$.

4.1 Relationship with Classical and Probabilistic Testing

We now show that the Markovian testing preorder is a strict refinement of both the classical testing preorder of [5] and the probabilistic testing preorder of [4]. As far as the classical testing preorder is concerned, this can be reformulated for $\mathrm{EMPA}_{\mathrm{ct}}$ terms by considering their functional semantics.

Definition 8. *Let* $E \in \mathcal{E}_{\mathrm{ct,ndiv}}$ *and* $T \in \mathcal{T}_{\mathrm{ct}}$.

- *An* $\mathcal{F}$*-configuration is a state of* $\mathcal{F}[\![E \parallel_{AType-\{\tau\}} T]\!]$.
- *An* $\mathcal{F}$*-configuration is successful iff its test component is success.*
- *An* $\mathcal{F}$*-computation is a maximal sequence*
$$E \parallel_{AType-\{\tau\}} T \equiv s_0 \xrightarrow{a_1}_{\mathcal{F}} s_1 \xrightarrow{a_2}_{\mathcal{F}} \dots \xrightarrow{a_n}_{\mathcal{F}} s_n$$
where $\mathcal{F}$*-configuration* s_i *is not successful for any* $0 \leq i \leq n-1$.
- *An* $\mathcal{F}$*-computation is successful iff so is its last configuration.*
- *We denote by* $\mathcal{C}_{\mathcal{F}}(E,T)$ *and* $\mathcal{S}_{\mathcal{F}}(E,T)$ *the set of* $\mathcal{F}$*-computations and successful* $\mathcal{F}$*-computations, respectively.* ■

Definition 9. *Let* $E, E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$ *and* $T \in \mathcal{T}_{\mathrm{ct}}$.

- $E\ \mathit{may}\ T$ *iff* $\mathcal{S}_{\mathcal{F}}(E,T) \neq \emptyset$.
- $E\ \mathit{must}\ T$ *iff* $\mathcal{S}_{\mathcal{F}}(E,T) = \mathcal{C}_{\mathcal{F}}(E,T)$.
- $E_1 \sqsubseteq_{\mathrm{may}} E_2$ *iff* $\forall T \in \mathcal{T}_{\mathrm{ct}}.\, E_1\ \mathit{may}\ T \Longrightarrow E_2\ \mathit{may}\ T$.
- $E_1 \sqsubseteq_{\mathrm{must}} E_2$ *iff* $\forall T \in \mathcal{T}_{\mathrm{ct}}.\, E_1\ \mathit{must}\ T \Longrightarrow E_2\ \mathit{must}\ T$.
- $E_1 \sqsubseteq_{\mathrm{T}} E_2$ *iff* $E_1 \sqsubseteq_{\mathrm{may}} E_2 \wedge E_1 \sqsubseteq_{\mathrm{must}} E_2$.
- $E_1 \sim_{\mathrm{T}} E_2$ *iff* $E_1 \sqsubseteq_{\mathrm{T}} E_2 \wedge E_2 \sqsubseteq_{\mathrm{T}} E_1$. ■

Since the classical testing preorder is based on the notion of passing a test, we need an alternative characterization for the Markovian testing preorder which is explicitly based on the quantification of the probability of passing a test within a given amount of time.

Definition 10. *Let* $E, E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$, $T \in \mathcal{T}_{\mathrm{ct}}$, $p \in \mathbb{R}_{[0,1]}$, *and* $t \in \mathbb{R}_+$.

- $E\ \mathit{pass}_{p,t}\ T$ *iff* $\mathit{prob}(\mathcal{S}_{\leq t}(E,T)) \geq p$.
- $E_1 \leq_{\mathrm{MT}} E_2$ *iff* $\forall T \in \mathcal{T}_{\mathrm{ct}}.\, \forall p \in \mathbb{R}_{[0,1]}.\, \forall t \in \mathbb{R}_+.\, E_1\ \mathit{pass}_{p,t}\ T \Longrightarrow E_2\ \mathit{pass}_{p,t}\ T$. ■

Lemma 1. *Let* $E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$. *Then* $E_1 \sqsubseteq_{\mathrm{MT}} E_2 \Longleftrightarrow E_1 \leq_{\mathrm{MT}} E_2$. ■

Lemma 2. *Let* $E \in \mathcal{E}_{\mathrm{ct,ndiv}}$ *and* $T \in \mathcal{T}_{\mathrm{ct}}$. *Then*

(*i*) $E\ \mathit{may}\ T \Longleftrightarrow \exists p \in \mathbb{R}_{[0,1]}.\, \exists t \in \mathbb{R}_+.\, E\ \mathit{pass}_{p,t}\ T$.
(*ii*) $E\ \mathit{must}\ T \Longleftrightarrow \exists t \in \mathbb{R}_+.\, E\ \mathit{pass}_{1,t}\ T$. ■

Theorem 1. *Let* $E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$. *Then* $E_1 \sqsubseteq_{\mathrm{MT}} E_2 \Longrightarrow E_1 \sqsubseteq_{\mathrm{T}} E_2$. ■

The converse of Thm. 1 does not hold, i.e. classical testing is strictly more abstract than Markovian testing. In fact, if we consider $E_1 \equiv {<}a, \lambda{>}.{<}b, \gamma{>}.\underline{0} + {<}a, \mu{>}.{<}c, \delta{>}.\underline{0}$ and $E_2 \equiv {<}a, \mu{>}.{<}b, \gamma{>}.\underline{0} + {<}a, \lambda{>}.{<}c, \delta{>}.\underline{0}$ where $\lambda \neq \mu$, then $E_1 \sim_{\rm T} E_2$ because $\mathcal{F}[\![E_1]\!]$ is isomorphic to $\mathcal{F}[\![E_2]\!]$, but $E_1 \not\sim_{\rm MT} E_2$ w.r.t. $T \equiv {<}a, *{>}.{<}b, *{>}.\mathit{success}$.

The probabilistic testing preorder of [4] is defined as follows.

Definition 11. *Let $E, E_1, E_2 \in \mathcal{E}_{\rm ct,ndiv}$, $T \in \mathcal{T}_{\rm ct}$, and $p \in \mathbb{R}_{[0,1]}$.*

- *$E \; pass_p \; T$ iff $prob(\mathcal{S}(E, T)) \geq p$.*
- *$E_1 \sqsubseteq_{\rm PT} E_2$ iff $\forall T \in \mathcal{T}_{\rm ct}. \forall p \in \mathbb{R}_{[0,1]}. E_1 \; pass_p \; T \Longrightarrow E_2 \; pass_p \; T$.*
- *$E_1 \sim_{\rm PT} E_2$ iff $E_1 \sqsubseteq_{\rm PT} E_2 \wedge E_2 \sqsubseteq_{\rm PT} E_1$.* ■

Theorem 2. *Let $E_1, E_2 \in \mathcal{E}_{\rm ct,ndiv}$. Then $E_1 \sqsubseteq_{\rm MT} E_2 \Longrightarrow E_1 \sqsubseteq_{\rm PT} E_2$.* ■

The converse of Thm. 2 does not hold, i.e. probabilistic testing is strictly more abstract than Markovian testing. In fact, if we consider $E_1 \equiv {<}a, \lambda{>}.\underline{0} + {<}b, \mu{>}.\underline{0}$ and $E_2 \equiv {<}a, 2 \cdot \lambda{>}.\underline{0} + {<}b, 2 \cdot \mu{>}.\underline{0}$, then we have $E_1 \sim_{\rm PT} E_2$ because both transitions with action type a have execution probability $\frac{\lambda}{\lambda+\mu}$ and both transitions with action type b have execution probability $\frac{\mu}{\lambda+\mu}$, but $E_1 \not\sim_{\rm MT} E_2$ w.r.t. $T \equiv {<}a, *{>}.\mathit{success}$.

4.2 Alternative Characterization

In order to ease the task of establishing relationships between Markovian processes, following [13] we define an alternative characterization of the Markovian testing preorder which does not require an analysis of process behavior in response to tests. To simplify the presentation, we consider only the restricted class $\mathcal{T}_{{\rm ct},\tau}$ of tests without internal, exponentially timed actions.

The characterization is based on the notion of extended trace, which is a sequence of pairs where the first component can be interpreted as the set of (passive) actions enabled by the environment (i.e., the test) and the second component can be interpreted as the action which is actually executed. Since a process can contain internal, exponentially timed transitions, there may be several different ways in which a process can execute an extended trace.

Definition 12. *An extended trace is an element of set $\mathit{ETrace}_\tau = \{(M_1, a_1) \dots (M_n, a_n) \in (2^{\mathit{AType}-\{\tau\}} \times (\mathit{AType} - \{\tau\}))^* \mid \forall i = 1, \dots, n. \, a_i \in M_i\}$.* ■

Definition 13. *Let $E \in \mathcal{E}_{\rm ct,ndiv}$ and $\sigma = (M_1, a_1) \dots (M_n, a_n) \in \mathit{ETrace}_\tau$.*

- *A computation of E compatible with σ is a sequence*
$$E \equiv E_0 \xrightarrow{(\tau,\lambda)^*} E'_0 \xrightarrow{a_1,\lambda_1} E_1 \dots E_{n-1} \xrightarrow{(\tau,\lambda)^*} E'_{n-1} \xrightarrow{a_n,\lambda_n} E_n$$
where $\xrightarrow{(\tau,\lambda)^*}$ is a shorthand for finitely many (possibly zero) internal, exponentially timed transitions.
- *We denote by $\mathcal{CC}(E, \sigma)$ the set of computations of E compatible with σ.* ■

Definition 14. *Let* $E \in \mathcal{E}_{\text{ct,ndiv}}$, $\sigma = (M_1, a_1) \dots (M_n, a_n) \in ETrace_\tau$, $c \in \mathcal{CC}(E, \sigma)$, *and* $C \subseteq \mathcal{CC}(E, \sigma)$.

$$- \; prob(c, \sigma) = \begin{cases} 1 & \text{if } c = E_n \\ \frac{\lambda}{\Lambda} \cdot prob(c', \sigma) & \text{if } c = E'_{i-1} \xrightarrow{a_i, \lambda_i} c' \text{ with} \\ & \quad \lambda = \lambda_i \wedge \Lambda = Rate(E'_{i-1}, M_i \cup \{\tau\}, 0, \mathcal{E}_{\text{ct}}) \\ & \text{or } c = E_{i-1} \xrightarrow{\tau, \lambda} c' \text{ with} \\ & \quad \Lambda = Rate(E_{i-1}, M_i \cup \{\tau\}, 0, \mathcal{E}_{\text{ct}}) \end{cases}$$

$$- \; time(c, \sigma) = \begin{cases} 0 & \text{if } c = E_n \\ \frac{1}{\Lambda} + time(c', \sigma) & \text{if } c = E'_{i-1} \xrightarrow{a_i, \lambda_i} c' \text{ with} \\ & \quad \Lambda = Rate(E'_{i-1}, M_i \cup \{\tau\}, 0, \mathcal{E}_{\text{ct}}) \\ & \text{or } c = E_{i-1} \xrightarrow{\tau, \lambda} c' \text{ with} \\ & \quad \Lambda = Rate(E_{i-1}, M_i \cup \{\tau\}, 0, \mathcal{E}_{\text{ct}}) \end{cases}$$

- $prob(C) = \sum_{c \in C} prob(c, \sigma)$.
- $C_{\leq t} = \{c \in C \mid time(c, \sigma) \leq t\}$. ■

Computing the aggregated rates w.r.t. $M_i \cup \{\tau\}$ instead of M_i in Def. 14 allows the internal transitions of a process to be correctly taken into account.

Definition 15. *Let* $E_1, E_2 \in \mathcal{E}_{\text{ct,ndiv}}$.

- $E_1 \ll_{\text{MT}} E_2$ *iff* $\forall \sigma \in ETrace_\tau . \forall t \in \mathbf{R}_+ . \, prob(\mathcal{CC}_{\leq t}(E_1, \sigma)) \leq prob(\mathcal{CC}_{\leq t}(E_2, \sigma))$.
- $E_1 \approx_{\text{MT}} E_2$ *iff* $E_1 \ll_{\text{MT}} E_2 \wedge E_2 \ll_{\text{MT}} E_1$. ■

We now prove that $\ll_{\text{MT}}$ coincides with $\sqsubseteq_{\text{MT}}$ in the case of tests without internal, exponentially timed actions. In other words, we prove that $\ll_{\text{MT}}$ is fully abstract w.r.t. $\sqsubseteq_{\text{MT}}$. In order to prove that $\sqsubseteq_{\text{MT}} \subseteq \ll_{\text{MT}}$, we construct a test from a trace in such a way that both of them have the same probabilistic and timing characteristics.

Definition 16. *We define the set of tests* $\mathcal{T}_{\text{ct},ETrace_\tau} = \{T(\sigma) \in \mathcal{T}_{\text{ct},\tau} \mid \sigma = (M_1, a_1) \dots (M_n, a_n) \in ETrace_\tau\}$ *as follows*

$$T(\sigma) \triangleq T_1(\sigma)$$
$$T_i(\sigma) \triangleq {<}a_i, *{>}.T_{i+1}(\sigma) + \sum_{b \in M_i - \{a_i\}} {<}b, *{>}.\underline{0}, \quad 1 \leq i < n$$
$$T_n(\sigma) \triangleq {<}a_n, *{>}.success + \sum_{b \in M_n - \{a_n\}} {<}b, *{>}.\underline{0}$$
■

Theorem 3. *Let* $E_1, E_2 \in \mathcal{E}_{\text{ct,ndiv}}$. *Then* $E_1 \sqsubseteq_{\text{MT}} E_2 \Longrightarrow E_1 \ll_{\text{MT}} E_2$. ■

In order to prove that $\ll_{\text{MT}} \subseteq \sqsubseteq_{\text{MT}}$, we observe that a test has some number of successful test executions leading from its initial state to one of its successful states. When a process interacts with a test, each successful computation exercises exactly one of the successful test executions. Because of internal, exponentially timed transitions of processes, there may be several successful computations exercising the same successful test execution.

Definition 17. *Let $T \in \mathcal{T}_{\mathrm{ct},\tau}$. An execution ξ of T is a sequence*

$$T \equiv T_0 \xrightarrow{a_1,*} \dots \xrightarrow{a_n,*} T_n$$

with associated extended trace $etrace(\xi) = (M_1, a_1) \dots (M_n, a_n)$ where $M_i = \{a \in AType \mid T_i \xrightarrow{a,*}\}$ for all $1 \le i \le n$. ■

Definition 18. *Let $E \in \mathcal{E}_{\mathrm{ct,ndiv}}$, $T \in \mathcal{T}_{\mathrm{ct},\tau}$, ξ be an execution of T, and $t \in \mathbb{R}_+$.*

- *We denote by $\mathcal{C}(E, T, \xi) = \{c \in \mathcal{C}(E, T) \mid \xi = proj_T(c)\}$ the set of computations exercising ξ, where*

$$proj_T(c) = \begin{cases} T' & \text{if } c = E' \parallel_{AType-\{\tau\}} T' \\ proj_T(c') & \text{if } c = E' \parallel_{AType-\{\tau\}} T' \xrightarrow{a,\lambda} c' \text{ with} \\ & \quad E'' \parallel_{AType-\{\tau\}} T' \text{ first config. of } c' \\ T' \xrightarrow{a,*} proj_T(c') & \text{if } c = E' \parallel_{AType-\{\tau\}} T' \xrightarrow{a,\lambda} c' \text{ with} \\ & \quad E'' \parallel_{AType-\{\tau\}} T'' \text{ first config. of } c' \\ & \quad \text{and } T'' \not\equiv T' \end{cases}$$

- $\mathcal{C}_{\le t}(E, T, \xi) = \{c \in \mathcal{C}_{\le t}(E, T) \mid \xi = proj_T(c)\}$. ■

Theorem 4. *Let $E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$. Then $E_1 \ll_{\mathrm{MT}} E_2 \Longrightarrow E_1 \sqsubseteq_{\mathrm{MT}} E_2$.* ■

Corollary 1. *Let $E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$. Then $E_1 \sqsubseteq_{\mathrm{MT}} E_2 \Longleftrightarrow E_1 \ll_{\mathrm{MT}} E_2$.* ■

Corollary 2. *Let $E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$. Then $E_1 \sqsubseteq_{\mathrm{MT}} E_2 \Longleftrightarrow E_1 \sqsubseteq_{\mathrm{MT}} E_2$ w.r.t. $\mathcal{T}_{\mathrm{ct},ETrace_\tau}$.* ■

The last result means that $\mathcal{T}_{\mathrm{ct},ETrace_\tau}$ is a class of essential tests, i.e. tests that expose all relevant aspects of process behavior.

We now show a proof technique based on the fully abstract characterization above which eases the task of proving $E_1 \sqsubseteq_{\mathrm{MT}} E_2$.

Definition 19. *Let $E \in \mathcal{E}_{\mathrm{ct,ndiv}}$ and $\sigma = (M_1, a_1) \dots (M_n, a_n) \in ETrace_\tau$.*

- *The functional trace semantics of E is the set $\mathcal{FT}[\![E]\!] = \{a_1 \dots a_n \in (AType - \{\tau\})^* \mid E \equiv E_0 \xrightarrow{\tau^*}_{\mathcal{F}} E'_0 \xrightarrow{a_1}_{\mathcal{F}} E_1 \dots E_{n-1} \xrightarrow{\tau^*}_{\mathcal{F}} E'_{n-1} \xrightarrow{a_n}_{\mathcal{F}} E_n\}$.*
- *The trace of σ is the sequence $trace(\sigma) = a_1 \dots a_n$.*
- *The functional extended trace semantics of E is the set $\mathcal{FET}[\![E]\!] = \{\sigma \in ETrace_\tau \mid trace(\sigma) \in \mathcal{FT}[\![E]\!]\}$.* ■

Theorem 5. *Let $E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$. Then $E_1 \ll_{\mathrm{MT}} E_2 \Longleftrightarrow \mathcal{FT}[\![E_1]\!] \subseteq \mathcal{FT}[\![E_2]\!] \wedge \forall \sigma \in \mathcal{FET}[\![E_2]\!]. \forall t \in \mathbb{R}_+. prob(\mathcal{CC}_{\le t}(E_1, \sigma)) \le prob(\mathcal{CC}_{\le t}(E_2, \sigma))$.* ■

On the basis of the theorem above and the full abstraction result of Cor. 1, we have the following proof technique for verifying whether $E_1 \sqsubseteq_{\mathrm{MT}} E_2$:

- Check if $\mathcal{FT}[\![E_1]\!] \subseteq \mathcal{FT}[\![E_2]\!]$.
- Check if $\forall \sigma \in \mathcal{FET}[\![E_2]\!]. \forall t \in \mathbb{R}_+. prob(\mathcal{CC}_{\le t}(E_1, \sigma)) \le prob(\mathcal{CC}_{\le t}(E_2, \sigma))$.

In the case of general tests containing internal, exponentially timed actions, the alternative characterization is given in terms of extended traces which keep track of internal, exponentially timed transitions performed by tests and therefore record not only action types but also action rates. See [2].

4.3 Relationship with Markovian Bisimulation Equivalence

The Markovian testing equivalence is strictly coarser than the Markovian bisimulation equivalence.

Theorem 6. *Let* $E_1, E_2 \in \mathcal{E}_{\mathrm{ct,ndiv}}$. *Then* $E_1 \sim_{\mathrm{EMB}} E_2 \Longrightarrow E_1 \sim_{\mathrm{MT}} E_2$. ■

The converse of Thm. 6 does not hold. In fact, if we consider terms

$$E_1 \equiv <a, \lambda_1>.<b, \mu>.<c, \gamma>.\underline{0} + <a, \lambda_2>.<b, \mu>.<d, \delta>.\underline{0}$$
$$E_2 \equiv <a, \lambda>.(<b, \mu_1>.<c, \gamma>.\underline{0} + <b, \mu_2>.<d, \delta>.\underline{0})$$

where $\lambda_1 = \lambda_2 = \lambda/2$, $\mu_1 = \mu_2 = \mu/2$, and $\gamma \leq \delta$, then it can be proved that $E_1 \sim_{\mathrm{MT}} E_2$ with the proof technique of the previous section, but $E_1 \not\sim_{\mathrm{EMB}} E_2$.

It can be shown that in general $\sim_{\mathrm{EMB}}$ is not a congruence for EMPA; in particular, the replacement of a parallel component by a Markovian bisimilar one will not in general preserve $\sim_{\mathrm{EMB}}$. When certain reasonable restrictions are imposed on the degree of internal nondeterminism among passive actions of the same type, however, this difficulty disappears [1]. It also turns out that essential tests characterizing $\sim_{\mathrm{MT}}$ obey this restriction; consequently, the testing theory of this paper is not affected by the technical shortcomings of $\sim_{\mathrm{EMB}}$ in EMPA.

5 An Alternative Approach to Markovian Testing

The notion of efficiency for Markovian processes developed in the previous sections is based on the probability of executing a successful computation whose average duration is not greater than a given amount of time. This quantification of the probability with which tests are passed within a given amount of time has been chosen because it is relatively easy to work with. On the other hand, it is not the most intuitive one, because it is different from the probability of reaching success within a given amount of time. The problem with this quantification is that it involves linear combinations of hypoexponential distributions [11] instead of numbers, so it is more difficult to deal with.

In this section we formalize the Markovian testing preorder arising from the latter quantification and we show that the Markovian testing equivalences induced by the two different quantifications coincide. This relationship between the two different quantifications, although partial in that it does not relate the two preorders, is important because it justifies the development of the testing theory for the less intuitive quantification and allows us to automatically derive that the properties we have proved for (the easier to work with) $\sim_{\mathrm{MT}}$ hold for the alternative Markovian testing equivalence as well.

Let us preliminarily redefine our Markovian testing preorder in the following equivalent way.

Definition 20. *Let* $E \in \mathcal{E}_{\mathrm{ct,ndiv}}$, $T \in \mathcal{T}_{\mathrm{ct}}$, $c \in \mathcal{C}(E, T)$, $C \subseteq \mathcal{C}(E, T)$, *and* $t \in \mathbf{R}_+$.

– $$\mathit{prob}(c) = \begin{cases} 1 & \text{if } c = s \\ \frac{\lambda}{\Lambda} \cdot \mathit{prob}(c') & \text{if } c = s \xrightarrow{a,\lambda} c' \text{ with } \Lambda = \mathit{Rate}(s, \mathit{AType}, 0, \mathcal{E}_{\mathrm{ct}}) \end{cases}$$

– $time(c) = \begin{cases} 0 & \text{if } c = s \\ \frac{1}{\Lambda} + time(c') & \text{if } c = s \xrightarrow{a,\lambda} c' \text{ with } \Lambda = Rate(s, AType, 0, \mathcal{E}_{\text{ct}}) \end{cases}$
– $prob(C, t) = \sum_{c \in C} prob(c) \cdot \Pr(time(c) \leq t)$. ■

Note that, for all computations c, $\Pr(time(c) \leq t) \in \{0, 1\}$. Thus $prob(C, t)$ represents the probability of executing a computation in C whose average duration is not greater than t.

Definition 21. *Let* $E_1, E_2 \in \mathcal{E}_{\text{ct,ndiv}}$.

– $E_1 \sqsubseteq_{\text{MT}} E_2$ *iff* $\forall T \in \mathcal{T}_{\text{ct}}. \forall t \in \mathbb{R}_+. prob(\mathcal{S}(E_1, T), t) \leq prob(\mathcal{S}(E_2, T), t)$.
– $E_1 \sim_{\text{MT}} E_2$ *iff* $E_1 \sqsubseteq_{\text{MT}} E_2 \wedge E_2 \sqsubseteq_{\text{MT}} E_1$. ■

The alternative Markovian testing preorder is defined as follows.

Definition 22. *Let us denote by* X_Λ *an exponentially distributed random variable with rate* Λ. *Let* $E \in \mathcal{E}_{\text{ct,ndiv}}$, $T \in \mathcal{T}_{\text{ct}}$, $c \in \mathcal{C}(E, T)$, $C \subseteq \mathcal{C}(E, T)$, *and* $t \in \mathbb{R}_+$.

– $time'(c) = \begin{cases} 0 & \text{if } c = s \\ X_\Lambda + time'(c') & \text{if } c = s \xrightarrow{a,\lambda} c' \text{ with } \Lambda = Rate(s, AType, 0, \mathcal{E}_{\text{ct}}) \end{cases}$
– $prob'(C, t) = \sum_{c \in C} prob(c) \cdot \Pr(time'(c) \leq t)$. ■

Observe that $prob'(C, t)$ is the probability of reaching the end of a computation in C within time t. This probability is obtained by summing the products stemming from the multiplication of the probability of performing a given computation in C by the probability of completing that computation within time t, where the latter quantity follows a hypoexponential distribution (see the definition of $time'$).

Definition 23. *Let* $E_1, E_2 \in \mathcal{E}_{\text{ct,ndiv}}$.

– $E_1 \preceq_{\text{MT}} E_2$ *iff* $\forall T \in \mathcal{T}_{\text{ct}}. \forall t \in \mathbb{R}_+. prob'(\mathcal{S}(E_1, T), t) \leq prob'(\mathcal{S}(E_2, T), t)$.
– $E_1 \simeq_{\text{MT}} E_2$ *iff* $E_1 \preceq_{\text{MT}} E_2 \wedge E_2 \preceq_{\text{MT}} E_1$. ■

Theorem 7. *Let* $E_1, E_2 \in \mathcal{E}_{\text{ct,ndiv}}$. *Then* $E_1 \simeq_{\text{MT}} E_2 \Longleftrightarrow E_1 \sim_{\text{MT}} E_2$. ■

6 Impossibility Result

Given a Markov chain, performance measures on it are typically defined through rewards [10], so it would be desirable to define a Markovian preorder consistently with reward based performance measures. A performance measure for a Markov chain can be defined (in a simplified form) as a weighted sum $R = \sum_{i \in S} \rho_i \cdot \pi_i$ of the state probabilities $\boldsymbol{\pi}$ of the Markov chain where weights $\boldsymbol{\rho}$ are expressed through real numbers, called rewards, associated with states. Unfortunately, the following counterexamples show that it is not possible to define a Markovian preorder which is consistent with reward based performance measures. Although we shall concentrate on steady state, instant-of-time performance measures, the impossibility result holds true also for transient state, instant-of-time performance measures as well as transient state, interval-of-time performance measures.

The first counterexample shows that, as one might expect, it is not possible to define a Markovian preorder which is consistent with all the reward based performance measures. Consider the two functionally equivalent terms E and F described below together with the related steady state probabilities:

$$E_1 \stackrel{\Delta}{=} {<}a, \lambda{>}.E_2 \quad E_2 \stackrel{\Delta}{=} {<}b, \mu{>}.E_1 \quad \pi_{E_1} = \mu/(\lambda + \mu) \quad \pi_{E_2} = \lambda/(\lambda + \mu)$$

$$F_1 \stackrel{\Delta}{=} {<}a, \gamma{>}.F_2 \quad F_2 \stackrel{\Delta}{=} {<}b, \delta{>}.F_1 \quad \pi_{F_1} = \delta/(\gamma + \delta) \quad \pi_{F_2} = \gamma/(\gamma + \delta)$$

and assume $\lambda \leq \gamma$ and $\mu \leq \delta$. Since the steady state probabilities of a Markov chain sum up to 1, we cannot have that $\pi_{E_1} \leq \pi_{F_1}$ and $\pi_{E_2} \leq \pi_{F_2}$. Suppose then $\pi_{E_1} \leq \pi_{F_1}$ and $\pi_{E_2} \geq \pi_{F_2}$. If we consider a performance measure defined through rewards $\rho_{E_1} = \rho_{F_1} = 1$ and $\rho_{E_2} = \rho_{F_2} = 0$, then $R_E = \pi_{E_1} \leq \pi_{F_1} = R_F$. If we consider instead a performance measure defined through rewards $\rho_{E_1} = \rho_{F_1} = 0$ and $\rho_{E_2} = \rho_{F_2} = 1$, then $R_E = \pi_{E_2} \geq \pi_{F_2} = R_F$.

The second counterexample shows that we do not succeed in defining a Markovian preorder even if we focus on a single reward based performance measure. Consider again terms E and F above and the performance measure defined through rewards $\rho_{E_1} = \rho_{F_1} = 1$ and $\rho_{E_2} = \rho_{F_2} = 0$. If $\lambda = 1$, $\gamma = 2$, $\mu = 3$, and $\delta = 4$, then $R_E = 0.75 > 0.\bar{6} = R_F$. If instead $\lambda = 1$, $\gamma = 1$, $\mu = 3$, and $\delta = 4$, then $R_E = 0.75 < 0.8 = R_F$.

The problem with the definition of a Markovian preorder which takes reward based performance measures into account is essentially that they are based on state probabilities and these are normalized to 1. This somehow causes state probabilities of different processes to be incomparable. This problem is further exacerbated by the fact that state probabilities depend on equations involving the global knowledge of all the rates, so it is not possible to rely on local information such as the relationships between rates of corresponding transitions. The only observation that can be made about E and F above is that, if we relate E_1 with F_1 and E_2 with F_2, then upon executing a the evolution of F_1 into F_2 is faster (on average) than the evolution of E_1 into E_2, because $\lambda \leq \gamma$, and upon executing b the evolution of F_2 into F_1 is faster (on average) than the evolution of E_2 into E_1, because $\mu \leq \delta$. This is exactly what is captured by our Markovian preorder based on a generic notion of efficiency.

7 Conclusion

In this paper we have developed a testing theory for Markovian processes and we have provided a fully abstract characterization of it which simplifies the task of establishing preorder relationships between Markovian processes. We have also justified why we have considered a generic notion of efficiency instead of reward based performance measures, and why we have characterized the generic notion of efficiency through average durations instead of duration distributions.

As far as future work is concerned, an open problem is to establish whether a result like Thm. 7 holds for preorders as well. Moreover, we would like to investigate compositionality related issues for the Markovian testing preorder as well as find a sound and complete axiomatization. Additionally, a more denotational characterization based entirely on the structure of Markovian processes should be studied. A good starting point may be [12] where the acceptance tree model of [5] is adapted to probabilistic processes. Such a characterization would help devising algorithms for determining preorder relationships between Markovian processes, like in the case of the classical testing theory [3].

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